R-curve and size effect in quasibrittle fractures: Case of notched structures

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Abstract

Within the framework of Bazant’s theory, the size effect on nominal strength of notched structures deduced from a size-dependent R-curve is proposed. It is shown that the expected size effect is more complicated than the one proposed in Bazant’s Size Effect Law (SEL) and especially in the crossover regime. As a function of the fracture parameters describing the R-curve, two kinds of size effect on the resistance at peak load are possible and lead to three different scalings on the nominal strength. We argue that these expected size effects are mainly driven by the value of the scaling exponent characterizing the size effect on the critical crack length increment and on the critical resistance assumed in the R-curve behavior. The three resulting size effects on the nominal strength are compared to Bazant’s SEL. It appears that, if Bazant’s SEL always underestimates nominal strength and consequently provides a safety design of structures, an optimal design should take into account the size effect on the R-curve and their consequences on the size effect on the nominal strength especially for large structures sizes.

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1. Introduction

In solid mechanics, an essential scaling problem is the effect of the structure size on its nominal strength. This effect is particularly important in the case of quasibrittle materials which are characterized by the existence of a large fracture process zone (FPZ) where various toughening mechanisms take place such as microcracking, crack branching or crack bridging. Materials as different as concretes, mortar and rocks, some composites and wood belong to this category. In notched structures, the fracture behavior of quasibrittle materials is usually characterized by a more or less pronounced rising resistance curve, commonly called R-curve (Lawn, 1993). This R-curve behavior emphasizes stress redistributions and stored energy release which take place in such large FPZ producing large stable crack growth before failure.

Since 1984, Bažant and co-workers (Bažant, 1984; Bažant, 1997a,b; Bažant, 2000) have shown that in the case of quasibrittle materials, contrary to what happens for Weibull’s statistics (Weibull, 1939), the size effect...
is linked to the very existence of the R-curve behavior. Within the framework of Bažant’s theory (Bažant, 1997a,b), the size effect can be described in the case of geometrically similar notched structures (with geometrically similar initial cracks) of different characteristic sizes \( D \) (dimension) by introducing a nominal stress:

\[
\sigma_N = c_N \frac{P}{D^2},
\]

where \( P \) is the external load applied to the structure (load independent of the displacement) and \( c_N \) is a coefficient introduced for convenience. When \( P = P_u \), which corresponds to the ultimate load or peak load, \( \sigma_N \) is called the nominal strength of the structure. From an energy-based asymptotic analysis founded on a single R-curve, i.e., independent of the specimen size, (Bažant (1997b)) has shown that, in a first order asymptotic approximation, the nominal strength \( \sigma_N \) can be estimated as a function the characteristic size \( D \) as:

\[
\sigma_N = \frac{Bf_t}{\sqrt{1 + \frac{D}{D_0}}},
\]

where \((Bf_t)\) has the dimension of a stress (Pa) and \( D_0 \) is the crossover size (m) between both asymptotic behaviors. In the case of small structure sizes, i.e., \( D \ll D_0 \), \( \sigma_N \approx Bf_t = \text{const} \): no size effect is expected. Indeed, for these small structure sizes, the fracture process zone is expected to occupy the whole volume of the structure, inducing no stress concentration. As a consequence, failure occurs with no crack propagation: this is the domain of strength theory. For large structures sizes, i.e., \( D \gg D_0 \), contrary to what happens for small sizes, \( \sigma_N \sim D^{-1/2} \) which is the size effect expected from linear elastic fracture mechanics (LEFM). A possible justification is that in large structures, the process zone is expected to lie within only an infinitesimal volume fraction of the body and hence, the stress and displacements fields surrounding the FPZ are the asymptotic elastic fields considered in LEFM.

However, despite the success of Bažant’s Size Effect Law (SEL) (Eq. (2)) to describe size effect of quasibrittle materials, the crossover regime between both asymptotic behaviors (estimated from the intermediate asymptotic theory) does not appear accurately defined. This point deserves some more thinking, especially since this is usually the range of the experimental values from which the SEL is entirely defined. On the other hand, it has been shown recently that the R-curve might be size-dependent (Morel et al., 2002a,b) contrary to what is assumed in Bažant’s SEL where any size effect on R-curve is considered.

In this study, within the framework of Bažant’s theory (Bažant, 1997a,b), the size effect on nominal strength is studied in the case of geometrically similar notched structures (with similar initial cracks) characterized by one dimension \( D \). We show that the size effect on nominal strength deduced from an analytical size-dependent R-curve appears more complicated than the one proposed in Bažant’s SEL and especially in the crossover regime. In Section 2, the more appropriate mathematical expression of the R-curve with respect to size effect is studied and discussed. An R-curve expression describing the size effects on the critical crack length increment and on the critical resistance is proposed. In Section 3, the implications on the size effect on the resistance at peak load are discussed in relation to the values of the scaling exponent describing the R-curve. The size effect on the nominal strength are then investigated in Section 4 as a function of the different scaling obtained for the resistance at peak load. Finally, the results are discussed in Section 5 and a comparison to the prediction of the Bažant’s SEL is performed.

### 2. R-curve and effective length of the FPZ

In notched structures, the fracture of quasibrittle materials can be successfully described within the framework of an equivalent linear elastic approach. Within this framework also called ‘equivalent LEFM’ the increase of the structure compliance due to the FPZ development is attributed to the propagation of an elastically equivalent crack (Bažant and Kazemi, 1990; Bažant, 1997a; Morel et al., 2005) which gives (according to LEFM) the same structure compliance as the actual crack with its fracture process zone. Thus, energy stored in the structure can be characterized by the complementary energy \( W^* \):

\[
W^* = \frac{P^2}{Eb} f\left(\frac{a}{D}\right),
\]
expressed as a function of the elastically equivalent crack length \( a \). In Eq. (3), \( P \) corresponds to the load applied on the structure, \( b \) is the structure thickness, the ratio \( a/D \) is the relative equivalent crack length, and, \( f \) is a dimensionless function characterizing the geometry of the structure.\(^1\) The modulus \( E' = E/(1 - v^2) \) for plane strain where \( E \) is the Young’s modulus of elasticity and \( v \) is Poisson’s ratio.

Thus, during the crack propagation (i.e., during the increase of the elastically equivalent crack length \( a \) induced by the increase of the specimen compliance due to the progressive damage of the structure), the elastic energy release rate \( G \) (obtained at a constant load \( P \) or displacement \( \delta \)) must be equal, according to LEFM, to the resistance to crack growth \( G_R \):

\[
G(a) = \frac{1}{b} \left[ \frac{\partial W^*(a)}{\partial a} \right] = G_R(a).
\]

Thus, within the framework of ‘equivalent LEFM’, quasibrittle fracture leads to an R-curve behavior (Lawn, 1993) corresponding to the dependence of the critical energy release rate required for fracture growth on the elastically equivalent crack length \( a \) (Eq. (4)). Generally, for large elastic equivalent crack lengths \( a \geq a_c \), the resistance to crack growth \( G_R \) becomes independent of the crack length defining a plateau value of the resistance also called critical resistance \( G_{Rc} \), i.e., \( G_{Rc} = G_R(a \geq a_c) = \text{const.} \), where \( a_c \) is called critical crack length. This critical resistance emphasizes that the influence of the toughening mechanisms is not indefinite and, in this sense, the critical crack length \( a_c \) gives an estimate of the effective size of the FPZ (or the effective length of the R-curve). For instance, a typical R-curve obtained in wood is shown in Fig. 1.

Unfortunately, the R-curve cannot be considered as an intrinsic fracture characteristic of the material. Indeed, it is well known that the R-curve is geometry-dependent (Bazant, 1997a,b; Morel et al., 2003), but, it has been shown recently that the R-curve might be also size-dependent, in the sense that the critical resistance \( G_{Rc} \) (plateau value of the resistance) and the critical crack length \( a_c \) can be both dependent on the structure size (Morel et al., 2002a,b). The dependence of the R-curve on the structure size is rarely addressed in literature. This lack is doubtless linked to the fact that most fracture experiments in quasibrittle materials exhibit monotonic rising R-curve and the absence of plateau value of the resistance leads to a difficult characterization of the size effect on the R-curve.\(^2\)

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\(^1\) The dimensionless function \( f \) is linked to the compliance \( j(\varepsilon) \) of the structure as \( f(\varepsilon) = E' b \ j(\varepsilon)/2 \).

\(^2\) The absence of plateau value of the R-curve is mainly due to a confined development of the FPZ linked to boundary conditions of fracture specimen. Indeed, in most fracture specimens used for quasibrittle materials, and especially in three-point bend fracture specimens usually used for concrete, the fact that the FPZ development is limited (in length) by the existence of the compression zone ahead of the FPZ is well known to lead to a monotonic increase of the macroscopic resistance. Note that, the choice of specimen geometry is often restricted for most quasibrittle materials.
Nevertheless, despite the fact that ‘equivalent LEFM’ leads to a geometry dependent estimate of fracture properties, the use of R-curve provides useful approximations of the quasibrittle fracture and especially allows to accurately estimate the crack length at peak load of a load–deflection curve (Morel et al., 2005). As a consequence, the connection between R-curve and size effect is worth thinking over.

### 2.1. R-curve independent of the structure size

The first idea consists to use a single expression for the R-curve, i.e., independent of the specimen size \( D \), as:

\[
G_R(\Delta a) = G_R(0)(1 + \phi \Delta a^\beta),
\]

where \( \Delta a = a - a_0 \) is the elastically equivalent crack length increment and \( a_0 \) corresponds to the initial crack length (or in others terms the length of the actual traction free portion of the crack). According to Eq. (5), when \( a = a_0 \) the resistance to crack growth leads to \( G_R(0) = G_{R0} \) which corresponds to the resistance at crack initiation. Thus, the dimensionless term in brackets in Eq. (5) can be seen as an increase factor of the resistance to crack growth, with respect to the equivalent crack length increment \( a \), related to the toughening phenomenon. Note that, the exponent \( \beta \) must lie between 0 and 1 (0 < \( \beta \) < 1), in order to ensure a correct curvature to the R-curve. Finally, \( \phi \) is a constant, i.e., a prefactor independent of the structure size \( D \).

Eq. (5) does not contain information about the critical crack length increment \( \Delta a_c = a_c - a_0 \), and the corresponding plateau value on the R-curve, \( G_{Rc} \). In order to resolve the problem of the scaling of \( \Delta a_c \), a possible way suggested by Bažant (1997a,b) is to consider that failure of a quasibrittle material is not only characterized by a critical energy dissipation such as \( G_{Rc} \), but also by a critical damage energy release rate \( G_d \) per unit volume of damaged material (1/m³), i.e., per unit volume of FPZ. Thus, the effective length of the FPZ \( \Delta a_c \) can be estimated from the balance between the energy needed to create the FPZ, linked to \( G \), and the energy released at the macroscopic scale, expressed from the R-curve \( G_R(\Delta a) \) (Eq. (5)), as:

\[
G_d V_{FPZ}(\Delta a_c) = b \int_0^{\Delta a_c} G_R(\Delta a) d\Delta a,
\]

where \( b \) is the thickness of the specimen, and \( V_{FPZ}(\Delta a_c) \) corresponds to the critical volume of the FPZ (Morel et al., 2002b). In the following, the critical volume of the FPZ is assumed to be equal to \( V_{FPZ}(\Delta a_c) = (b/n)\Delta a_c^2 \), where the length of the FPZ in the direction of crack propagation corresponds to the effective length of the FPZ\( \Delta a_c \). the height of the FPZ is assumed to be a fraction of its length: \( \Delta a_c/n \) (where \( n \) is a constant), and its thickness corresponds to the specimen thickness \( b \).

From Eqs. (5) and (6), it can be shown numerically that the effective length of the FPZ \( \Delta a_c = \text{const.} \) and this, whatever the specimen size \( D \). Thus, the relative critical length of the FPZ defined as \( \theta = \Delta a_c/D \) evolves as a power law in function of the structure size \( D \): \( \theta \sim D^{-1} \). For small structure sizes \( D \), i.e., when \( D \rightarrow 0 \), the relative critical length of the FPZ tends to infinity, \( \theta \rightarrow \infty \), which emphasizes that the FPZ occupies the entire ligament of the structure. For large structure sizes, i.e., when \( D \rightarrow \infty \), then \( \theta \rightarrow 0 \); in this case, the FPZ lies only within an infinitesimal volume fraction of the body. Note that both asymptotic behaviors of the relative critical length of the FPZ are in agreement with those assumed in Bažant’s SEL (Bažant, 1997a,b).

On the other hand, the fact that \( \Delta a_c(D) = \text{const.} \) whatever \( D \) leads, according to Eq. (5), to a constant critical resistance to crack growth \( G_{Rc} \), i.e., independent of the specimen size \( D \). As a consequence, the R-curve expression defined in Eq. (5) is analogous to the one assumed in Bažant’s SEL (Bažant, 1997a,b) where the R-curve is assumed to be unique, i.e., there is a single internal length (length of the FPZ) \( \Delta a_c = \text{const.} \) and hence, there is no size effect on the R-curve.

However, such an invariability of \( G_{Rc} \) disagrees with experimental evidence (Morel et al., 2002a,b), where, as previously mentioned, a size effect on this critical resistance has been observed. As a consequence, an R-curve such as defined by Eq. (5) is not able to take into account the size effect on the critical crack length increment \( \Delta a_c \) and its consequence on the critical resistance to crack growth \( G_{Rc} \) experimentally observed.

### 2.2. Size dependent R-curve

The second idea consists in modifying Eq. (5) and especially the prefactor \( \phi \), which is now considered as size dependent:
\[ G_R(\Delta a) = G_{R0}[1 + \phi(D)\Delta a^\beta]. \]  

However, in Eq. (7), there are now two unknown parameters: \( \phi(D) \) and as previously \( \Delta a_c(D) \). As a consequence, the single energy balance defined by Eq. (6) appears insufficient to solve the problem. Additional information is needed to estimate the critical evolution of the FPZ as a function of the structure size \( D \). As a consequence, in the following, one assumes that the critical resistance \( G_{Rc} \) evolves as a function of the structure size \( D \) as:

\[ G_{Rc}(D) = G_{R0}\left[1 + \left(\frac{D}{D_c}\right)^{\beta_c}\right], \]  

where \( D_c \) is the crossover length (or crossover size) between two asymptotic behaviors: for large structure sizes \( D \) (i.e., when \( D/D_c \gg 1 \)) the critical resistance evolves as a power law of size \( D \), \( G_{Rc} \sim D^{\beta_c} \), while, for small structure sizes (i.e., when \( D/D_c \ll 1 \)), the critical resistance tends to a constant, \( G_{Rc} \to G_{R0} \). Note that the scaling exponent: \( 0 < \beta_c < 1 \), as explained in the following. The size effect on the critical resistance defined by Eq. (8) is plotted in Fig. 2.

Thus, from Eqs. (7) and (8), the energy balance defined by Eq. (6) leads to a prefactor \( \phi(D) \):

\[ \phi(D) = \frac{\left(\frac{D}{D_c}\right)^{\beta_c}}{\left\{c_0\left[1 + \left(\frac{D}{D_c}\right)^{\beta_c}\right]\right\}^{\beta}}, \]  

where \( D_{ca} \) is a crossover size defined as \( D_{ca} = (1 + \beta_c)^{-1/\beta_c}D_c \) and \( c_0 = nG_{R0}/G_d \) corresponds to an internal length of the material, or, more exactly, to the minimum internal length. Moreover, from Eqs. (7) and (8), the critical crack length increment \( \Delta a_c \) can be expressed as:

\[ \Delta a_c = \left[\left(D/D_c\right)^{\beta_c}/\phi(D)\right]^{1/\beta}. \]  

In latter expression, substituting \( \phi(D) \) by its expression defined in Eq. (9) leads to a scaling relation on \( \Delta a_c \), or in other terms, to the size effect on the internal length (critical length of the FPZ):

\[ \Delta a_c(D) = c_0\left[1 + \left(\frac{D}{D_{ca}}\right)^{\beta_c}\right]. \]  

The size effect on the internal length \( \Delta a_c \) defined by Eq. (10) is plotted in Fig. 3. As shown in Fig. 3, the size effect on the critical size of the FPZ \( \Delta a_c \) (Eq. (10)) is transitional between two asymptotic behaviors: for large structure sizes \( D \), i.e., \( D/D_{ca} \gg 1 \), the critical size of the FPZ evolves as a power law of \( D \), \( \Delta a_c \sim D^{\beta_c} \) while, for small structure sizes, i.e., \( D/D_{ca} \ll 1 \), this critical size tends toward the minimum internal length \( c_0 \), \( \Delta a_c \to c_0 \).

![Fig. 2. Size effect on the critical resistance \( G_{Rc}(D) \) according to Eq. (8).](image-url)
Note that the size effect obtained on the critical crack length increment $\Delta a_c$ (Eq. (10)) is analogous to the one assumed on the critical resistance $G_{ Rc }$ (Eq. (8)) but with different crossover sizes.

Moreover, according to Eq. (10) for large structure sizes, i.e., when $D/D_{ca} \gg 1$, the relative critical length of the FPZ $\theta = \Delta a_c(D)/D$ evolves in power law $\theta \sim D^{\beta_c-1}$, and, since $0 < \beta_c < 1$ then $\theta \to 0$ and so the FPZ in large structures lies within only an infinitesimal volume fraction of the body. In the case of small structures sizes ($D/D_{ca} \ll 1$), the critical length of the FPZ being constant ($\Delta a_c \to c_0$) then $\theta \to \infty$: the FPZ is expected to occupy the entire ligament. Indeed, as shown in Fig. 3, for size $D < D_{min}$, the theoretical size of the FPZ becomes greater than the ligament length $D$ which is impossible. Thus, $D_{min}$ defines the minimum size below which the FPZ occupies the entire ligament of the structure, a situation where failure occurs with no crack propagation: this is the expected domain of strength theory (Bažant, 1997a).

However, if both asymptotic behaviors of the relative length of the FPZ are in agreement with those assumed in Bažant’s SEL (Bažant, 1997a,b), the fact that, for large structure sizes, the critical resistance evolves as a power law $G_{ Rc } \sim D^{\beta_c}$ disagrees with the condition of LEFM (where a constant resistance to crack growth is expected). Indeed, the behaviors of $G_{ Rc }$ and $\Delta a_c$ being strongly linked, it seems particularly unreasonable to consider that the critical size of the FPZ (but also the corresponding critical resistance) increases indefinitely for large structures sizes. In order to obtain an upper bound, one can assume that there exists a maximum critical size $\Delta a_{c_{max}}$ of the FPZ which corresponds to a second internal length of the material, or in other terms, to a maximum internal length. According to Eq. (10), it is easy to estimate $D_{max}$ for which the maximum critical size of the FPZ $\Delta a_{c_{max}}$ is reached:

$$D_{max} = \left[ \frac{\Delta a_{c_{max}}}{c_0} - 1 \right]^{1/\beta_c} D_{ca},$$

as well as, according to Eq. (8), the corresponding maximum critical resistance to crack growth $G_{ Rc_{max}}$:

$$G_{ Rc_{max}} = G_{ Rc}(D_{max}) = G_{ R0} \left[ 1 + \left( \frac{D_{max}}{D_c} \right)^{\beta_c} \right].$$

Thus, for structure sizes $D > D_{max}$, it is assumed that $G_{ Rc}(D)$ and $\Delta a_c(D)$ do not follow Eqs. (8) and (10) but are equal to $G_{ Rc_{max}}$ (Eq. (12)) and $\Delta a_{c_{max}}$, as shown in Figs. 2 and 3. This upper bound implies also that, for structure sizes $D > D_{max}$, the R-curve (Eq. (7)) becomes independent of the structure size $D$ because the pre-factor $\phi$ defined in Eq. (9) becomes constant, i.e., $\phi(D) > D_{max} = \phi(D_{max}) = \phi_{max} = const.$

To sum up, for structure sizes $D < D_{max}$, the R-curve can be expressed as:
where $G_R(c(D), D_{ac}(D))$ and $G_R(c_{max})$ are respectively defined from Eq. (8), (10) and (9), while, for structure sizes $D > D_{max}$, the R-curve becomes independent of the structure size $D$ and leads to:

$$G_R(\Delta a, D > D_{max}) = \begin{cases} G_{R0}(1 + \phi(D)\Delta a^\beta) & \text{if } \Delta a < \Delta a_{c}(D) \\ G_{Rc}(D) & \text{if } \Delta a \geq \Delta a_{c}(D) \end{cases},$$

where $\phi_{max} = \phi(D_{max}) = \text{const.}$ (Eq. (9)) and $G_{Rc}(D)$ is defined in Eq. (12). In order to illustrate the size effect on R-curve induces by the scaling of $\phi(D)$ (Eq. (9)) and of $\Delta a_c(D)$ (Eq. (10)), the R-curves related to different characteristic sizes $D$ are plotted in Fig. 4.

3. Size effect on the resistance at peak load

Within the framework of Bažant’s theory (Bažant, 1997a,b), when the nominal strength $\sigma_N$ (Eq. (1)) of the structure is reached, i.e., when the external load $P$ applied to the structure (load independent of the displacement) corresponds to the ultimate or peak load $P_u$, the resistance to crack growth can be deduced from the elastic energy release rate $G$ (Eq. (4)) as:

$$G(a_u) = \frac{1}{b} \left[ \frac{\partial W^*(a_u)}{\partial a} \right]_{P_u} = \frac{P_u^2}{E' b^2 D} g\left(\frac{a_u}{D}\right) = G_R(a_u),$$

where $a_u$ corresponds to the elastic equivalent crack length at peak load, and, $G_R(a_u)$ and $g(a_u/D)$ define respectively the resistance to crack growth and the dimensionless energy release rate function $g$ at peak load. If $x_u = a_u/D$ denotes the relative crack length at peak load, it is easy to show, from Eqs. (15) and (1), that the nominal strength is linked to the corresponding resistance to crack growth at peak load such as:

$$\sigma_N = \sqrt{\frac{E' G_R(x_u)}{Dg(x_u)}},$$

where $G_R(x_u)$ correspond to the resistance at peak load deduced from Eq. (7) but expressed with respect to the relative crack length at peak load $x_u = a_u/D$ (instead of the crack length $a_u$). Note that $x_u = a_u/D$.

$\text{The dimensionless energy release rate function } g(z) = f'(z) \text{ where } f'(z) = \frac{\partial f(z)}{\partial z}.$

Fig. 4. R-curves obtained for different characteristic sizes $D$, $D_1 \ll D_2 \ll D_{max}$, according to Eqs. (13) and (14). The size effect on R-curve is linked to the scalings of the prefactor $\phi(D)$ (Eq. (9)) and of the internal length $\Delta a_c(D)$ (Eq. (10)).
$D = (a_0 + \Delta a_0)/D = x_0 + \Delta a_0/D$ where $a_0$ is the initial crack length, $x_0$ the relative length of the initial crack and $\Delta a_0$ is the crack length increment at peak load.

As a consequence, knowing the R-curve $G_R(x)$, defined from Eqs. (13) and (14), and the dimensionless energy release rate function $g(x)$, the study of the size effect on the nominal strength $\sigma_N$ consists to estimate the scaling of the relative crack length at peak load $x_u$ with respect to the structure size $D$.

The condition of peak load is well known (Bazant and Cedolin, 1991; Morel et al., 2005): the relative crack length corresponding to the peak load of a load–deflection curve $x_u$ is solution of the equation:

\[
\frac{G'_R(x)}{G_R(x)} = \frac{g'(x)}{g(x)},
\]

where $G'_R(x) = \partial G_R(x)/\partial x$ and $g'(x) = \partial g(x)/\partial x$. Note that Eq. (17) is valid for load or displacement-controlled fracture tests (Morel et al., 2005). Moreover, in the case of a load-controlled test (which is generally the case for actual structures), one has: $g'(x)/g(x) = G'(x)/G(x)$. In this case, Eq. (17) is in agreement with the two well known conditions at peak load: (i) the energy release rate is equal to the resistance to crack growth $G(x_u) = G_R(x_u)$ and (ii) the curves relative to the energy release rate and to the resistance must be tangent.

\[
G'_R(x) = \frac{\partial G_R(x)}{\partial x} = \frac{g'(x)}{g(x)}.
\]

Moreover, it can be seen in Fig. 5 that the relative crack length at peak load $x_u$ does not correspond to the critical length $a_c = a_0 + \Delta a_c$ as shown in Fig. 1 where $a_u < a_c$ in the case of the notched flexure specimen tested in (Morel et al. 2005). According to Eq. (17) and from the size dependent R-curve defined in Eq. (7), the ratio $G'_R(x)/G_R(x)$ can be expressed as:

\[
\frac{G'_R(x)}{G_R(x)} = \frac{\beta}{\alpha - x_0} \Omega(x, x_0, D),
\]

where

\[
\Omega(x, x_0, D) = \frac{\phi(D)D^\beta(x - x_0)^\beta}{1 + \phi(D)D^\beta(x - x_0)^\beta}.
\]

The ratios $G'_R(x)/G_R(x)$ (Eq. (18)) and $g'(x)/g(x)$ (corresponding here to a SENB specimen) are plotted in Fig. 5 for various structure sizes $D$. It can be seen in Fig. 5 that the relative crack length at peak load $x_u$ evolves with respect to the structure size $D$. The first consequence of the scaling of $x_u$ is the size effect induced on the resistance at peak load $G_R(x_u)$. Moreover, it can be seen in Fig. 5 that, for large structures sizes $D$, the ratio $G'_R(x)/G_R(x)$ tends toward an asymptotic curve which corresponds to $G'_R(x)/G_R(x) = \beta/(\alpha - x_0)$ as defined in Eq. (18). Indeed, for large structure sizes, $\Omega \rightarrow 1$ and, as a consequence, for large sizes, the relative crack length at peak load $x_u$ leads to a single solution, i.e., independent on the structure size $D$, noted here as $x_u = x_{um}$ (Fig. 5).

Thus, for large structure sizes, if the relative crack length at peak load $x_u$ tends toward the constant $x_{um}$, it is possible to estimate the corresponding resistance to crack growth as:

\[
G_R(x_{um}, D) = G_{R0}\left[1 + \phi(D)D^\beta(x_{um} - x_0)^\beta\right] = G_{R0}\left[1 + \left(\frac{D}{D_{cu}}\right)^{\beta + \beta_c(1-\beta)}\right],
\]

where

\[
D_{cu} = \left\{D_c\right\}^{-\beta} \left\{\frac{1}{\phi(D)D^\beta(x_{um} - x_0)^\beta + 1}\right\}^\beta_c(1-\beta).
\]

From Eq. (20) it is easy to show that the resistance to crack growth $G_R(x_{um}, D)$ is transitional between two asymptotic behaviors: for small structures sizes $D/D_{cu} \ll 1$, the resistance tends toward the initial resistance

\footnote{It is well known that an R-curve estimated from a load-control test is truncated to the crack length $a_u$ at peak load while from a displacement-control one, the R-curve might develop in the post-peak regime of the load–deflection curve up to the critical crack length $a_c$ (Morel et al., 2005).}
G_{R0}$ while, for large structure sizes, $D/D_{cu} \gg 1$, the resistance is expected to evolve as a power law $G_R(x_{\text{un}}, D) \sim D^{\beta_c/(1-\beta_c)}$. Note that the first asymptotic behavior, i.e., $G_R(x_{\text{un}}, D \ll D_{cu}) \sim G_{R0}$, is purely theoretical because the solution $x_u = x_{\text{un}}$ is only valid for large structure sizes as shown in Fig. 5 and especially for sizes $D \gg D_{cu}$. However, it appears interesting to know this theoretical behavior as shown in the following.

Nevertheless, as previously mentioned, the relative critical length of the FPZ $\theta = \Delta a_c(D)/D$ evolves as $\theta \sim D^{\beta_c/(1-\beta_c)}$ and so, since the scaling exponent $0 < \beta_c < 1$, when $D \rightarrow \infty$ then $\theta \rightarrow 0$. Thus, it is expected that, for a particular size $D_{cc}$, the solution $x_u = x_{\text{un}} = x_0 + \theta$. Hence, according to the expression of $\Delta a_c(D)$ defined in Eq. (10), the size $D_{cc}$ can be expressed as:

$$D_{cc} = \left[ \frac{\beta + 1}{c_0} (x_{\text{un}} - x_0) D_c \right]^{1/(1-\beta_c)}.$$  \hspace{1cm} (22)

However, the size $D_{cc}$ appears mainly driven by the scaling exponent $\beta_c$, i.e., the scaling exponent of $G_{Rc}$ (Eq. (8)) and of $\Delta a_c$ (Eq. (10)). Especially, for large $\beta_c$ values, $D_{cc}$ becomes greater than the upper bound $D_{\text{max}}$. This phenomenon translates the fact that, for large $\beta_c$ values, size $D$ can reach and exceed $D_{\text{max}}$ when the solution $x_u = x_{\text{un}}$ is active. As a consequence, two kinds of size effects on the resistance at peak load are possible as a function of the value of the scaling exponent $\beta_c$.

### 3.1. Size effect on the resistance at peak load in the case of small $\beta_c$ values

In the case of small $\beta_c$ values, the size $D_{cc}$ defined in Eq. (22), i.e., the size for which the solution $x_{\text{un}} = x_0 + \theta$, is smaller than $D_{\text{max}}$. As a consequence, for structures sizes $D \gg D_{cc}$, the resistance at peak load does not follow Eq. (20) but evolves as defined by Eq. (8) since $G_R(x_u = x_0 + \theta, D) = G_{Rc}(D)$. Moreover, it is easy to show from Eq. (22) that $D_{cc} \gg D_c$ and hence, for large structure sizes $D \gg D_{cc}$ (or $D \gg D_c$), $G_{Rc}$ is expected to evolve as a power law $G_{Rc}(D \gg D_c) \sim D^{\beta_c}$ as shown in Fig. 2. Nevertheless, the latter scaling is only valid for structures sizes $D_{cc} \ll D < D_{\text{max}}$ since for sizes $D > D_{\text{max}}$, the resistance to crack growth reaches the upper bound $G_{Rc}(D > D_{\text{max}}) = G_{Rc_{\text{max}}}$ (Eq. (12)). Thus, in the case of small $\beta_c$ values, the resistance to crack growth at peak load is expected to scale with respect to the structure size $D$ as:
where the crossover sizes $D_{cu}$, $D_{cc}$ and $D_{max}$ are respectively defined in Eqs. (21), (22) and Eq. (11). The size effect on the resistance at peak load $G_{R}(\alpha_{u})$ described in Eq. (23) is plotted in Fig. 6.

3.2. Size effect on the resistance at peak load in the case of large $\beta_{c}$ values

As previously mentioned, for large $\beta_{c}$ values, the size $D_{cc}$, which defines the crossover between the asymptotic regimes $G_{R}(\alpha_{uun}, D >> D_{cu})$ and $G_{Re}(D >> D_{c})$, becomes greater than the upper bound $D_{max}$. As a consequence the asymptotic regime $G_{Re}(D) \sim D^{\beta_{c}}$ vanishes to the benefit of a new regime. As a matter of fact, in the case of large $\beta_{c}$ values, size $D$ can reach and exceed $D_{max}$ when the solution $\alpha_{u} = \alpha_{uun}$ is active. Thus, according to Eq. (9), $\phi(D > D_{max}) = \phi(D_{max}) = \phi_{max} = \text{const.}$, and, as a consequence, according to Eq. (7), the new regime which can be expressed as:

$$G_{R}(\alpha_{uun}, D > D_{max}) = G_{R0} \left[ 1 + \phi_{max} D^{\beta_{c}} (\alpha_{uun} - \alpha_{0})^{\beta_{c}} \right]$$

$$= G_{R0} \left[ 1 + \left( \frac{D}{D_{cu}^{*}} \right)^{\beta_{c}} \right],$$

where

$$D_{cu}^{*} = \frac{\phi_{max}^{-1/\beta_{c}}}{\alpha_{uun} - \alpha_{0}}.$$  \hfill (25)

The new regime defined in Eq. (24) is valid for structure sizes $D_{max} \ll D \ll D_{max}^{*}$ where $D_{max}^{*}$ corresponds to the crossover size with the asymptotic regime $G_{R_{max}}$ (Eq. (12)). Indeed, when the solution $\alpha_{u} = \alpha_{uun}$ is valid, it is expected that the crack length increment reaches $\Delta a_{c_{max}}$ for the size $D_{max}^{*}$. The crossover size $D_{max}^{*}$ can be deduced from the equation $\alpha_{uun} - \alpha_{0} = \Delta a_{c_{max}} / D_{max}^{*}$ as:

$$D_{max}^{*} = \frac{\Delta a_{c_{max}}}{\alpha_{uun} - \alpha_{0}}.$$  \hfill (26)
As a consequence, in the case of large $\beta_c$ values, the size effect on the resistance at peak load does not follow Eq. (23) but is expected to scale as:

$$G_R(\alpha_u) \sim \begin{cases} G_{R0} = \text{const.} & \text{if } D \ll D_{cu} \\ G_R(\alpha_u, D) \sim D^{\beta_c (1-\beta)} & \text{if } D_{cu} \ll D \ll D_{max} \\ G_R(\alpha_u, D > D_{max}) \sim D^\beta & \text{if } D_{max} \ll D \ll D_{max}^* \\ G_{Rc_{max}} = \text{const.} & \text{if } D \gg D_{max}^* \end{cases}$$  \( \text{Eq. (27)} \)

where the crossover sizes $D_{cu}$, $D_{max}^*$ and $D_{max}^*$ are respectively defined in Eqs. (21), (11) and (26). The size effect defined in Eq. (27) is plotted in Fig. 7.

Thus, two kinds of size effects on the resistance at peak load are possible as a function of the scaling exponent $\beta_c$ which appears to be the relevant parameter for this scaling problem. Small $\beta_c$ values induce a size effect such as the one defined in Eq. (23), while, large $\beta_c$ values lead to a scaling on the resistance at peak load described in Eq. (27). In other terms, small $\beta_c$ values favor the asymptotic regime $G_R(\alpha_u, D \gg D_{cu}) \sim D^\beta_c$ to the detriment of the regime $G_R(\alpha_u, D \approx D_{cu}) \sim D^{\beta_c (1-\beta)}$, while, large $\beta_c$ values favor the asymptotic regime $G_R(\alpha_u, D \gg D_{cu})$ (to the detriment of the regime $G_{Rc}$) and induce a new asymptotic regime at large sizes $G_R(\alpha_u, D > D_{max}) \sim D^\beta$.

4. Size effect on the nominal strength

The scalings of the relative crack length $\alpha_u$ and of the resistance $G_R(\alpha_u)$ at peak load having been defined in the previous section, it appears relatively easy to estimate the corresponding size effect on the nominal strength $\sigma_N$ on the basis of Eq. (16). Two cases must be studied corresponding respectively to small $\beta_c$ values (Eq. (23)) and large $\beta_c$ values (Eq. (27)).

4.1. Size effect on $\sigma_N$ in the case of small $\beta_c$ values

4.1.1. Size effect on $\sigma_N$ linked to the asymptotic regime $G_{Rc_{max}}$ (Eq. (23))

According to Eq. (23), for structures sizes $D > D_{max}$, the resistance to crack growth is expected to stay constant and equal to the upper bound $G_{Rc_{max}}$ while the relative crack length $\alpha_u$ can be expressed as $\alpha_u = \alpha_0 + \Delta \alpha_{c_{max}}/D$. Thus, when $D \rightarrow \infty$ then $\Delta \alpha_{c_{max}}/D \rightarrow 0$ and hence, the energy release rate function $g(\alpha_u)$ at peak load can be expanded in Taylor series around $\alpha_u = \alpha_0$. On this basis, Eq. (16) thus yields:
Note that $\sigma_{\text{max}}$ (Eq. (29)) corresponds to a stress [Pa] while $D_{N_{\text{max}}}$ (Eq. (30)) leads to a length (m). Eq. (28) provides a large-size asymptotic series expansion of the size effect on nominal strength $\sigma_N$. Indeed, the terms containing nonzero powers of $D$ in denominator vanish when $D \to \infty$ while Eq. (28) is expected to diverge for structure sizes $D \to 0$ as shown in Figs. 8–10 where the size effect on $\sigma_N$ obtained from Eq. (28) is plotted. Moreover, the asymptotic behavior of Eq. (28) at large sizes (i.e., $D > D_{N_{\text{max}}}$) leads to $\sigma_N \sim D^{-1/2}$ which is the size effect expected from LEFM. Note that the large-size asymptotic series expansion of the size effect defined in Eq. (28) is in agreement with the one proposed in Bazănt’s SEL (Bazănt, 1997a,b).

4.1.2. Size effect on $\sigma_N$ linked to the asymptotic regime $G_{R_{\text{c}}}(D \gg D_{c})$ (Eq. (23))

According to Eq. (23), for structure sizes $D_{\text{ec}} \ll D \ll D_{\text{max}}$, the resistance to crack growth is expected to scale as $G_{R_{\text{c}}}(D \gg D_{c}) \sim D^{\beta_c}$ (Eq. (8)) while the relative crack length $z_u$ can be expressed as $z_u = z_0 + \Delta a(D) / D = z_0 + \theta$ where the critical size of the FPZ $\Delta a_{\text{crit}}(D)$ is obtained from Eq. (10). Moreover, when $D \to \infty$, then, according to Eq. (10), $\theta \to 0$ and hence, the energy release rate function $g(z_u)$ in Eq. (16) can be expanded in Taylor series around $z_u = z_0$ and the corresponding size effect on $\sigma_N$ thus leads to:

$$\sigma_N = \sqrt[\beta_c]{ \frac{E G_{R_{\text{c}}}}{D} \left[ g(z_0) + g_1(z_0) \theta + \frac{g_2(z_0)}{2!} \theta^2 + \cdots + \frac{g_3(z_0)}{3!} \theta^3 + \cdots \right]^{-1/2} }$$

$$= \sigma_{\text{eq}} \left[ 1 + \frac{D}{D_{\text{ec}}} \right]^{1/\beta_c} \left[ \frac{D_{\text{ec}}}{D_{\text{ca}}} + \frac{D}{D_{\text{c}} \theta} + \frac{D_{\text{ec}}}{D_{\text{ca}}} \left( \frac{D}{D_{\text{c}}} \right)^{2/\beta_c} + \frac{D_{\text{ec}}}{D_{\text{ca}}} \left( \frac{D}{D_{\text{c}}} \right)^{3/\beta_c} + \cdots \right]^{-1/2},$$

(31)

![Graph](image_url)

**Fig. 8.** Size effect on the nominal strength $\sigma_N$ (Eq. (37)) deduced from the scaling on the resistance $G_{R_{\text{c}}}(z_u,D)$ (Eq. (23)) obtained in the case of small $\beta_c$ values. $\beta_c = \beta = 0.3$, $D_{c} = 1$ mm, $z_0 = 2$ mm and $\Delta a_{\text{crit}} = 60$ mm.
where the constants $g_i(x_0)$ and $b_i$ have been already defined in the previous section, and,

$$\sigma_{Mc} = \frac{E'G_{R0}}{g(x_0)D_{Nc}},$$

$$D_{Nc} = \frac{g_i(x_0)}{g(x_0)}c_0. \tag{33}$$

Eq. (31) provides another large-size asymptotic series expansion of the size effect and its asymptotic behavior at large sizes (i.e., $D \gg D_{Nc}$) as $\sigma_N \sim D^{-1/2+b/2}$. Note that this asymptotic behavior disagrees with the LEFM size effect because here the resistance to crack growth increases as a function of the structure size as $G_{Rc}(D \gg D_c) \sim D^{b_c}/2$ (Eq. (8)), while in LEFM, this resistance is assumed to be constant (as described in the previous section).

On the other hand, the crossover size between the large size asymptotic behaviors defined from Eq. (28), $\sigma_N \sim D^{-1/2}$, and Eq. (31), $\sigma_N \sim D^{-1/2+b/2}$, corresponds to $D_{max}$. Indeed, the expansions in Taylor series in Eqs. (28) and (31) are performed around the same relative crack length $x_u = x_0$ and, as a consequence, the
crossover size between both asymptotic regimes corresponds to the one defined from the resistance to crack growth, i.e. $D_{\text{max}}$ as previously described in Eq. (23).

4.1.3. Size effect on $\sigma_N$ linked to the asymptotic regime $G_R(z_{\text{uun}}, D \gg D_{\text{cu}})$ (Eq. (23))

According to Eq. (23), when $D_{\text{cu}} \ll D \ll D_{\text{cc}}$, the resistance to crack growth scales as $G_R(z_{\text{uun}}, D \gg D_{\text{cu}})$ (Eq. (20)) while the relative crack length at peak load is expected to stay constant and equal to $z_u = z_{\text{uun}}$. Hence, it is easy to show from Eq. (16) that the nominal strength in this case scales as:

$$\sigma_N = \sqrt{\frac{E' G_R(z_{\text{uun}}, D \gg D_{\text{cu}})}{D g(z_{\text{uun}})}} = \sigma_{M_u} \sqrt{\frac{1 + \left(\frac{D}{D_{\text{cu}}}\right)^{\beta_1(1-\beta)}}{\frac{D}{D_{\text{cu}}}}},$$

(34)

where

$$\sigma_{M_u} = \sqrt{\frac{E' G_{R0}}{g(z_{\text{uun}}) D_{\text{cu}}}},$$

(35)

and $D_{\text{cu}}$ has been previously defined in Eq. (21). According to Eq. (34), the asymptotic behavior of the nominal strength at large sizes (i.e., $D \gg D_{\text{cu}}$) scales as $\sigma_N \sim D^{-1/2+|\beta-\beta_1(1-\beta)|/2}$. However, contrary to what is defined in Eq. (23), the crossover size between the asymptotic regimes corresponding to large sizes in Eq. (31), $\sigma_N \sim D^{-1/2+\beta_1/2}$, and in Eq. (34), $\sigma_N \sim D^{-1/2+\beta_1(1-\beta)/2}$ differs from $D_{\text{cc}}$. Indeed, Eq. (31) results from a Taylor expansion around the relative crack length $z_0$, while Eq. (34) is obtained from the relative crack length $z_{\text{uun}}$. The crossover size between the two asymptotic regimes can be estimated as:

$$\begin{align*}
D'_{\text{cc}} &= \left[\frac{g(z_{\text{uun}})}{g(z_0)}\right]^{1/\beta(1-\beta)} D_{\text{cc}}.
\end{align*}$$

(36)

Finally, for structure sizes $D < D_{\text{min}}$, the critical crack length increment being expected to be larger than the ligament of the structure, there is no stress concentration and hence, failure occurs with no crack propagation: this is the expected domain of the strength theory as shown by Bazant (1997a,b). Moreover, the size $D_{\text{min}}$ is generally greater than the crossover size $D_{\text{cu}}$ and, as a consequence, the crossover size between the asymptotic behaviors $\sigma_N \sim D^{-1/2+\beta_1(1-\beta)/2}$ deduced from Eq. (34) and $\sigma_N \simeq \text{const.}$, corresponding to the strength theory, is here assumed to be $D_{\text{min}}$.

Thus, in the case of the scaling on the resistance to crack growth defined in Eq. (23) obtained in the case of small values of the relevant parameter $\beta_1$, it is expected that the corresponding size effect on the nominal strength $\sigma_N$ scales, according to Eqs. (28), (31) and (34), as:

$$\sigma_N \simeq \begin{cases} 
\text{const.} & \text{if } D \ll D_{\text{min}} \\
D_{\text{cc}}^{\frac{1}{2} - \frac{\beta_1(1-\beta)}{2}} & \text{if } D_{\text{min}} \ll D \ll D'_{\text{cc}} \\
D_{\text{cc}}^{\frac{1}{2} - \frac{\beta_1}{2}} & \text{if } D'_{\text{cc}} \ll D \ll D_{\text{max}} \\
D_{\text{max}}^{\frac{1}{2}} & \text{if } D \gg D_{\text{max}},
\end{cases}$$

(37)

where $D_{\text{max}}$ and $D'_{\text{cc}}$ are respectively defined in Eqs. (11) and (36). The size effect on the nominal strength $\sigma_N$ defined in Eq. (37) is shown in Fig. 8.

Nevertheless, it may happen that the crossover size $D'_{\text{cc}}$ (Eqs. (36) and (37)) is greater than $D_{\text{max}}$. This implies that the asymptotic regime which theoretically takes place between $D'_{\text{cc}}$ and $D_{\text{max}}$, i.e., the regime associated to the resistance $G_R(D \gg D_{\text{cc}})$ which corresponds to the third regime in Eq. (37), vanishes. This is essentially linked to the value of the scaling exponent $\beta_1$. Indeed, it can be observed that the ratio $D'_{\text{cc}}/D_{\text{max}}$ strongly increases when $\beta_1$ increases. Thus, if $D'_{\text{cc}}/D_{\text{max}} > 1$, the asymptotic regime associated to the resistance $G_R$ vanishes and hence, only the asymptotic regimes related to Eqs. (34) and (28), i.e., the second and the fourth regimes in Eq. (23), remain. In this case, the new crossover size $D'_{\text{max}}$ between the asymptotic regimes $\sigma_N \sim D^{-1/2+\beta_1(1-\beta)/2}$ Eq. (34) and $\sigma_N \sim D^{-1/2}$ Eq. (28) can be expressed as:
4.2. Size effect on \( r \) final strength as:

\[
D'_{\text{max}} = \left[ \frac{G_{R_{\text{max}}} g(z_{\text{un}})}{G_{R_0}} \right]^{1/\beta + \beta_c(1-\beta)} D_{\text{cu}}.
\]  

As a consequence, in the case of the scaling of the resistance to crack growth defined in Eq. (23) in the case of small \( \beta_c \) values, if the ratio \( D'_{\text{ce}}/D_{\text{max}} > 1 \) which appears in the case median values of \( \beta_c \) (i.e. \( \beta_c \approx 0.5 \)), the nominal strength \( \sigma_N \) is expected to scale as:

\[
\sigma_N \simeq \begin{cases} 
\text{const.} & \text{if } D \ll D_{\text{min}} \\
D^{-1/2} & \text{if } D_{\text{min}} \ll D \ll D'_{\text{max}} \\
D^{-1} & \text{if } D \gg D'_{\text{max}},
\end{cases}
\]  

where \( D'_{\text{max}} \) is obtained from Eq. (38). The size effect on nominal strength defined by Eq. (39) is shown in Fig. 9. As shown in Figs. 8 and 9, related respectively to Eqs. (37) and (39), for the same scaling of the resistance to crack growth defined in Eq. (23), two kinds of size effects on the nominal strength are possible and this phenomenon appears to be mainly a function of the value of the scaling exponent \( \beta_c \). The size effect on the nominal strength defined by Eq. (37) is obtained in the case of small \( \beta_c \) values, while, the size effect defined by Eq. (39) corresponds to median \( \beta_c \) values, i.e., \( \beta_c \approx 0.5 \).

4.2. Size effect on \( \sigma_N \) in the case of large \( \beta_c \) values (Eq. (27))

In the case of the scaling of the resistance to crack growth defined by Eq. (27), two asymptotic behaviors have been already estimated from Eqs. (28) and (34). As a consequence, only the size effect on \( \sigma_N \) associated to the scaling on resistance to crack growth \( G_R(z_{\text{un}}, D > D_{\text{max}}) \) (Eq. (24)) must be estimated. In this case, the resistance to crack growth scales as defined in Eq. (24), while the relative crack length at peak load is expected to stay constant and equal to \( z_u = z_{\text{un}} \). Hence, from Eq. (16), it is easy to estimate the size effect on the nominal strength as:

\[
\sigma_n = \sqrt{\frac{E'G_R(z_{\text{un}}, D > D_{\text{max}})}{Dg(z_{\text{un}})}} = \sigma_{M_u} \left[ 1 + \left( \frac{D}{D_{\text{cu}}} \right)^\beta \right],
\]  

where

\[
\sigma_{M_u} = \sqrt{\frac{E'G_{R_0}}{g(z_{\text{un}})D_{\text{cu}}}},
\]  

and \( D_{\text{cu}}^u \) has been previously defined in Eq. (25). The asymptotic behavior of Eq. (40) at large sizes, i.e., \( D \gg D_{\text{cu}}^u \), scales as \( \sigma_N \sim D^{-1/2+\beta/2} \). This asymptotic regime takes place between the crossover size \( D_{\text{max}} \) and a new crossover size \( D_{\text{max}}'' \). The size \( D_{\text{max}} \) corresponds to the crossover with the asymptotic regime defined from Eq. (34) while, \( D_{\text{max}}'' \) defines the crossover with the asymptotic regime defined from Eq. (28). Note that the new crossover size \( D_{\text{max}}'' \) is different to \( D_{\text{max}} \) defined in Eq. (27) because the relative crack length of both asymptotic regimes are different: \( z_u = z_{\text{un}} \) in Eq. (40) while \( z_u \rightarrow z_0 \) in Eq. (28). The crossover size \( D_{\text{max}}'' \) can be expressed as:

\[
D_{\text{max}}'' = \left[ \frac{G_{R_{\text{max}}} g(z_{\text{un}})}{G_{R_0}} \right]^{1/\beta} D_{\text{cu}}',
\]  

where \( D_{\text{cu}}' \) has been already defined by Eq. (25).

Thus, in the case of the scaling on the resistance to crack growth defined in Eq. (27) for large \( \beta_c \) values, it is expected that the corresponding size effect on the nominal strength \( \sigma_N \) scales, according to Eqs. (28), (40) and (34), as:
The various asymptotic behaviors expected from Eqs. (23) and (27) for the resistance at peak load, and from Eqs. (37), (39) and (43) for the nominal strength, have been compared to the exact size effects obtained from the numerical resolution of Eq. (17). It can be seen in the corresponding Figs. 6–10 that the actual size effects are very close to the expected asymptotic behaviors with the exception of structures sizes close to \( D_{\text{min}} \). However, this disagreement was expected since, for these small structure sizes, the FPZ should occupy the whole volume of the structure but such behavior cannot be described from a theoretical approach as the one developed in this study. Especially, in Fig. 8–10, for a structure size close to \( D_{\text{min}} \), the nominal strength \( \sigma_N \) should tend toward an horizontal asymptote, which is the behavior expected according to the strength theory (Bazant, 1997a,b).

On the other hand, the actual size effects, i.e., those obtained from the numerical resolution of Eq. (17), are very close to the expected asymptotic behaviors (Figs. 6–10) and especially in the crossover zones because the analytical R-curve (Eq. (7)) considered in this study does not provides a nil slope of the curve when the crack length increment \( \Delta a \) reaches the critical crack length increment \( \Delta a_c \). Indeed, as shown in Fig. 4, the transition between the part related to the rising resistance (i.e., when \( \Delta a < \Delta a_c \)) and the plateau of the R-curve (i.e., when \( \Delta a > \Delta a_c \)) is not smooth, and this implies strong transitions between the asymptotic regimes linked to both behaviors. Moreover, the fact that the R-curve becomes suddenly independent of the structure size \( D \) when \( D > D_{\text{max}} \) induces also strong transitions around the crossover sizes with the asymptotic regimes related to the maximum critical resistance \( G_{R_{\text{max}}} \).

In this study, six fracture parameters describe the R-curve behavior defined in Eqs. (13) and (14): (1) the exponent \( \beta \) driving the curvature of the R-curve, (2) the scaling exponent \( \beta_c \) which drives the size effects on the critical crack length increment \( \Delta a_c(D) \) and on the critical resistance \( G_{R_0}(D) \), (3) the crossover size \( D_c \) between both asymptotic behaviors of the crack length increment \( \Delta a_c(D) \), (4) the resistance \( G_{R_0} \) at crack initiation, (5) the minimum internal length \( c_0 \) and (6) the maximum internal length \( \Delta a_{\text{max}} \) of the FPZ. Among these fracture parameters only the scaling exponent \( \beta_c \) is actually a relevant parameter with regards to the scaling of fracture. Indeed, it appears that the influence of the exponent \( \beta \) on the scaling of fracture is very weak, as well as, the value of the resistance at crack initiation which is generally found very lesser than the critical resistance \( G_{R_0} \) (for instance, \( G_{R_0} \) has been found around 10 J/m² in (Morel et al. 2002a, 2003)) while the ratio \( G_{R_c}/G_{R_0} \) which depends on the geometry and on the size of the structure is usually greater than 10–20.

On the other hand, the minimum internal length \( c_0 \) (i.e., the minimum size of the FPZ which can be estimated as the minimum size of the material microstructure relevant for fracture), as well as, the crossover size \( D_c \), should be considered small, to the order of mm or less. However, despite the fact that both parameters are difficult to estimate, they do not appear really relevant for the scaling of fracture. As a matter of fact, the size \( D_{\text{min}} \), i.e., the size for which the FPZ occupies the entire volume of the structure and where begins the domain of the strength theory, is generally reached before the FPZ reaches the minimum internal length \( c_0 \). Finally, note that the maximum internal length \( \Delta a_{\text{max}} \), i.e., the maximum critical size of the FPZ, must be also considered as a relevant parameter for the scaling of fracture. Indeed, this maximum internal length is directly linked to the proximity of the expected domain of LEFM, i.e., the domain where the maximum size effect on the nominal strength takes place and scales as \( \sigma_N \sim D^{-1/2} \).

On the other hand, the asymptotic behaviors defined from Eqs. (37), (39) and (43) have been respectively plotted in Fig. 11–13 in order to be compared with Bazant’s SEL (Eq. (2)). SEL has been fitted from the exact size effects on nominal strength obtained from the numerical resolution of Eq. (17) and for structure sizes:
20 mm ≤ D ≤ 400 mm. Note that the ratio between the largest and the smallest sizes is equal to 20 which is much larger than the ratio generally reached experimentally. It can be seen in Figs. 11–13 that, with the exception of Fig. 13 where SEL is in very good agreement with the expected size effect, SEL does not describe accurately the evolution of the nominal strength especially for large structure sizes where the asymptote related to the size effect of LEFM is expected to be located more to the right. Nevertheless, the first global approximation of the size effect provided by SEL has the advantage to always underestimate the nominal strength, especially at large sizes, and as a consequence, a nominal strength estimated from SEL should warrant a safety design of structures.

Finally, another theories, and especially the cohesive crack model, can be used nowadays to model the evolution of the FPZ and the associated size effect on the nominal strength. (Bažant (2002)) has recently performed a comparison between the size effect on nominal strength obtained from cohesive crack model and the classical SEL (Eq. (2)) but also the 'broad-range' SEL (cf. Fig. 11 of the study, Bažant (2002) reported here as Fig. 14). It appears in Fig. 14 that the asymptote related to the size effect of LEFM, deduced from the evolution of the cohesive crack model results, is located more to the right compared to the classical SEL (Eq. (2)). Nevertheless, a good fit is obtained from the 'broad-range' SEL which is mainly obtained from numerical data.  

Fig. 11. Comparison of the asymptotic behaviors of the nominal strength σN (Eq. (37)) and of the SEL (Eq. (2)) in the case of small βc values. βc = β = 0.3, Dc = 1 mm, c0 = 2 mm and Δa_max = 60 mm.

Fig. 12. Comparison of the asymptotic behaviors of the nominal strength σN (Eq. (39)) and of the SEL (Eq. (2)) in the case of median βc values. βc = β = 0.5, Dc = 1 mm, c0 = 2 mm and Δa_max = 60 mm.
a modification of the crossover size $D_0$ (Bažant, 2002).\textsuperscript{5} Note that the shift to the right of the LEFM’s asymptote observed in Fig. 14 (compared to classical SEL) is in agreement with the corresponding shifts observed in Fig. 11 and in Fig. 12. Indeed, in cohesive crack model, the fracture energy (characterizing the area under the softening function) is usually considered as constant and hence leads to size-independent R-curve. On this basis, one can observed in Fig. 12 (median $\beta_c$ values) and especially in Fig. 11 (small $\beta_c$ values) that, more the $\beta_c$ value decreases, i.e., more the size effect on the R-curve is small, more the LEFM’s asymptote is located to the right.

Note however that a comparison with size effect obtained from cohesive crack model will be necessary in the future to asses the degree of approximation of the present theory as well as a comparison with experimental results.

\textsuperscript{5} Good agreements between size effects obtained from cohesive crack model (using the eigenvalue approach) and SEL have been recently obtained by Bažant et al. (2002\textsuperscript{b}) for various test geometries excepted for small sizes for which deviations are observed.

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Fig. 13. Comparison of the asymptotic behaviors of the nominal strength $\sigma_N$ (Eq. (43)) and of the SEL (Eq. (2)) in the case of large $\beta_c$ values. $\beta_c = 0.7$, $\beta = 0.3$, $D_c = 1$ mm, $c_0 = 2$ mm and $\Delta a_{\text{max}} = 60$ mm.

Fig. 14. Numerical results (data points) of Hillerborg (left) and Jirášek (right) obtained with the cohesive crack model and their fits with the broad-range size effect law – figure extracted from Bažant (2002), Fig. 11, p. 182.
6. Conclusion

In this study, within the framework of Bažant’s theory (Bažant, 1997a,b), we have shown that the size effect on the nominal strength deduced from a size-dependent R-curve (analytical), is more complicated than the one proposed in Bažant’s SEL, especially in the crossover regime. As a function of the fracture parameters describing the R-curve, especially the scaling exponent $\beta_c$ which drives the size effects on the critical crack length increment and on the critical resistance, two kinds of size effects on the resistance are possible. Moreover, it has been shown that both size effects on the resistance lead to three possible size effects on the nominal strength obtained respectively for small, median and large $\beta_c$ values. The three resulting size effects on the nominal strength have been compared to Bažant’s SEL. If in some cases SEL does not describe accurately the evolution of the nominal strength, it has the advantage to always underestimate the nominal strength. Thus, if Bažant’s SEL always provides a safety design of structures, an optimal design should take into account the size effect on the R-curve and their implications on the size effect on the nominal strength especially for large structure sizes.

Finally, the analytical form of the present theory (with the exception of the single solution $z_{un}$ which needs a numerical solving of the Eq. (17)) allows to study easily the influence of each fracture parameters describing the R-curve on the size effect on the nominal strength of structures.

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