Separable partitions

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Abstract

An ordered partition of a set of \( n \) points in the \( d \)-dimensional Euclidean space is called a \textit{separable partition} if the convex hulls of the parts are pairwise disjoint. For each fixed \( p \) and \( d \) we determine the maximum possible number \( r_{p,d}(n) \) of separable partitions into \( p \) parts of \( n \) points in real \( d \)-space up to a constant factor. Of particular interest are the values \( r_{p,3}(n) = \Theta(n^{p-1}) \) for every fixed \( p \) and \( d \geq 3 \), and \( r_{p,2}(n) = \Theta(n^{p-1}) \) for every fixed \( p \geq 3 \). We establish similar results for spaces of finite Vapnik-Chervonenkis dimension and study the corresponding problem for points on the moment curve as well. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

A \textit{separable} \( p \)-partition of a set of \( n \) points in the \( d \)-dimensional Euclidean space \( \mathbb{R}^d \) is an ordered tuple \( \pi = (\pi_1, \ldots, \pi_p) \) of \( p \) nonempty sets whose disjoint union is \( S \), where the convex hulls of the sets \( \pi_j \) are pairwise disjoint. Let \( r_{p,d}(n) \) denote the maximum possible number of separable \( p \) partitions of a set of \( n \) points in \( \mathbb{R}^d \). It is easy to see that \( r_{p,1}(n) = p! \binom{n-1}{p-1} = \Theta(n^{p-1}) \). The following theorem determines the asymptotic behavior of \( r_{p,d}(n) \) for every fixed \( p, d \), when \( n \) is large.

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Theorem 1.1.

- For every fixed \( p \geq 2 \), \( r_{p,1}(n) = \Theta(n^{p-1}) \).
- \( r_{2,2}(n) = \Theta(n^2) \) and for every fixed \( p \geq 3 \), \( r_{p,2}(n) = \Theta(n^{bp-12}) \).
- For every fixed \( p \geq 2 \) and \( d \geq 3 \),
  \[
  r_{p,d}(n) = \Theta(n^{d(\frac{1}{p})}).
  \]

We also obtain similar estimates for the maximum possible number of separable partitions in spaces of finite Vapnik-Chervonenkis dimension [16]. Here are the relevant definitions. A space is a pair \( (X, \mathcal{H}) \) with \( \mathcal{H} \) a collection of subsets of a set \( X \). A \( p \)-partition \( \pi = (\pi_1, \ldots, \pi_p) \) of a subset \( S \subseteq X \) in a space \( (X, \mathcal{H}) \) is an ordered partition of \( S \) into \( p \) pairwise disjoint parts. It is separable if for each pair \( 1 \leq r < s \leq p \) there is a member \( H_{r,s} \in \mathcal{H} \) such that \( \pi_r \subseteq H_{r,s} \) and \( \pi_s \subseteq X \setminus H_{r,s} \). A subset \( S \subseteq X \) is shattered if all 2-partitions of \( S \) are separable. A space \( (X, \mathcal{H}) \) has finite VC-dimension \( d \) if \( X \) contains a shattered \( d \)-subset but not a shattered \( (d + 1) \)-subset. An important example, which is of major concern in this article, is provided by real Euclidean \( d \)-space, which is the space \( (X, \mathcal{H}) \) with \( X = \mathbb{R}^d \) and \( \mathcal{H} \) the collection of closed halfspaces in \( \mathbb{R}^d \). In this space, a partition is separable if and only if the convex hulls of the parts are pairwise disjoint, as defined earlier. The VC-dimension of this space is \( d + 1 \).

Let \( v_{p,d}(n) \) denote the maximum possible number of separable \( p \)-partitions of a set of \( n \) points in a space of finite VC-dimension \( d \). We provide an upper bound on \( v_{p,d}(n) \) which, together with the lower bound on \( r_{p,d}(n) \) from Theorem 1.1, gives the following statement.

Theorem 1.2. For every fixed \( p \geq 2 \) and \( d \geq 4 \),
  \[
  \Omega(n^{d-1}(\frac{1}{p})) \leq v_{p,d}(n) \leq O(n^{d(\frac{1}{p})}).
  \]

The special case \( p = 2 \) of 2-partitions in the real Euclidean space had been considered 30 years ago by Harding [7], who proved that \( r_{2,d}(n) = 2 \sum_{i=1}^{d} \binom{n-1}{i-1} = \Theta(n^d) \) for all \( d \). The case \( p = d = 2 \) of 2-partitions in the real plane under additional size constraints has been extensively studied ever since [9]. The upper bound in Theorem 1.1 for real Euclidean space with arbitrary \( p \) and \( d \geq 3 \) has been recently derived in [8] in the course of the study of a broad class of hard optimization problems over partitions. For \( p = 2 \) and an arbitrary space of finite VC-dimension \( d \), an upper bound is available through the so-called Sauer’s lemma [14].

The article is organized as follows. In the next section we provide our lower bounds for the numbers \( r_{p,d}(n) \). In Section 3 we describe the upper bounds in Theorems 1.1 and 1.2, and discuss the related problem for spaces of finite VC dimension. In Section 4 we study the problem of estimating the number of separable partitions of sets of points on the moment curve in \( \mathbb{R}^d \) and show that it is closely related to the study of long Davenport–Schinzel sequences. The final Section 5 contains some concluding remarks.
2. The lower bounds

Throughout this section we restrict attention to the real Euclidean space. We assume some familiarity with the rudiments of convex polytopes theory (such as in [6, 17]), but include a compact description of all notions and facts that we use.

An orientation of a hyperplane $H$ in $\mathbb{R}^d$ is a designation of the closed and open halfspaces $H^-, H^+$ below it and the closed and open halfspaces $H^+, H^-$ above it. A presentation of $H$ as the zero set of a linear form $H = \{x \in \mathbb{R}^d : h_0 + h^T x = 0\}$ gives an orientation of $H$ in the obvious way. The hyperplane spanned by an ordered list $(v^1, \ldots, v^d)$ of $d$ affinely independent points will be oriented by the linear form $\det[z, v^1, \ldots, v^d]$, where $z$ is used to denote the vector in $\mathbb{R}^{d+1}$ obtained by appending a first coordinate 1 to the vector $v \in \mathbb{R}^d$. We need the following building block. Let $H = \{x \in \mathbb{R}^d : h_0 + h^T x = 0\}$ be an oriented hyperplane in $\mathbb{R}^d$, let $v$ be a point in $H$, let $m$ be a positive integer and let $\epsilon > 0$. A set $S \subset \mathbb{R}^d$ will be called an $(H, v, m, \epsilon)$-set if $S$ is contained in the ball $B(v, \epsilon)$ of radius $\epsilon$ about $v$ and is of the form

$$S = \bigcup_{j=1}^{d} \{v_j + j \cdot \delta \cdot h : j = 1, \ldots, m\},$$

where $(v_1, \ldots, v_d)$ is an ordered list of affinely independent points which span the oriented hyperplane $H$, and $\delta$ is a positive real. Thus, $S$ consists of $md$ points $\epsilon$-close to $v$ and above $H$, evenly spread on $d$ parallel lines orthogonal to $H$. The canonical partition of $S$ associated with $1 \leq j_1, \ldots, j_d \leq m$ is defined to be the 2-partition $(\pi_1, \pi_2)$ of $S$ with $\pi_1 = \bigcup_{j=1}^{d} \{v_j + j \cdot \delta \cdot h : 1 \leq j < j_1\}$ and $\pi_2 = S \setminus \pi_1$. The canonical hyperplane of $S$ associated with $1 \leq j_1, \ldots, j_d \leq m$ is defined to be the oriented hyperplane spanned by the list $(v_1 + j_1 \cdot \delta \cdot h, \ldots, v_d + j_d \cdot \delta \cdot h)$. The verification of the following simple proposition is left to the reader.

**Proposition 2.1.** Let $S$ be an $(H, v, m, \epsilon)$-set and let $H$ and $(\pi_1, \pi_2)$ be, respectively, the canonical hyperplane and the canonical partition of $S$ associated with $1 \leq j_1, \ldots, j_d \leq m$. Then $\pi_1 \subset H^\leq$ and $\pi_2 \subset H^\geq$. In particular, every canonical partition of $S$ is separable.

A polytopal complex is a nonempty finite collection $\mathcal{P}$ of convex polytopes in some $\mathbb{R}^d$ such that the face of any member of $\mathcal{P}$ is also in $\mathcal{P}$ and such that the intersection of any two members of $\mathcal{P}$ is a common face of both. A polytopal complex is pure $d$-dimensional if all (inclusion) maximal polytopes of $\mathcal{P}$ have the same dimension $d$. Two maximal polytopes in a pure $d$-dimensional polytopal complex are adjacent if their intersection is $(d-1)$ dimensional, i.e. a facet of both. The graph $G(\mathcal{P})$ of a pure polytopal complex is the graph whose vertices are the maximal polytopes of $\mathcal{P}$ and whose edges are the pairs of adjacent maximal polytopes. We define a $(p, d)$-complex to be a pure $d$-dimensional polytopal complex embedded in $\mathbb{R}^d$ and containing $p$ maximal polytopes.
Lemma 2.2. For any fixed $p,d,k$, if the graph of some $(p,d)$-complex contains $k$ edges then $r_{p,d}(n) = \Omega(n^{dk})$.

**Proof.** Let $\mathcal{P}$ be a $(p,d)$-complex whose graph contains $k$ edges. Let $P_1,\ldots,P_p$ be the maximal polytopes in $\mathcal{P}$. For each pair $1 \leq r < s \leq p$, let $F_{rs} = F_{sr} := P_r \cap P_s$ which is a common face of $P_r$ and $P_s$, and let $H_{rs} = H_{sr}$ be a hyperplane such that $H_{rs} \cap P_r = H_{rs} \cap P_s = F_{rs}$ (note that if $P_r, P_s$ are adjacent, then $F_{rs}$ is a common facet of both and $H_{rs}$ is uniquely defined). Let $H_{rs}$ be the orientation of $H_{rs}$ with $P_r$ below $H_{rs}$ and $P_s$ above it and let $H_{sr}$ be the opposite orientation.

For each $r$ let $N_r := \{ s : P_r, P_s$ adjacent$\}$ be the set of indices of neighbors of $P_r$. For each adjacent pair $P_r, P_s$, choose a point $v_{rs} = v_{sr}$ in the relative interior of $F_{rs}$. Then for all $1 \leq r < s \leq p$ the following hold: every point $v_{rs}$ with $i \in N_r \setminus \{s\}$ lies strictly below $H_{rs}$, and every point $v_{js}$ with $j \in N_s \setminus \{r\}$ lies strictly above $H_{rs}$. For any given $m$ it is therefore possible to choose an $\varepsilon > 0$ sufficiently small, and an $(H_{rs}, v_{rs}, m, \varepsilon)$-set $S_{rs} = S_{sr}$ for each adjacent pair $P_r$ and $P_s$ with $r < s$, such that the following hold for all $1 \leq r < s \leq p$:

- Every set $S_{ri}$ with $i \in N_r \setminus \{s\}$ lies strictly below $H_{rs}$, and every set $S_{js}$ with $j \in N_s \setminus \{r\}$ lies strictly above $H_{rs}$.
- If $P_r$ and $P_s$ are adjacent and $\tilde{H}$ is the canonical hyperplane of $S_{rs}$ associated with any $1 \leq j_1, \ldots, j_d \leq m$ then every set $S_{ri}$ with $i \in N_r \setminus \{s\}$ lies strictly below $\tilde{H}$ and every set $S_{js}$ with $j \in N_s \setminus \{r\}$ lies strictly above $\tilde{H}$.

Now, let $S$ be the union of all the $S_{rs}$. So $S$ consists of $mdk$ points. For each adjacent pair $P_r, P_s$ with $r < s$ choose a canonical partition $\pi^{r,s} = (\pi_{1}^{r,s}, \pi_{2}^{r,s})$ of $S_{rs}$ and let $\tilde{H}_{r,s}$ be the corresponding canonical hyperplane which separates it. Now define a $p$-partition $\pi = (\pi_1, \ldots, \pi_p)$ of $S$ as follows: for $i = 1, \ldots, p$ let

$$\pi_i := \left( \bigcup_{j \in N_i, j > i} \pi_1^{j,i} \right) \cup \left( \bigcup_{j \in N_i, j < i} \pi_2^{j,i} \right).$$

We claim that $\pi$ is a separable partition of $S$. Indeed, it follows from the discussion above together with Proposition 2.1 that, for each pair $r < s$, we have that if $c$ is sufficiently small $\pi_r$ and $\pi_s$ are separated by $H_{rs}$ if $P_r$ and $P_s$ are not adjacent, and by $\tilde{H}_{rs}$ if $P_r$ and $P_s$ are adjacent.

As there are $d^k$ canonical partitions for each $S_{rs}$, we obtain this way $d^k$ separable $p$-partitions of the $mdk$-set $S$.

Now, given any positive integer $n$, let $m := \lceil n/dk \rceil$. Then, as claimed,

$$r_{p,d}(n) \geq r_{p,d}(mdk) \geq m^{dk} \geq \left( \frac{n}{2dk} \right)^{dk} = \Omega(n^{dk}).$$

We proceed to describe a simple construction of $(p,d)$-complexes with dense graphs for all $p$ and $d$. In particular, for every $p$ and every $d \geq 3$ there is a $(p,d)$-complex whose graph is complete, a fact first established half a century ago by Rado [12].
In what follows, $G(P)$ denotes the graph of 0-faces and 1-faces of a convex polytope $P$.

**Lemma 2.3.** Let $P$ be a $(d+1)$-polytope with $p+1$ vertices and let $v$ be an arbitrary vertex in its graph $G(P)$. Then there exists a $(p,d)$-complex $\mathcal{P}$ with $G(\mathcal{P})$ isomorphic to $G(P) - v$.

**Proof.** Assume without loss of generality, that $P$ is full dimensional, let $a$ be any interior point of $P$, and let $Q$ be a polar of $P$.

$$Q := (P - a)^* = \{x \in \mathbb{R}^{d+1} : (y - a)^T x \leq 1 \text{ for all } y \in P\}.$$  

Then the face lattice of $Q$ is the poset-dual of the face lattice of $P$. In particular, there is a bijection $v \mapsto F_v$ from the vertices of $P$ to the facets of $Q$.

Let $v$ be an arbitrary vertex of $P$ and let $F_v$ be the facet of $Q$ corresponding to $v$. Define the polytopal complex $\mathcal{P}$ to be a Schlegel diagram of $Q$ at $F_v$ (see [6, 17] for details). Briefly, it is defined as follows. Let $H = \text{aff}(F_v) = \{x : h_0 + h^T x = 0\}$ be the hyperplane supporting $Q$ at $F_v$ oriented so that $Q \subset H^\subseteq$. Choose any point $b$ in the relative interior of $F_v$ and let $u := b + \varepsilon \cdot h \in H^\geq$, with $\varepsilon > 0$ sufficiently small, so that for every point $x \in Q \setminus F_v$, the intersection point $x'$ of the line segment $[u, x]$ with $H$ is in the relative interior of $F_v$. The Schlegel diagram of $Q$ at $F_v$ is the $(p,d)$-complex $\mathcal{P}$ whose polytopes are the images of all proper faces of $Q$ but $F_v$ under the radial projection $x \mapsto x'$ (transformed by an affine map taking $H$ onto $\mathbb{R}^d$). The face poset of $\mathcal{P}$ is then isomorphic to the poset of all proper faces of $Q$ but $F_v$. Since the face lattice of $P$ determines its graph $G(P)$ and the face poset of $\mathcal{P}$ determines its graph $G(\mathcal{P})$, it follows that $G(\mathcal{P})$ is isomorphic to $G(P) - v$ as desired. $\Box$

We can now obtain our lower bounds.

**Lemma 2.4.** For all $d \geq 3$ we have $r_{p,d}(n) = \Omega(n^d \binom{p}{2})$.

**Proof.** Let $C(p,d) := \text{conv}\{M_d(1), \ldots, M_d(p)\}$, with $M_d(i) := [i, i^2, \ldots, i^d]$, denote the cyclic polytope with $p$ vertices in $\mathbb{R}^d$. It is well known (cf. [17]) that the graph of $C(p,d)$ is complete for all $d \geq 4$. Now, given any $p \geq 1$ and $d \geq 3$, let $P := C(p + 1, d + 1)$, and let $v$ be any vertex in $G(P)$. By Lemma 2.3, there is a $(p,d)$-complex $\mathcal{P}$ whose graph $G(\mathcal{P})$ is isomorphic to $G(P) - v$ and hence is complete and has $\binom{p}{2}$ edges. The bound follows by Lemma 2.2. $\Box$

**Lemma 2.5.** For all $p \geq 3$ we have $r_{p,2}(n) = \Omega(n^b p^{12})$.

**Proof.** First, we note that for every $p \geq 3$ there is a graph $G_p$ with the following properties: it is planar; it is 3-connected; it is simplicial (i.e., all its faces are triangles in every planar embedding); it has $p+1$ vertices and $3(p-1)$ edges; and it has a vertex of degree 3. Clearly $G_3 := K_4$, the 4-clique, satisfies these properties. Proceeding by induction, suppose $G_p$ has been constructed and embedded in the plane. Choose any
triangular face, insert a new vertex and connect it to each of the three vertices of that triangle. Clearly, this new graph $G_{p+1}$ has again all desired properties. Now, each 3-connected planar graph is isomorphic to the graph of a 3-polytope by Steinitz' Theorem (see e.g. [10]). Given $p \geq 3$, let $P$ be a 3-polytope with $p + 1$ vertices whose graph $G(P)$ is isomorphic to $G_p$, and choose a vertex $v$ of degree 3 in that graph. By Lemma 2.3, there is a $(p, 2)$-complex $\mathcal{P}$ whose graph $G(\mathcal{P})$ is isomorphic to $G(P) - v$. Since $G_p$, hence $G(P)$, have $3(p - 1)$ edges and $v$ has degree 3 in $G(P)$, the graph of the $(p, 2)$-complex $\mathcal{P}$ has $3p - 6$ edges. The bound follows by Lemma 2.2.

3. The upper bounds

We now derive our upper bounds on $r_p,d(n)$ and $v_p,d(n)$. We start with real Euclidean space.

Lemma 3.1. For any fixed $p,d,k$, if for every collection $\mathcal{S}$ of $p$ compact, convex, pairwise disjoint sets in real $d$-space, there is a collection $\mathcal{H}$ of $k$ hyperplanes such that any two members of $\mathcal{S}$ are separated by at least one hyperplane of $\mathcal{H}$, then $r_p,d(n) = O(n^d)$. 

Proof. Suppose that the hypothesis holds for $p,d,k$. Consider any set $S$ of $n$ points in $\mathbb{R}^d$, and consider any separable $p$-partition $\pi = (\pi_1, \ldots, \pi_p)$ of $S$. By the hypothesis, there is a collection of $k$ hyperplanes that separate each pair among $\text{conv}(\pi_1), \ldots, \text{conv}(\pi_p)$. Clearly, we may assume that each of these hyperplanes contains no point of $S$. Thus, by Harding's theorem mentioned above there are only $O(n^d)$ choices for each hyperplane. Suppose, now, that we are given $S$ and the collection $\mathcal{H}$ of the $k$ hyperplanes together with the information, for each of the convex hulls $\text{conv}(\pi_i)$ and each of the hyperplanes $H$ in $\mathcal{H}$, if $\text{conv}(\pi_i)$ intersects $H$ and in case it does not, in which side of $H$ it lies. Then we can easily reconstruct the whole partition, as each $\pi_i$ is simply the intersection of $S$ with all the corresponding half spaces supported by members of $\mathcal{H}$ which contain it. It thus follows that the total number of separable $p$-partitions of $S$ is bounded by $3^{\binom{k}{d}}$ (which is fixed for fixed $p,k$) times the number of distinct collections $\mathcal{H}$ of $k$ hyperplanes, which is bounded by $O(n^d)$, by Harding's Theorem.

Since $3^{\binom{k}{d}}$ hyperplanes always suffice to pairwise separate $p$ compact, convex, pairwise disjoint sets in any real space, an immediate consequence of this lemma is the following upper bound, which was first proved (by a slightly different argument) in [8, Lemma 4.1].

Lemma 3.2. For all $d$ and $p \geq 2$ we have $r_p,d(n) = O(n^d(\binom{p}{d}))$.

We now turn to derive the following tighter upper bound for the real plane.
Lemma 3.3. For all fixed \( p \geq 3 \) we have \( r_{p,2}(n) = O(n^{6p-12}) \).

This lemma will follow at once from Lemma 3.1 and the following result, which is proved implicitly in [4, 5] (see also [11]), and also follows from Lemma 8 of [2]. For the sake of completeness we sketch a proof.

Lemma 3.4. For every collection \( \mathcal{S} \) of \( p \) compact, convex, pairwise disjoint sets in the plane, there is a collection \( \mathcal{K} \) of \( 3p - 6 \) lines such that any two sets of \( \mathcal{S} \) are separated by at least one line of \( \mathcal{K} \).

Proof. We begin by constructing pairwise disjoint circumscribing polygons around the sets in \( \mathcal{S} \) and then we circumscribe a triangle \( T \) around all these polygons. Let \( A \) denote the set of these \( p \) polygons. It is convenient to assume that all directions of the sides of \( A \) and \( T \) have pairwise distinct slopes. Next we grow the polygons in \( A \) so as to maximize their area, thus obtaining polygons with overlapping boundaries but disjoint interiors. This is done by moving the sides of the polygons, one by one, until each polygon is of maximal area subject to the interiors of the polygons being disjoint, and subject to staying within the triangle \( T \). The precise expansion process is done by choosing, arbitrarily, a side of a polygon and moving the corresponding half-plane in the direction perpendicular to the side and away from the polygon’s interior. The side stops moving further only when it touches another polygon’s corner or it reaches the boundary of \( T \) or when it shrinks to a point and vanishes.

Once this process is finished, observe that the set of all lines containing all sides of the polygons without the lines containing the edges of \( T \) can serve as our separating set \( \mathcal{K} \). Thus it suffices to show that the total number of such lines is at most \( 3p - 6 \). To this end we define, following [4], a graph whose vertices are all polygons, where two are adjacent iff a side of one of them intersects the boundary of another, where the side is considered here as a relatively open set (i.e., it does not contain its endpoints). As proved in Lemma 1 of [4] this graph is planar and hence its number of edges is at most \( 3p - 6 \). Moreover, as shown in Lemma 2 of [4] there is a simple way to embed this graph in the plane so that each line containing a side of a polygon which is not part of a side of \( T \) is crossed by at least one edge of this graph, and each edge of the graph crosses only one such line. This supplies the desired bound and completes the proof of the lemma.

Another interesting consequence of Lemmas 2.4, 2.5, 3.1 and 3.4, is the following result.

Proposition 3.5. Let \( s(p,d) \) be the smallest number \( k \) such that for every collection \( \mathcal{S} \) of \( p \) compact, convex, pairwise disjoint sets in real \( d \)-space, there is a collection \( \mathcal{K} \) of \( k \) hyperplanes such that any two members of \( \mathcal{S} \) are separated by at least one hyperplane of \( \mathcal{K} \). Then

- For every fixed \( p \geq 2 \), \( s(p,1) = p - 1 \).
\begin{itemize}
\item $s(2,2) = 1$ and for every fixed $p \geq 3$, $s(p,2) = 3p - 6$.
\item For every fixed $p \geq 2$ and $d \geq 3$, $s(p,d) = \binom{p}{2}$.
\end{itemize}

Note that the construction in Section 2 is not really needed in order to prove that for $p \geq 2$ and $d \geq 3$, $s(p,d) = \binom{p}{2}$. Indeed, it is easier to observe that a collection of $p$ lines in general position in $R^3$ cannot be separated, in the sense of Lemma 3.1, by less than $\binom{p}{2}$ planes, and it is a simple matter to replace the lines with compact sets and to extend the result for higher dimensions as well. Note, also, that the assertion of Lemma 3.1 can be strengthened, with essentially the same proof, yielding the following result.

**Proposition 3.6.** We say that a collection \( \mathcal{H} \) of hyperplanes separates a collection \( \mathcal{S} \) of compact, convex pairwise disjoint sets, if for any $S \in \mathcal{S}$ the intersection of all half spaces bounded by an element of \( \mathcal{H} \) which contain $S$ contains no point of any other member of \( \mathcal{S} \). If for some fixed $p,d,k$, every collection \( \mathcal{S} \) of $p$ compact, convex, pairwise disjoint sets in real $d$-space, is separated by a collection \( \mathcal{H} \) of $k$ hyperplanes, then $r_{p,d}(n) = O(n^{dk})$.

Therefore, the construction in Section 2 provides examples of $p$ compact, convex, pairwise disjoint sets in $R^d$ which cannot be separated by less than $\binom{p}{2}$ hyperplanes even according to the separation as defined in the last proposition.

We proceed to derive an upper bound on $r_{p,d}(n)$. We use the following construction, which is similar to the one used in [8] for the real Euclidean space. Let $S$ be any set and let $p \geq 2$. With each list $(\pi_{r,s})_{1 \leq r < s \leq p}$ of $(\binom{p}{2})$ 2-partitions of $S$ associate a $p$-tuple $\pi = (\pi_1, \ldots, \pi_p)$ of subsets of $S$ as follows: for $i = 1, \ldots, p$ put

\[ \pi_i := \left( \bigcap_{j=1}^{p} \pi_{i,j} \right) \cap \left( \bigcap_{j=1}^{i-1} \pi_{i,j} \right). \]

Since $\pi_r \subseteq \pi_{r,s}$ and $\pi_s \subseteq \pi_{r,s}$ for all $1 \leq r < s \leq p$, the $\pi_i$ are pairwise disjoint. If moreover $\bigcup_{i=1}^{p} \pi_i = S$ then $\pi$ will be called the $p$-partition of $S$ associated with the given list. The following lemma extends the one provided in [8] to any space.

**Lemma 3.7.** For any space $(X, \mathcal{H})$ and $p \geq 2$, the set of separable $p$-partitions of any subset $S$ of $X$ equals the set of $p$-partitions associated with lists of $(\binom{p}{2})$ separable 2-partitions of $S$.

**Proof.** Fix a subset $S$ of $X$. Consider any $p$-partition $\pi$ associated with a list of $(\binom{p}{2})$ separable 2-partitions of $S$. For each pair $1 \leq r < s \leq p$, since $\pi_{r,s}$ is a separable 2-partition, there is an $H_{r,s} \in \mathcal{H}$ such that $\pi_r \subseteq \pi_{r,s} \subseteq H_{r,s}$ and $\pi_s \subseteq \pi_{r,s} \subseteq X \setminus H_{r,s}$. It follows that $\pi$ is separable. Conversely, let $\pi = (\pi_1, \ldots, \pi_p)$ be any separable $p$-partition. Consider any pair $1 \leq r < s \leq p$. By definition, there exists an $H_{r,s} \in \mathcal{H}$ such that $\pi_r \subseteq H_{r,s}$ and $\pi_s \subseteq X \setminus H_{r,s}$. Let $\pi_{r,s} := (\pi_{1,r,s}, \pi_{2,r,s})$ be the separable 2-partition of $S$ defined by $\pi_{1,r,s} := S \cap H_{r,s}$ and $\pi_{2,r,s} := S \setminus H_{r,s}$. Let $\pi'$ be the $p$-tuple associated with
the list of \( \pi_{i,j} \) obtained that way. Then the \( \pi_{i,j}' \) are pairwise disjoint and for \( i = 1, \ldots, p \) we have

\[
\pi_i \subseteq \left( \bigcap_{j=i+1}^{p} \pi_{i,j} \right) \cap \left( \bigcap_{j=1}^{i-1} \pi_{2,j} \right) = \pi_i'.
\]

Since

\[
S = \bigcup_{i=1}^{p} \pi_i \subseteq \bigcup_{i=1}^{p} \pi_i' \subseteq S,
\]

it follows that equality holds hence \( \pi = \pi' \) is the \( p \)-tuple associated with the constructed list of \( \left( \frac{p}{2} \right) \) separable 2-partitions. □

This lemma allows to extend upper bounds on the number of 2-partitions in any space to upper bounds on the number of \( p \)-partitions in that space. For instance, it implies that any set in any real space has at most one separable \( p \)-partition per each list of \( \left( \frac{p}{2} \right) \) separable 2-partitions, hence \( r_{p,d}(n) \leq r_{2,d}(n)^{\left( \frac{p}{2} \right)} \). This together with the known bound \( r_{2,d}(n) = O(n^d) \) gives a second proof (which is the one provided in [8]) of Lemma 3.2 above.

For spaces of finite VC-dimension, the so-called Sauer’s lemma, suitably rephrased, provides the following upper bound on the number of separable 2-partitions.

**Lemma 3.8** (Sauer [14]). Let \((X, \mathcal{F})\) be a space of VC-dimension \( d \). Then the number of separable 2-partitions of any set of \( n \) points in \( X \) is at most \( \sum_{i=0}^{d} \binom{n}{i} = O(n^d) \).

We obtain the following upper bound.

**Lemma 3.9.** For every \( p,d \), the maximum number of separable \( p \)-partitions of any set of \( n \) points in any space of VC-dimension \( d \) satisfies

\[
v_{p,d}(n) = O(n^{d\left( \frac{p}{2} \right)})
\]

**Proof.** By Lemma 3.7, any \( n \)-subset of \( X \) has at most one separable \( p \)-partition per each list of \( \left( \frac{p}{2} \right) \) separable 2-partitions. Since the number of separable 2-partitions of any \( n \)-subset is \( O(n^d) \) by Lemma 3.8, the bound follows. □

We can now combine all the necessary ingredients and obtain Theorems 1.1 and 1.2.

**Proof of Theorem 1.1.** Follows from Lemmas 2.4, 2.5, 3.2 and 3.3. □

**Proof of Theorem 1.2.** Follows from Lemma 2.4, the fact that real Euclidean \( d \)-space has VC-dimension \( d + 1 \), and Lemma 3.9. □
4. Separable partitions on the moment curve

Recall that the moment curve in $\mathbb{R}^d$ is the image of the map

$$M_d : \mathbb{R} \to \mathbb{R}^d : t \mapsto \begin{bmatrix} t \\ t^2 \\ \vdots \\ t^d \end{bmatrix}.$$ 

The moment curve is totally ordered in the obvious way. Therefore, we can and will identify any set $S$ of $n$ points on the moment curve with $[n] = \{1, \ldots, n\}$, and partitions of $S$ with partitions of $[n]$. We will regard a $p$-partition $\pi$ of $[n]$ also as the function $\pi : [n] \to [p]$ defined by $\pi(j) = i$ for all $j \in \pi_i$ and as the sequence $\pi = [\pi(1), \ldots, \pi(n)]$. We shall move freely among these representations of $\pi$.

Let $m_{p,d}(n)$ denote the maximum number of separable $p$-partitions of a set of $n$ points on the moment curve in $\mathbb{R}^d$. In this section, we first provide a combinatorial characterization of separable partitions of point sets on the moment curve, and then use it in estimating $m_{p,d}(n)$. Our characterization implies that, in fact, any set of $n$ points on the moment curve admits the same number $m_{p,d}(n)$ of separable $p$-partitions.

A $(p,d)$-sequence is a sequence $\pi - [\pi(1), \ldots, \pi(n)]$ with $\pi(i) \in [p]$ for all $i \in [n]$, and with the property that for any pair $1 \leq r < s \leq p$, the length of any subsequence $[r,s,r,s,\ldots]$ of $\pi$ whose elements alternate between $r$ and $s$ is at most $d + 1$. For instance,

$$[1,2,1,2,1,3,1,3,1,2,3,2,3,1]$$

is a $(3,5)$-sequence with a (non-unique) longest alternating subsequence $[1,2,1,2,1,2]$. The following lemma links the geometric property of a separable $p$-partition on the moment curve in $\mathbb{R}^d$ to the combinatorial property of a $(p,d)$-sequence.

**Lemma 4.1.** A $p$-partition $\pi = (\pi_1, \ldots, \pi_p)$ of a set of $n$ points on the moment curve in $\mathbb{R}^d$ is separable if and only if $[\pi(1), \ldots, \pi(n)]$ is a $(p,d)$-sequence.

**Proof.** Let $\pi = (\pi_1, \ldots, \pi_p)$ be a $p$-partition of a set of $n$ points on the moment curve in $\mathbb{R}^d$. The claim being trivial for $p = 1$ assume that $p \geq 2$. Now, $\pi$ is separable if and only if for each pair $1 \leq r < s \leq p$ the 2-partition $(\pi_r, \pi_s)$ is separable. It suffices then to show that for each such pair, $(\pi_r, \pi_s)$ is separable if and only if the subsequence of $\pi$ consisting of all occurrences of $r,s$ does not contain an alternating subsequence $[r,s,r,s,\ldots]$ of length $d + 2$. This is essentially known, see, e.g., [1]. For completeness, we include the short proof.

Consider any such pair $1 \leq r < s \leq p$, and let $V := \pi_r \cup \pi_s$. Then $V = \{v_1, \ldots, v_m\}$ with $v_i = M_d(t_i)$ for some $t_1 < \cdots < t_m$. Let $\mu := [\mu(1), \ldots, \mu(m)]$ denote the sequence which is the restriction of $\pi$ to $V$, so that $\mu(i) = r$ if $v_i \in \pi_r$ and $\mu(i) = s$ if $v_i \in \pi_s$. If $\mu$ contains an alternating subsequence of length $d + 2$, then it is obvious that $\pi_r$ and $\pi_s$ cannot be separated, since any hyperplane intersects the moment curve in at most $d$
points. On the other hand, if there is no such subsequence, then there are \( d \) real numbers \( y_1 < y_2 < \cdots < y_d \) whose images on the moment curve split it into \( d + 1 \) intervals, so that each element of \( \pi_r \) lies in one of the even intervals and each element of \( \pi_s \) lies in one of the odd intervals. Put \( \prod_{i=1}^{d} (t - y_i) = c_0 + c_1 t + \cdots + c_{d-1} t^{d-1} + t^d \). It is easy to check that the hyperplane \( H = \{ (x_1, x_2, \ldots, x_d) : c_0 + c_1 x_1 + c_2 x_2 + \cdots + c_{d-1} x_{d-1} + x_d = 0 \} \) separates \( \pi_r \) and \( \pi_s \).

Concluding, we see that for each pair \( 1 < r \neq s \leq p \), the 2-partition \( (\pi_r, \pi_s) \) is separable if and only if the length of any alternating subsequence \( [r, s, r, s, \ldots] \) of \( \pi \) is at most \( d + 1 \), and so \( \pi = (\pi_1, \ldots, \pi_p) \) is separable if and only if \( [\pi(1), \ldots, \pi(n)] \) is a \( (p, d) \)-sequence. \( \square \)

Lemma 4.1 implies that the number of \( (p, d) \)-sequences of length \( n \) is equal to \( m_{p,d}(n) \). We proceed to estimate the asymptotics of this number.

A DS\((p, d)\)-sequence, or Davenport–Schinzel \((p, d)\)-sequence (termed so in [13] and introduced in [3]), is a \((p, d)\)-sequence \( \pi = [\pi(1), \ldots, \pi(n)] \) in which \( \pi(i+1) \neq \pi(i) \) for all \( i \). For example, the sequence in Eq. (1) above is a DS\((3, 5)\)-sequence of maximum possible length 14. A \( k \)-composition of \( n \) is a \( k \)-tuple \( x = (x_1, \ldots, x_k) \) of positive integers summing up to \( n \). Clearly, each \((p, d)\)-sequence of length \( n \) is uniquely obtainable as the replication \( [\mu(1)^x, \ldots, \mu(k)^x] \) of a DS\((p, d)\)-sequence \( [\mu(1), \ldots, \mu(k)] \) by a \( k \)-composition \( (x_1, \ldots, x_k) \) of \( n \). Therefore, an estimate on the total number of DS\((p, d)\)-sequences of each length will lead to an estimate on the number \( m_{p,d}(n) \) of \((p, d)\)-sequences of length \( n \). But for our purpose here it suffices to consider the replications of a single long DS\((p, d)\)-sequence, since the number of such sequences is a function of \( p \) and \( d \) (and does not depend on \( n \)). Let \( \lambda_d(p) \) denote the maximum length of any DS\((p, d)\)-sequence. It is easy to see that \( \lambda_d(p) \) is finite and satisfies \( \lambda_d(p) \leq d \binom{p}{2} + 1 \).

We have the following simple estimate on \( m_{p,d}(n) \), where \( p \) and \( d \) are regarded as fixed parameters.

**Proposition 4.2.** For every fixed \( p, d \), the number of \((p, d)\)-sequences of length \( n \) satisfies

\[
m_{p,d}(n) = \Theta(n^{\lambda_d(p)-1}).
\]

**Proof.** Let \( \mu \) be any DS\((p, d)\)-sequence of length \( k = \lambda_d(p) \). This sequence gives rise to \( \binom{k-1}{k-1} \) replications, one for each \( k \)-composition of \( n \). This way, \( \binom{n-1}{k-1} \) distinct \((p, d)\)-sequences of length \( n \) are obtained. The upper bound follows in a similar manner, using the fact that the number of DS\((p, d)\)-sequences is a function of \( p \) and \( d \) only. \( \square \)

A \((p, d)\)-sequence is normal if the first occurrence of \( i \) precedes that of \( i+1 \) for all \( i \). In [13], the following recursive construction of a long normal DS\((p, d)\)-sequence \( \mu(p,d) \) was described for all \( d \geq p \). For \( p = 1 \) and any \( d \) set \( \mu(p,d) = [1] \). For \( d \geq p \geq 2 \) construct \( \mu(p, d) \) as follows: start with a sequence of length \( (p-1)(d-1)/2 \) + 1 of 1's; next, insert \( (p-1)((d-1)/2) \) symbols so as to separate each pair of 1's, where the first \( (d-1)/2 \) of the new symbols are \( p \), the
next \([(d-1)/2]\) are \(p-1\), and so on; then, append to the right the normal sequence 
\(\mu(p-1,d-1)\) on the symbols \(\{2,3,\ldots,p\}\), and, if \(d\) is even, an additional last symbol 
1 to its right; finally, apply the necessary permutation of \([p]\) to the elements of the
sequence so as to make it normal. For example, the sequence in Eq. (1) above is
precisely the normal DS(3,5)-sequence \(\mu(3,5)\) obtained this way. With some care, the
results of [13] can be shown to imply the following bound on the length of \(\mu(p,d)\)
and hence on the value of \(\lambda_d(p)\).

**Proposition 4.3.** The bound \(\lambda_d(p) = (d - \frac{1}{2} p) \left(\frac{p}{2}\right) + 1\) holds for all \(d \geq p\).

Propositions 4.2 and 4.3 give the following lower bound on \(m_{p,d}(n)\).

**Proposition 4.4.** For every fixed \(p, d\) with \(d \geq p\),

\[m_{p,d}(n) = \Omega(n^{(d-(1/2)p)}(\frac{p}{2})).\]

**Theorem 4.5.** Let \(0 < \varepsilon \leq \frac{1}{2}\) be any constant. For every \(p,d\) with \(d \geq (1/2\varepsilon)p\), the
number \(m_{p,d}(n)\) of separable p-partitions of any set of \(n\) points on the moment curve
in real d-space satisfies

\[\Omega(n^{(1-\varepsilon)d}(\frac{p}{2})) \leq m_{p,d}(n) \leq O(n^{d(\varepsilon)}).\]

**Proof.** If \(0 < \varepsilon \leq \frac{1}{2}\) and \(d \geq (1/2\varepsilon)p\) then \(d \geq p\) and \(d - \frac{1}{2} p \geq (1 - \varepsilon)d\). The lower
bound then follows from Proposition 4.4. The upper bound follows from Lemma 3.2
and the inequality \(m_{p,d}(n) \leq \tau_{p,d}(n)\).

5. Conclusions and remarks

Theorem 1.1 determines the asymptotic behavior of \(\tau_{p,d}(n)\) for all fixed admissible \(p\)
and \(d\) up to a constant factor. However, the bounds for spaces of finite VC-dimension
\(d\) given in Theorem 1.2 are not that tight. It might be interesting to close the gap
between the lower and upper bounds here.

For points on the moment curve, the asymptotic behavior of \(m_{p,d}(n)\) is reduced, in
Proposition 4.2, to the well-studied problem of determining or estimating the max-
imum possible length \(\lambda_d(p)\) of a DS(\(p,d\))-sequence. The known bounds for this
function can be found in [15]. In particular, \(\lambda_2(p) = 2p - 2\), \(\lambda_3(p) = \Theta(p \zeta(p))\) and
\(\lambda_d(p) = \Theta(p 2^{\zeta(p)})\), where \(\zeta(p)\) is the inverse of Ackermann’s function. Thus \(\lambda_3(p)\) is
already a superlinear function of \(p\).

**References**

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