Finiteness Questions in Quasi-Symmetric Designs

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Quasi-symmetric designs with block intersection numbers 0 and \( y \geq 2 \) are considered. It is shown that the number of such designs is finite under any one of the following two restrictions: (1) The block size \( k \) is fixed. (2) The integer pair \((\tilde{e}, z)\), with the following property is fixed: the number of blocks disjoint from a given block is at most \( \tilde{e} \) and the positive block intersection number \( y \) is at most \( z \). The connection of these results with a well-known conjecture on symmetric designs is discussed.

1. Introduction

Let \( D \) be a design (2-design or a BIBD) with the standard parameters. \( D \) is called proper quasi-symmetric if it has exactly two block intersection numbers \( x \) and \( y \) with \( x < y \); this paper restricts itself to \( x = 0 \) and \( y \geq 2 \). Construct the block graph \( \Gamma \) on the blocks of \( D \) by joining two vertices if the corresponding blocks meet (i.e., intersect in \( y \) points). As is well known, \( \Gamma \) is strongly regular and the block intersection number \( y \) divides the block size \( k \) (see, for example, [5]). We shall denote the integer \( k/y \) by \( m \). Observe that this result is false for a improper quasi-symmetric design which is simply a symmetric design.

Proper quasi-symmetric designs with no three mutually disjoint blocks have been the objects of recent interest [1, 4]. It was shown in [1] that there are finitely many such designs for a fixed value of \( y \), a result that was later strengthened in [4] to show an analogous assertion for the other
parameters (for example, $m$) of $D$. It was also shown [4, Theorem 3.12] that the parameters of such designs are expressible in terms of $m$ and $y$ alone.

Translated in terms of block graph $\Gamma$ of $D$ the condition imposed above means the following. In the complement $\overline{\Gamma}$ of $\Gamma$ there are no triangles. Let $D$ be any (proper) quasi-symmetric design and for any edge of $\overline{\Gamma}$, let $\tilde{c}$ denote the number of triangles containing that edge. That is, the number of blocks disjoint from both of two disjoint blocks is $\tilde{c}$. The purpose of this paper is to drop the "triangle-freeness" condition (i.e., $\tilde{c} = 0$) of [1, 4] and to study proper quasi-symmetric designs in general. In fact, we have been able to show that all the finiteness results of [1, 4] also hold in this general set-up.

With the exception of a couple of standard known results (see [1, 5]), this paper is self-contained. All of our results depend on the main theorem (Theorem 2.1) proved in Section 2. This theorem asserts the existence of a quadratic equation (2) in $\lambda$ with coefficients in $m$, $y$, and $\tilde{c}$. In Section 3 we exploit Eq. (2) to show that for a fixed pair $(m, y)$, $y \geq 2$, there are finitely many quasi-symmetric designs (Theorem 3.1) with $k = my$ and with block intersection numbers 0 and $\tilde{c}$. Corollary 3.2 immediately obtains the following: For a fixed value of the block size $k$, there are finitely many quasi-symmetric designs with block intersection numbers 0 and $\tilde{c} \geq 2$. Theorem 3.3 makes a similar assertion for a fixed pair $(\tilde{c}, z)$ of positive integers with $z \geq 2$. Let $S_{\tilde{c}z}$ denote the set of all quasi-symmetric designs with the properties that any two intersecting blocks meet in $y$ points for some $y$ with $2 \leq y \leq z$ and the number of blocks $\tilde{n}$ disjoint from a given block satisfies $1 \leq \tilde{n} \leq \tilde{c}$. Then $S_{\tilde{c}z}$ is finite. The limit argument used in Theorem 3.3 allows an alternative restatement (Theorem 3.4). With everything as above, there exists a number $k_0 = k_0(\tilde{c}, z)$ such that any quasi-symmetric design with $k > k_0$ is, in fact, a symmetric design.

Section 4 discusses two conjectures (presumably due to N. M. Singhi and M. Hall, Jr.) related to quasi-symmetric and symmetric designs in the light of the main results of our paper. We first show (Theorem 4.1) that for a fixed $\lambda \geq 2$, there are finitely many proper quasi-symmetric designs $D$ with any point pair on $\lambda$ blocks. Conjecture $S$ asserts the existence of only finitely many quasi-symmetric designs (proper or improper) with a given value of $\lambda \geq 2$. It clearly implies conjecture $H$ which states that for a fixed $\lambda \geq 2$ there are only finitely many symmetric designs. Our Theorem 4.1 shows the equivalence of these two conjectures (Corollary 4.2). This essentially proves that the harder part in proving or disproving conjecture $S$ is facing conjecture $H$; no counterexample to the latter seems to be known.
2. The Main Theorem

In this section and also in Section 3 assume that $D$ is a proper (not symmetric) quasi-symmetric design with standard parameters $v, b, r, k = my, y \geq 2$, and $\lambda$ with block intersection numbers 0 and $y$. The following equation is easily established [1, Lemma 2.3]:

$$(r-1)(y-1)=(\lambda-1)(k-1)=(\lambda-1)(my-1).$$

(1)

Let $\bar{c}$ be the number of triangles containing a given edge in the graph $\Gamma$ (complement of the block graph $\Gamma$ of $D$), i.e., given two disjoint blocks $X$ and $Y$, $\bar{c}$ is the number of blocks not meeting both of them. Our main theorem is the following:

**Theorem 2.1.** With everything as above the integers $m, y, \bar{c},$ and $\lambda$ satisfy the equation:

$$(m-1)y[m(y+1)-m^2-1]\lambda^2-[(m-1)y(2m(y^2+1)-2m^2y-\bar{c}(y+1))-\bar{c}(y-1)^2my^2(m-1)^2=0$$

(2)

**Proof.** Fix any two disjoint blocks $X$ and $Y$ of $D$ and let $U = \{(z, Z) : z \in Z, z \notin X \cup Y, Z$ meets both $X$ and $Y\}$.

We do a two-day counting of $|U|$. For any $i, 0 \leq i \leq \bar{c}$, let $a_i = |\{z : z \notin X \cup Y$ and there are exactly $i$ blocks through $z$ missing both $X$ and $Y\}|$.

Suppose $z \notin X \cup Y$ and $z$ is on $i$ blocks missing both $X$ and $Y$. Let $S$ be the set of those blocks through $z$ meeting $X$ or $Y$ or both. Then $|S| = r - i$. Let $S_X$ (resp. $S_Y$) be that subset of $S$ consisting of those blocks that do not meet (are disjoint from) $X$ (resp. do not meet $Y$). Then $|S_X| = |S_Y| = (r - i) - (k\lambda/y) = (r - i) - m\lambda$. But $S_X \cap S_Y = \phi$ and so $|S_X \cup S_Y| = 2(r - i - m\lambda)$. But the blocks through $z$ meeting both $X, Y$ are just the members of $S \setminus (S_X \cup S_Y)$. Hence $z$ is on exactly $(r - i) - 2(r - i - m\lambda) = 2m\lambda - r + i$ blocks meeting both $X$ and $Y$. Thus

$$|U| = \sum (2m\lambda - r + i) a_i.$$  

(3)

Now consider a $Z$ meeting both $X$ and $Y$. Such a $Z$ has $k - 2y = y(m - 2)$ points not in $X \cup Y$ and the number of such $Z$'s is $(k^2\lambda)/y^2 = m^2\lambda$. Hence

$$|U| = (m - 2)m^2\lambda y.$$  

(4)
Equating Eq. (3) and Eq. (4) obtains

\[(2m\lambda - r)\sum a_i + \sum ia_i = (m - 2)m^2 \lambda y.\]  

(5)

Obviously, \(\sum a_i = v - 2my\) and \(\sum ia_i = \bar{c}my\). Substitution of these values and the use of the standard relation \(v = (r(my - 1) + \lambda)/\lambda\) in Eq. (5) gives

\[(m - 2)m^2 \lambda^2 = (2m\lambda - r)[r(my - 1) + \lambda - 2my\lambda] + \bar{c}my\lambda.\]  

(6)

Finally using Eq. (1) and making a straightforward but tedious simplification, we obtain Eq. (2). This completes the proof of Theorem 2.1.

Remark 2.2. Equation (2) is a generalization of [1, Theorem 3.2], where \(\bar{c}\) was assumed to be zero. It can be easily seen, using (1) and the standard relations, that the parameters of a quasi-symmetric design \(D\) are expressible in terms of \(m, y,\) and \(\bar{c}\) (cf. [4, Theorem 3.12]).

3. CONSEQUENCES OF THE MAIN THEOREM

Referring to our earlier observation (Remark 2.2), \(m, y,\) and \(\bar{c}\) are the basic parameters of a (proper) quasi-symmetric design. If we fix two of these three at a time, can the possibilities of the third be bounded by a finite number so that we have at most finitely many feasible designs with the stipulated parameters? The results of Section 3 answer this question in the affirmative for the pairs \((m, y)\) and \((y, \bar{c}), \ y \geq 2.\) We also show by a counterexample that a similar assertion is false for the pair \((m, \bar{c}).\)

**THEOREM 3.1.** For a fixed \((m, y), \ y \geq 2\) there are finitely many quasi-symmetric designs with block size \(k = my\) and with block intersection numbers 0 and \(y.\)

**Proof.** Writing Eq. (2) in a different form, we obtain

\[A\lambda^2 + [\alpha + B] \lambda + C = 0,\]  

(7)

where \(\alpha = (y - 1)^2 my\bar{c}\) and \(A, B, C\) are appropriate integral polynomial functions of the (fixed) integers \(m\) and \(y.\) If a quasi-symmetric design \(D\) with relevant parameters does exist, then Eq. (7) must have an integral solutions and hence the discriminant \(\Delta\) of Eq. (7) is a square of some integer, i.e.,

\[\Delta = N^2 - (\alpha + B)^2 - 4AC.\]  

(8)

In Eq. (8), \(4AC = f(m, y) \neq 0\) as can be easily checked. Hence, \(f(m, y) = (\alpha + B)^2 - N^2 = (\alpha + B + N)(\alpha + B - N).\) It follows that the number of
possibilities of the integer $x + B$ is limited by the factors of (the constant) $f(m, y)$. But $m, y$ determine $B$ uniquely and hence $x$ has finitely many possibilities. Since $x = (y - 1)^2 my\tilde{c}$, it is clear that there are at most finitely many feasible values of $\tilde{c}$. But then using Eq. (7) or (2) (also see Remark 2.2), $\lambda$ has finitely many possibilities. Use of (1) and the standard necessary conditions now complete the proof of Theorem 3.1.

**Corollary 3.2.** For a fixed value of the block size $k$, there are finitely many quasi-symmetric designs with block intersection numbers 0 and $y \geq 2$.

*Proof.* Write $k = my$ and use Theorem 3.1.

**Theorem 3.3.** For a positive integer pair $(\tilde{e}, z)$ let $S(\tilde{e}, z)$ be the set of quasi-symmetric designs $D$ with the following properties: Given any block, the number $\tilde{n}$ of blocks disjoint from it satisfies $1 \leq \tilde{n} \leq \tilde{e}$ and any two blocks intersect in 0 or $y$ points, where $2 \leq y \leq z$. Then $S(\tilde{e}, z)$ is finite.

*Proof.* For $0 \leq \tilde{c} < \tilde{n} \leq \tilde{e}$ and $2 \leq y \leq z$, let $T(\tilde{n}, \tilde{c}, y)$ be the set of all the quasi-symmetric designs with the following properties:

(a) Given any block, the number of blocks disjoint from it is $\tilde{n}$.

(b) Given a pair of disjoint blocks, the number of blocks disjoint from both of them is $\tilde{c}$.

(c) Any two blocks intersect in 0 or $y$ points.

Since $S(\tilde{e}, z)$ is a union of finitely many $T$s, it suffices to show that each $T(\tilde{n}, \tilde{c}, y)$ is finite. By Theorem 3.1, it suffices to show that given $\tilde{c}$ and $y$ there are finitely many possible $m$'s.

Suppose $m$ has infinitely many possibilities. Consider an equivalent form of Eq. (2) obtained as follows: Divide all the terms of Eq. (2) by $m(m - 1)y$ to get a function $g(m, \lambda)$ (with coefficients in $y$ and $\tilde{c}$) whose value must be zero for the feasibility of a design. Since all quasi-symmetric designs considered are proper, (1) implies $\lambda > y$ since $r > k$. However,

$$g(m, \lambda) = \left[ \frac{y + 1}{m} - 1 - \frac{1}{m^2} \right] \lambda^2 - \left[ \frac{2(y^2 + 1)}{m} - 2y - \frac{(y + 1)}{m^2} \right] \lambda - \frac{(my - 1)(m - 1)y}{m(m - 1)}$$

and $\lim_{m \to \infty} g(m, \lambda) = -\lambda^2 + 2y\lambda - y^2 = 0$, which implies that $\lambda \to y$ as $m \to \infty$, a contradiction. Thus the number of feasible $m$'s is finite. Hence the proof.

In fact, the limit argument in the above proof has proved the following alternative restatement of Theorem 3.3.
Theorem 3.4. For a fixed pair of integers \((\tilde{e}, z)\), \(z \geq 2\), there exists a number \(k_0 = k_0(\tilde{e}, z)\) with the following property: let \(D\) be a quasi-symmetric design (proper or improper) with block size \(k > k_0\) and with the property that any two blocks intersect in \(0\) or \(y\) points with \(2 \leq y \leq z\) and given any block the number of blocks disjoint from it is at most \(\tilde{e}\). Then \(D\) is a symmetric design.

Remark 3.5. A similar assertion for the pair \((m, \tilde{e})\) is false. Consider the examples of affine designs [6, 3]. These are special kinds of quasi-symmetric designs in which parallelism is an equivalence relation. In particular, if \(m = \tilde{e} + 2\) is a prime power and \(y\) is a power of \(m\) then affine designs with these parameters can always be constructed (see [6 or 3]). Hence it is possible to fix the pair \((m, \tilde{e})\), \(\tilde{e} = m - 2\), and make \(y\) (and therefore \(k\)) arbitrarily large to obtain a quasi-symmetric design with the stipulated parameters.

Remark 3.6. The assumption \(y \geq 2\) is crucial for all of our results and cannot be dropped. To see this, it is enough to consider Steiner triple systems (see [2]) which give infinitely many examples. To get many more examples, one can use Wilson theory.

4. Further Results Related to the Main Theorem

In this section, we do not insist that \(D\) is a proper quasi-symmetric design (i.e., both \(0\) and \(y\) need not occur as block intersection numbers). As remarked earlier, an improper quasi-symmetric design is just a symmetric design with \(\lambda = y\).

Theorem 4.1. For a fixed \(\lambda \geq 2\) let \(S_\lambda\) be the class of those proper quasi-symmetric designs \(D\) in which any point-pair occurs on \(\lambda\) blocks. Then \(S_\lambda\) is finite.

Proof. Since \(D\) is proper, \(y < \lambda\). Let \(S_{\nu, \lambda}\) denote that subset of \(S_\lambda\) consisting of these designs that have block intersection numbers \(0\) and \(y\). Observe that \(\lambda \geq 2\) implies \(y \geq 2\) and hence \(S_{1, \lambda} = \emptyset\). Thus \(S_\lambda\) is a union of \(S_{\nu, \lambda}\)'s with \(2 \leq y \leq \lambda\). It suffices to show that each \(S_{\nu, \lambda}\) is finite. Consideration of Eq. (2) modulo \(m\) leads to \(\theta(y, \lambda) = y\lambda^2 - y(y + 1)\lambda + y^2 \equiv 0\) (modulo \(0\)). But \(\lambda \geq y + 1\) implies that \(\theta(y, \lambda)\) is positive and \(m\) divides \(\theta(y, \lambda)\). Hence a fixed pair \((y, \lambda)\) determines finitely many \(m\)'s. Use of Theorem 3.1 then completes our proof.

Conjecture S. For a fixed \(\lambda \geq 2\) there are finitely many quasi-symmetric designs in which any point pair occurs on \(\lambda\) common blocks.
Professor N. M. Singhi, in an informal discussion with the authors went on to make a conjecture much stronger than conjecture S. No counterexample seems to be known. Clearly conjecture S implies the following outstanding conjecture on symmetric designs (presumably due to Hall).

Conjecture H. For a fixed \( \lambda \geq 2 \), there are finitely many symmetric designs in which any point pair is on \( \lambda \) common blocks.

An immediate consequence of Theorem 4.1 is

Corollary 4.2. Conjecture H and conjecture S are equivalent.

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