Asymptotic Behavior of Solutions of a Stefan Problem

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1. Introduction

This note might well bear the title "How long to thaw a chicken?" It is concerned with the asymptotic behavior of

$$\theta(x, t) = \int_0^t \left[ \int_{T_m}^{T(x, t)} K(s) \, ds \right] \, dt,$$

for fixed $x$ and large $t$, where $T(x, t)$ is the temperature at a point $x$ at time $t$ of a homogeneous conducting medium of thermal conductivity $K(T)$ subject to a change of phase at a certain temperature $T = T_m$. The temperature satisfies equations of the form (see [1, 2]),

$$\text{div}[K(T) \text{ grad } T] = \rho c(T) (\partial T/\partial t) \quad (1.1)$$

in the time dependent regions where $T < T_m$ and $T > T_m$, and conditions

$$T = T_m, \quad \rho L (\partial S/\partial t) = [K(T^+) \text{ grad } T^+ - K(T^-) \text{ grad } T^-] \cdot \text{ grad } S \quad (1.2)$$

on the melting surface $S = 0$ separating the regions of differing phase. Here, $L$ is the increased enthalpy per unit mass due to the phase change, $c(T)$ is the specific heat per unit mass, and $\rho$ the density of the homogeneous isotropic medium. The functions $K(T), c(T)$ are assumed to be positive, bounded away from zero and continuous in $T$ except at $T_m$. The energy content $e(x, t)$ per unit volume of the medium above that of its "frozen" state at the "melting" temperature $T_m$ is given by

$$e(x, t) = \int_{T_m}^T \rho c(s) \, ds \quad \text{for } T < T_m$$

$$= \rho L + \int_{T_m}^T \rho c(s) \, ds \quad \text{for } T > T_m. \quad (1.3)$$
When $T = T_m$, $\varepsilon$ is zero if the neighborhood of the point is frozen and it is $\rho L$ if the neighborhood has thawed. In Eq. (1.2) $K(T^+)$ grad $T^+$ denotes the limiting value of $K(T)$ grad $T$ as the point of $S = 0$ in question is approached from inside the region $T > T_m$ and $K(T^-)$ grad $T^-$ is defined as a similar limit from the frozen region $T < T_m$.

Initially, $\varepsilon$ is given as $\varepsilon_0(x)$ in the region $G$ of finite diameter $l$, occupied by the medium, and Dirichlet type boundary conditions

$$T(x, t) = f(x, t), \quad x \in \partial G, \quad t > 0$$

are assumed given on the boundary $\partial G$ of $G$.

The function $\theta(x, t)$ is shown to satisfy the nonlinear parabolic equation

$$\nabla^2 \theta = \varepsilon - \varepsilon_0 = F(\partial \theta/\partial t),$$

where $F(y)$ is a strictly increasing function of $y$ defined by Eq. (1.3) and

$$y = \int_{T_m}^T K(s) \, ds, \quad F(y) = \varepsilon - \varepsilon_0,$$

for $y \neq 0$ and $F(0) = 0$. Initially, $\theta = 0$, and on $\partial G$,

$$\theta(x, t) = \int_0^t \left( \int_{T_m}^{f(x, t)} K(s) \, ds \right) dt.$$

It is assumed that $f(x, t)$ is uniformly continuous in $x$ on $\partial G$ and tends to $f_\infty(x) > T_m$ as $t$ tends to infinity in a manner compatible with

$$|f_\infty(x) - f(x, t)| < A/(1 + t)^\alpha$$

for some $A > 0$ and $\alpha > 1$.

It is shown that $\theta$ is asymptotic to $\theta_\infty$ where

$$\theta_\infty(x, t) = t\psi_\infty(x) - \phi_\infty(x)$$

and $\psi_\infty$ and $\phi_\infty$ are given by the boundary value problems;

$$\nabla^2 \psi_\infty = 0, \quad \nabla^2 \phi_\infty + F(\psi_\infty) = 0 \text{ in } G.$$

$$\psi_\infty(x) = \int_{T_m}^{f_\infty(x)} K(s) \, ds, \quad \phi_\infty(x) = \int_0^\infty \int_{f(x, t)} K(s) \, ds \, dt, \quad x \in \partial G.$$

Since $F(y)$ is a strictly increasing function of $y$, the parabolic operator

$$E(\theta) = \nabla^2 \theta - F(\partial \theta/\partial t)$$

(1.9)
is associated with a comparison theorem of the usual form [1, p. 52], despite
the discontinuity in $F(y)$ at $y = 0$. Solutions of Eq. (1.9) for which $\theta = 0$
at $t = 0$ and which have bounded derivatives $\partial^2 \theta / \partial x^2$, $\partial \theta / \partial t$ are therefore
uniquely specified by given Dirichlet type boundary conditions

$$\theta(x, t) = \int_0^t \left( \int_{T_m} f(x, s) K(s) ds \right) dt \quad \text{for} \quad x \in \partial G, \quad t > 0. \quad (1.10)$$

In certain problems, the asymptotic form (1.7) of $\theta$ can be used to give
an estimate of the time taken to change the phase of $G$ under conditions
(1.10). Thus, suppose $G$ is initially frozen so that $e_0 \leq 0$, and that $f(x, t) \leq f_\infty(x)$ for all $t > 0$, $x \in \partial G$. Then, since

$$\theta(x, t) = \int_0^t \int_{T_m} f(x, s) K(s) ds dt - \int_0^t \int_{f(x, t)} f(x, s) K(s) ds dt$$

we have

$$\theta(x, t) \geq \theta_\infty(x, t), \quad x \in \partial G, \quad t > 0.$$ 

Also, $\psi_\infty(x) > 0$ on $\partial G$ and hence in $G$, so that $F(\psi_\infty) = e_\infty - e_0 > 0$ in $G$
and since $\phi_\infty(x) \geq 0$ on $\partial G$ then $\phi_\infty(x) > 0$ in $G$. Hence $\theta_\infty(x, 0) = -\phi_\infty(x) \leq 0$
and therefore $\theta(x, 0) = 0 \geq \theta_\infty(x, 0)$. Finally, $E(\theta_\infty) = F(\psi_\infty) - F(\phi_\infty) = 0 = E(\theta)$ and the comparison theorem implies $\theta(x, t) \geq \theta_\infty(x, t)$ in $G$ for
t $> 0$. An upper bound for the time $t_\infty$ at which a complete phase change is
attained in $G$ is therefore given by $t_\infty$, the time for which $\min_{x \in G} \theta_\infty(x, t_\infty) = 0$,
$\theta_\infty(x, t_\infty) \geq 0$ in $G$, evidently given by the expression

$$t_\infty = \sup_{x \in G} \left[ \frac{\phi_\infty(x)}{\psi_\infty(x)} \right].$$

A lower bound for $t_\infty$ may sometimes be obtained as follows: Let $\theta^*$
be the solution of an analogous melting problem differing from that describing
$\theta$ only in that the specific heat $c^*(T)$ is reduced for all $T$. Then $\theta^* = \theta$
at $t = 0$ and on $\partial G$ for $t > 0$ and

$$E^*(\theta^*) = 0 = E(\theta^*) + \int_{T_m} [c(s) - c^*(s)] ds$$

so that

$$E(\theta^*) \leq 0 = E(\theta) \quad \text{for} \quad x \text{ in } G, \quad t > 0.$$
and hence $\theta^* \geq \theta$ in $G$ for $t > 0$ and $t_m^* \leq t_m$. The special limiting case $c^*(T) \equiv 0$ may give a value for $t_m^*$ as in the example below.

Consider the special problem where $T = T_m$ and $e_0 = 0$ in $G$ at $t = 0$, $T = T_1$ on $\partial G$ for all $t > 0$ and $K(T)$ and $c(T)$ are constant. This can be transformed to a dimensionless form in new variables defined as follows:

$$x = a\xi, \quad t = [\rho La^2/K(T_1 - T_m)]\tau, \quad T = T_m + (T_1 - T_m)\psi, \quad \theta = \psi, \quad \epsilon = c(T_1 - T_m)/L.$$ 

If $\nabla^2$ denotes the Laplacian in the new variables and $G$ the dimensionless region, $\bar{\theta}$ satisfies

$$\nabla^2 \bar{\theta} = 0 \text{ in } G \quad \text{where} \quad (\partial \bar{\theta}/\partial \tau) = 0$$

$$= 1 + \epsilon(\partial \bar{\theta}/\partial \tau) \text{ in } G \quad \text{where} \quad (\partial \bar{\theta}/\partial \tau) > 0.$$ 

On the boundary $\partial G$, $\bar{\theta} = \tau$ and at $\tau = 0$, $\bar{\theta} = 0$ in $\bar{G}$. When $\epsilon \ll 1$, $\theta$ is given approximately by the system

$$\nabla^2 \bar{\theta} = 0 \text{ in } G \quad \text{where} \quad \bar{\theta} = 0$$

$$= 1 \text{ in } G \quad \text{where} \quad \bar{\theta} > 0.$$ 

$\bar{\theta} = 0$ in $G$ at $\tau = 0$ and $\bar{\theta} = \tau$ on $\partial G$ for $\tau > 0$. At $\tau = \tau_m$, $\nabla^2 \bar{\theta} = 1$ in $G$, $\bar{\theta} = \tau_m$ on $\partial G$, and $\min_{x \in G} \bar{\theta} = 0$.

Consider as an example, the case where $G$ is the ellipsoid

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1.$$ 

At the moment $\tau = \tau_m$ of thawing

$$\bar{\theta} = A + B(x^2/a^2 + y^2/b^2 + z^2/c^2)$$ 

in $G$ say. Since $\nabla^2 \bar{\theta} = 2B(1/a^2 + 1/b^2 + 1/c^2) = 1$, we have

$$B = 1/2(1/a^2 + 1/b^2 + 1/c^2).$$

On $\partial G$, $\bar{\theta} = A + B = \tau_m$ and since $\min_{x \in G} \theta = A = 0$ we have $\tau_m = B = 1/2(1/a^2 + 1/b^2 + 1/c^2)$.

2. MAIN RESULTS

The Eqs. (1.1) and (1.2) of our moving boundary problem derive from a more fundamental result describing the conservation of energy in an
arbitrary volume $V \subset G$. The Energy content $E_v$ of this volume at time $t$ is given by

$$E_v = \int_V e \, dv. \quad (2.1)$$

On the other hand, the flux of energy into the region through the boundary of $V$ is given by the surface integral

$$\oint_V K(T) \, \nabla T \cdot d\sigma.$$ 

Since energy is conserved we have

$$\frac{\partial E_v}{\partial t} = \oint_V K(T) \, \nabla T \cdot d\sigma = \frac{\partial}{\partial t} \int_V e \, dv. \quad (2.2)$$

Equations (1.1) and (1.2) are a consequence of this basic formulation. If Eq. (2.2) is integrated from $0$ to $t$ we get

$$\int_V [e(x, t) - e(x)] \, dv = \oint_V \int_0^t K(T) \, \nabla T \cdot d\sigma = \int_V \nabla^2 \theta \, dv,$$ 

where $\theta = \int_0^t \{\int_{T_m} K(s) \, ds\} \, dt$. The volume $V$ is arbitrary and so

$$\nabla^2 \theta = e - e_0, \quad \frac{\partial \theta}{\partial t} = \int_{T_m} K(s) \, ds. \quad (2.3)$$

The energy $e$ is given by

$$e(x, t) = \int_{T_m}^{T(x, t)} \rho c(s) \, ds \quad \text{if} \quad T < T_m$$

$$= \rho L + \int_{T_m}^{T(x, t)} \rho c(s) \, ds \quad \text{if} \quad T > T_m. \quad (2.5)$$

The relations (2.4) and (2.5) define $e - e_0$ as a strictly increasing function of $\partial \theta / \partial t$ except where $\partial \theta / \partial t$ vanishes and $T = T_m$. However if $x, t_1$ is a point where $T = T_m$ and it is also in the interior of a thawed or frozen region where $T = T_m$, then the strong maximum principle [1, p. 39] for Eq. (1.1) implies $T = T_m$ and $e$ is constant in this region for $t < t_1$ so that $e - e_0 = 0$. If we write $e - e_0 \equiv F(\partial \theta / \partial t)$ where $F(x)$ is obtained by eliminating $T$ from the equations

$$x = \int_{T_m}^T K(s) \, ds, \quad F(x) = \int_{T_m}^T \rho c(s) \, ds - e_0, \quad T < T_m$$

$$F(x) = \rho L + \int_{T_m}^T \rho c(s) \, ds - e_0 \quad \text{for} \quad T > T_m \text{ and } F(0) \equiv 0,$$ 

(2.6)
then
\[ \nabla^2 \theta = F(\dot{\theta}; \dot{t}) , \tag{2.7} \]
except possibly at certain points on the boundary between the frozen and thawed zones. These points are strictly speaking really neither frozen nor thawed and the energy per molecule located at these boundary points is intermediate between energies per molecule in the interior of frozen states and that in the interior of thawed states. The problem can be seen to be a limiting case of that where
\[
e(x, t) = \int_{T_m}^{T} \rho c(s) \, ds \quad \text{if } T \leq T_m
\]
\[
= (\rho L / \epsilon)(T - T_m) \quad \text{if } T_m < T \leq T_m + \epsilon
\]
\[
= \rho L + \int_{T_m + \epsilon}^{T} \rho c(s) \, ds \quad \text{for } T > T_m + \epsilon
\]
i.e.,
\[
e(x, t) = \int_{T_m}^{T} \rho c^*(s) \, ds \quad \text{for all } T,
\]
where \( c^*(s) = L / \epsilon \) in the range \( (T_m, T_m + \epsilon) \).

It will best serve the limited objectives of this note to assume the temperature distributions in the problems considered give rise to functions \( \theta(x, t) \) defined in the closure of \( \Omega \), \( \{ x, t: x \in G, 0 < t < \tau \} \) for any \( \tau > 0 \) and which are continuous in \( \overline{Q}_r \), and have bounded derivatives of the second order in the spatial coordinates and of the first order in \( t \) satisfying Eq. (2.7) in \( \Omega_r \) for all \( \tau > 0 \).

It is assumed \( c(s) \) and \( K(s) \) are bounded in the following manner
\[
0 < k \leq K(s) \leq K, \quad 0 < c \leq c(s) \leq C, \quad 0 < m \leq [\rho c(s)/K(s)] \leq M \tag{2.8}
\]
for all \( s \), so that for \( x \neq 0 \),
\[
dF/dx = \rho[C(T)/K(T)] \quad \text{where } x = \int_{T_m}^{T} K(s) \, ds \tag{2.9}
\]
and hence \( F(x) \) has a bounded derivative for \( x \neq 0 \) satisfying
\[
m \leq dF/dx \leq M.
\]

Boundary conditions of the following form are assumed to be given for \( \theta \);
\[
\theta = 0 \text{ in } G \text{ at } t = 0, \quad \theta(x, t) = \int_0^t \int_{T_m}^{f'(x,t)} K(s) \, ds \, dt, \quad x \in \partial G, \quad t > 0. \tag{2.10}
\]
For simplicity, \( f(x, t) \) is assumed to be uniformly continuous in \( \partial G \) and tending to \( f(x, t) \) as \( t \) tends to infinity in such a way that
\[
|f(x) - f(x, t)| \leq A(1 + t)^\alpha, \quad \alpha > 1, \quad A > 0, \quad x \in \partial G, \quad t > 0, \quad (2.11)
\]
for some constants \( A \) and \( \alpha \).

For \( x \in \partial G \) and \( t > 0 \) we may write
\[
\theta(x, t) = \int_0^t \int_{\Gamma_m} K(s) ds \, dt - \int_0^t \int_{f(x, t)} K(s) ds \, dt
\]
\[
= t \psi_\infty(x) - \phi(x, t) \quad \text{say.}
\]

Now for all \( t_1, t_2 > T \) say, \( \phi(x, t) \) is subject to
\[
|\phi(x, t_1) - \phi(x, t_2)| \leq \int_{t_1}^{t_2} [KA/(1 + t)^\alpha] \, dt < [2KA/(\alpha - 1)(1 + T)^{\alpha-1}].
\]
Thus the uniformly continuous \( \phi(x, t) \) converges uniformly to a continuous function \( \phi_\infty(x) \) on \( \partial G \) given by
\[
\phi_\infty(x) = \lim_{t \to \infty} \phi(x, t) = \psi_\infty(x).
\]

Thus, on \( \partial G \) \( \theta \) is asymptotic to \( t \psi_\infty - \phi_\infty \). We show this behavior also holds in \( G \) where \( \psi_\infty(x) \) and \( \phi_\infty(x) \) are extended by the boundary value problems,
\[
\begin{align*}
\nabla^2 \psi_\infty &= 0 \text{ in } G, & \psi_\infty(x) &= \int_{\Gamma_m} K(s) \, ds \text{ on } \partial G \\
\nabla^2 \phi_\infty &= -F(\phi_\infty) \text{ in } G, & \phi_\infty(x) &= \int_0^\infty \int_{f(x, t)} K(s) \, ds \, dt \text{ on } \partial G.
\end{align*}
\]

Despite the discontinuity at \( x = 0 \) in \( F(x) \), the solutions of Eq. (2.7) with bounded derivatives satisfy the comparison theorem that follows.

**Theorem 2.1.** If \( \theta_1, \theta_2 \) have bounded derivatives \( \partial^i \theta_1/\partial x_2^i, \partial \theta_1/\partial t, i = 1, 2, \ldots, n \) where \( x = (x_1, x_2, \ldots, x_n) \), in \( \Omega_\tau \) for all \( \tau > 0 \), are continuous in \( \Omega_\tau \), and satisfy the inequalities
\[
\theta_1 \leq \theta_2 \quad \text{at } t = 0 \quad \text{and on } \partial G \quad \text{for } \quad 0 \leq t \leq \tau,
\]
\[
E(\theta_1) \geq E(\theta_2) \quad \text{in } \Omega_\tau \quad \text{where } \quad E(y) = \nabla^2 y - F(\partial y/\partial t),
\]
then
\[
\theta_1 \leq \theta_2 \quad \text{in } \Omega_\tau.
\]
Proof. If $\theta_1 = \theta_2 + \lambda$ at a point $x_1 \in G$ at $t_1 \in (0, \tau]$ where $\lambda > 0$, then $\theta_1 \geq \theta_2 + \lambda e^{(t-t_1)} = \theta_2^*$ at least at $x_1$ in $\Omega_{t_1}$, while $\theta_1 < \theta_2^*$ in $G$ at $t = 0$ and on $\partial G \times [0, t_1]$.

Hence there is a point $x^* \in G$ and a $t^* \in (0, t_1]$ such that

$$\theta_1 \leq \theta_2^* \quad \text{in} \quad \Omega_{t^*} \quad \text{and} \quad \theta_1(x^*, t^*) = \theta_2^*(x^*, t^*).$$

At $(x^*, t^*)$,

$$\frac{\partial \theta_1}{\partial t} \geq \frac{\partial \theta_2^*}{\partial t} = \left(\frac{\partial \theta_2}{\partial t} + \lambda e^{(t-t_1)}\right), \quad \nabla^2 \theta_1 \leq \nabla^2 \theta_2^* = \nabla^2 \theta_2$$

$$E(\theta_1) - E(\theta_2) = (\nabla^2 \theta_1 - \nabla^2 \theta_2^*) - [F(\frac{\partial \theta_1}{\partial t}) - F(\frac{\partial \theta_2^*}{\partial t}) - \lambda e^{(t-t_1)}] < 0$$

since $F(x_1) > F(x_2)$ for $x_1 > x_2$. This violates the assumptions and so $\theta_1 \leq \theta_2$ in $\Omega_\tau$. Q.E.D.

This theorem is the basic tool used in deriving the asymptotic result below.

**Theorem 2.2.** If $\theta(x, t)$ is a solution of Eq. (2.7) with bounded derivatives $\frac{\partial \theta}{\partial x \alpha^2}, \frac{\partial \theta}{\partial t}$ (in $\Omega_\tau$ for all $\tau > 0$ satisfying boundary conditions of the form (2.10), (2.11), then

$$\theta(x, t) \sim t \psi_\infty(x) - \phi_\infty(x), \quad x \in G,$$

as $t$ tends to infinity where $\psi_\infty(x), \phi_\infty(x)$ are given by (2.13).

Proof. Let

$$B = \sup_{x \in G} (|\psi_\infty(x)|, KA/(\alpha - 1)), \quad b = \inf_{x \in G} \psi_\infty(x).$$

The constant $b > 0$ since $f_\infty(x)$ is assumed bounded away from $T_m$ and so $\psi_\infty(x) > 0$ and bounded away from zero on $\partial G$. If $l$ is the diameter of $G$ and $0 < m \leq \rho c/K \leq M$, set

$$p = \min(b/2B, \pi^2/4l^2M)$$

and define $v(x)$ by $\nabla^2 v + M p v = 0$ in $G$, $v = 1$ on $\partial G$. Comparing $v$ with $\omega = \cos(\pi x_1/2l)/\cos \pi/4$ where the origin has been chosen so that $\{x: x_1 = \pm l/2\}$ lie outside $G$, we see that $\omega \geq v = 1$ on $\partial G$ and since

$$\nabla^2 \omega + \frac{\pi^2}{4l^2} \omega = 0 \quad \text{in} \quad G \quad \text{and} \quad M p \leq \pi^2/4l^2,$$

then $\omega \geq v$ in $G$ and so $v < 2$ in $G$. Define the function;

$$u(x, t) = h(t) v(x) \quad \text{where} \quad h(t) = \max[Be^{-pt} \times [KA/(\alpha - 1)(1 + t)^{x-1}]], \quad x \in G, \quad t > 0$$
and observe that $h(0) = B$, $dh/dt = -q(t)h$ where $q \leq p$, and $|dh/dt| \leq Bp < b/2$. Thus $|\partial u/\partial t| < b$ in $G$ for all $t > 0$.

Consider the comparison function

$$\bar{\theta}(x, t) \equiv t\psi_\infty - \phi_\infty + u.$$ 

On $\partial G$, $t > 0$;

$$\bar{\theta}(x, t) = t \cdot \int_{T_m}^{f_\infty(x)} K(s) \, ds - \int_0^t \int_{f(x,t)}^{f_\infty(x)} K(s) \, ds \, dt + h(t)$$

$$\geq \int_0^t \int_{T_m}^{f_\infty(x)} K(s) \, ds - \int_t^{f_\infty(x)} K(s) \, ds \, dt + \frac{KA}{(x-1)(1+t)^{x-1}}$$

$$\geq \theta(x, t).$$

At $t = 0$, $x \in G$;

$$\bar{\theta}(x, 0) = -\phi_\infty + Bv$$

$$\geq B - \phi_\infty \geq 0 = \theta(x, 0).$$

Finally for $t > 0$ and $x \in G$,

$$E(\bar{\theta}) = -V^2\phi_\infty + V^2u - F(\psi_\infty + \partial u/\partial t)$$

$$\leq V^2u - M(\partial u/\partial t),$$

since $|\partial u/\partial t| < b$ and hence $\psi_\infty + \partial u/\partial t > 0$. Thus

$$E(\bar{\theta}) \leq -hMpv + Mqvh = -Mhv(p - q) \leq 0 = E(\theta)$$

in $\Omega$, for any $\tau > 0$, and Theorem 2.1 shows

$$\bar{\theta} \geq \theta \text{ in } \Omega, \text{ for all } \tau > 0.$$

Next, consider the comparison function $\bar{\theta}$ defined by

$$\bar{\theta} = t\psi_\infty - \phi_\infty - u.$$ 

It can be shown in similar fashion that $\bar{\theta} \leq \theta$ in $\Omega$, for any $\tau > 0$, so that

$$t\psi_\infty - \phi_\infty - u \leq \theta(x, t) \leq t\psi_\infty - \phi_\infty + u \text{ in } \Omega, \tau > 0.$$ 

Now $\sup_{x \in G} |u(x, t)|$, tends to zero as $t$ tends to infinity and so $\theta(x, t)$ is asymptotic to $t\psi_\infty - \phi_\infty$.

Q.E.D.
As in a previous paper concerning conduction problems with no phase change [3], the time $t^*$ defined as

$$t^* = \sup_{x \in G} [\phi_e(x); \psi_e(x)]$$

may be regarded as a measure of when the action occurs in the process of attaining final equilibrium where $e = e_\infty$ in $G$.

REFERENCES