1. Introduction

In this paper, we mainly investigate the $L^p - L^q$ estimates of the solution for the following dispersive equation:

\[
\begin{aligned}
\partial_t u(t, x) &= i P(D) u(t, x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d, \\
 u(0, x) &= u_0(x), \quad x \in \mathbb{R}^d,
\end{aligned}
\]

(1.1)

where $D = -i(\partial_1, \ldots, \partial_n)$, $n \geq 1$, and $P : \mathbb{R}^d \to \mathbb{R}$ is a real elliptic polynomial of order $m \geq 2$ ($m$ must be even as $n \geq 2$). Without loss of generality, we may assume that $P_m(\xi) > 0$ for $\xi \neq 0$, where $P_m(\xi)$ is the principal part of $P(\xi)$. As we know, for every initial value $u_0 \in S(\mathbb{R}^d)$ (the Schwartz function space), the solution of Cauchy problem (1.1) is given by

\[
u(t, \cdot) = e^{i P(D)} u_0 := \mathcal{F}^{-1} (e^{i P}) * u_0,
\]

where $\mathcal{F}$ (or *) denotes Fourier transform, $\mathcal{F}^{-1}$ its inverse and $\mathcal{F}^{-1}(e^{i P})$ is understood in the distributional sense.

In order to treat with the $L^p - L^q$ estimates of the solution of Eq. (1.1), it is a key to estimate the fundamental solution $\mathcal{F}^{-1}(e^{i P}) (x)$ which is a kind of oscillatory integral. Since $P$ is elliptic, by integrations by parts one easily checks that $\mathcal{F}^{-1}(e^{i P}) (x)$ is an infinitely differential function in $x$ variable for each $t \neq 0$ (cf. [17]). To get the pointwise estimates of $\mathcal{F}^{-1}(e^{i P}) (x)$ in $(t, x)$-variable, some further assumptions are needed. We first recall that $P(\xi)$ is nondegenerate if the Hessian matrix $(\partial_i \partial_j P_m(\xi))_{n \times n}$ of $P_m$ satisfies

\[
HP_m(\xi) := \det \left( \frac{\partial^2 P_m(\xi)}{\partial \xi_i \partial \xi_j} \right)_{n \times n} \neq 0, \quad \xi \neq 0.
\]

(1.2)
Since $HP_m(\xi)$ is exactly the principal part of the polynomial $HP(\xi) = \det(\partial_i \partial_j P(\xi))_{n \times n}$, the condition (1.2) actually implies that $HP(\xi)$ is an elliptic polynomial of order $n(m - 2)$. Moreover, it was well known that all one-dimensional polynomials of order $m \geq 2$ are nondegenerate, and as $n \geq 2$, the condition (1.2) is also equivalent to that (cf. [32,41])

(\Sigma): the level hypersurface $\Sigma := \{\xi \in \mathbb{R}^n; P_m(\xi) = 1\}$ of $P_m$ has nonzero Gaussian curvature everywhere.

It was well known that these nondegenerate type conditions not only play a key role in some topics of harmonic analysis (cf. [37,39]), but also are very important in the study of Eq. (1.1). More explicitly, if $P$ is homogeneous and satisfies the condition (\Sigma), then the optimal $L^p-L^q$ estimates for the evolution operator $e^{itP(D)}$ ($t \neq 0$) can be deduced from [31, Theorem 4.1]. Actually, Miyachi [31] mainly considered the $H^p-H^q$ boundedness of a class of singular multipliers $\psi(\xi)|\xi|^{b} e^{it|\xi|^p}$ ($a > 0$, $b \in \mathbb{R}$) where $\psi(\xi)$ is equal to 1 for large $\xi$ and 0 near origin. But from [31, Remark 4.2] one knows that his results even hold for a positive homogeneous phase function $\phi$ of degree $m$ with the same assumption as $P$. Moreover, dropping homogeneity of $P$, based on one or another equivalently nondegenerate conditions on $P$, the $L^p-L^q$ estimates and some related topics have also been extensively studied by several papers (see e.g. [2,5,6,10,11,16,17,27,29]). In particular, all one-dimensional cases have been covered by these papers.

On the other hand, as $n \geq 2$, there exist many elliptic polynomials which are degenerate, such as

$$\xi_1^6 + 2\xi_1^2 \xi_2^2 + \xi_2^6$$

and

$$\xi_1^m + \xi_2^m + \cdots + \xi_n^m \quad (m = 4, 6, \ldots)$$

eq etc. Since lacking of the nondegenerate conditions like (1.2) for these degenerate $P$, it would be more difficult to estimate the oscillatory integral $\mathcal{F}^{-1}(e^{itP(\xi)})$ ($t \neq 0$). Indeed, such difficulty is essentially due to the failure of the principle of stationary phase, and also shared by many other degenerate oscillatory integrals arisen in other problems (cf. [25, Chapter VII], [39, Chapters VIII–IX]).

Motivated by these examples, based on a powerful result of [8], recently Zheng et al. [43] have made an interesting work on the $L^p-L^q$ estimates of $e^{itP(D)}$ ($t \neq 0$) if $P$ is homogeneous and the level hypersurface $\Sigma$ is convex. For instance, it is easy to check that the simple degenerate homogeneous polynomials

$$\xi_1^6 + \xi_2^6 \quad (m = 4, 6, \ldots)$$

and

$$\xi_1^4 + 6\xi_1^2 \xi_2^2 + \xi_2^4$$

meet the requirement of convexity (see [20, Proposition 2.1]). However, for a degenerate nonhomogeneous elliptic $P$ like as

$$\xi_1^6 + 2\xi_1^2 \xi_2^2 + \xi_2^6$$

in general, because its principal part $P_m$ cannot dominate the lower terms of $P$ in the oscillatory integral $\mathcal{F}^{-1}(e^{itP(\xi)})$ in contrast to nondegenerate case, there exists an essential difficult to use the method in [43] to study general degenerate nonhomogeneous polynomials. Therefore, in order to study a class of degenerate nonhomogeneous cases, a kind of condition (H_b) was introduced by Yao and Zheng [41]:

$$(H_b): 1/\lambda_k(\xi) = 0 (|\xi|^{-bn_2 \|b\|}) \quad \text{as} \quad |\xi| \to \infty \quad \text{for each} \quad k \in \{1, \ldots, n\}, \quad \text{where} \quad b \in (0, 1], \quad \lambda_k(\xi) \quad \text{is the kth-eigenvalue of the Hessian matrix} \quad (\partial_i \partial_j P(\xi))_{n \times n} \quad \text{which is a positive definite matrix for sufficiently large} \quad |\xi|.$$ 

When $b = 1$, it follows from [41, Proposition 2.1] that the condition (H_1) is exactly equivalent to the nondegeneracy of $P$, i.e. the condition (1.2). Hence, when $b < 1$, (H_b) must be a degenerate condition where $b$ is an important index which reflects the degeneracy of $P$. For example, by a direct calculation we can check that all these elliptic polynomials

$$P(\xi) = \xi_1^m + \xi_1^2 \xi_2^2 -4 + \xi_2^2 \xi_1^{-4} + \xi_2^2 \quad (m = 6, 8, \ldots)$$

degenerate and satisfy the condition (H_b) with $b = \frac{m-4}{m-1} \in (0, 1)$.

Assume that $P$ satisfies the condition (H_b) for some $b \in [\frac{1}{2}, 1]$, the authors of [41] have showed the following estimate:

$$|\mathcal{F}^{-1}(e^{itP(\xi)})| \leq C(|t|^{-\sigma} + |t|^p), \quad t \neq 0.$$ 

(1.3)

Because of lacking of the decay estimate in $x$ variable, it is obvious that the estimate (1.3) is not complete. Thus to overcome the deficiency, under the same assumption (H_b), this paper is mainly devoted to obtain a new pointwise estimate for $\mathcal{F}^{-1}(e^{itP(\xi)})$ in $(t, x)$-variable (see Theorem 2.2), which is exactly consistent with the nondegenerate case if $b = 1$. More generally, in this paper we prove a pointwise estimate for $\mathcal{F}^{-1}(a(\xi)e^{itP(\xi)})$ where $a(\xi)$ belongs to a smooth symbol class $S^d(R^n)$ (see Section 2 for its definition). Moreover, we also establish the $L^p-L^q$ estimates of $\langle D \rangle^{d} e^{itP(D)}(t \neq 0)$ where $\langle D \rangle^{d} := (1 - \Delta)^{d/2}$ denotes the Bessel potential of order $-d$ (see [38, p. 131]). For $d > 0$, this kind of the $L^p-L^q$ estimates yield so-called global smoothing effects of Eq. (1.1), as emphasized in [22,27] on $L^p$ for nondegenerate polynomials.

Furthermore, as an application of the $L^p-L^q$ estimates of $\langle D \rangle^{d} e^{itP(D)}(t \neq 0)$, another goal of this paper is to show that higher-order Schrödinger operator $iP(D) + V(x, D)$ in $L^p(R^n)$ generates a fractionally integrated group where $V(x, D)$ can be a differential operator of lower-order than $P$ with some (real or complex) integrable coefficients (see Theorem 4.4).

It is well known that the semigroup of operator is an abstract tool to study Cauchy problems. However, the elliptic operator $iP(D)$ in Eq. (1.1) cannot generate a classical $C_0$-semigroup on $L^p(R^n)$ ($p \neq 2$) (see [24]). Since then, several generalizations of $C_0$-semigroup were introduced, such as distribution semigroup, integrated semigroup and regularized
semigroup, etc., as well as applied to general differential operators (including \(iP(D)\) and even nonelliptic class) and associated Cauchy problems (see e.g. [1,4,19,18,22,23,42]).

In this paper, we also consider the following Cauchy problem corresponding to \(iP(D) + V(x, D)\) (i.e. generalized Schrödinger equation):

\[
\begin{align*}
\partial_t u(t, x) &= iP(D)u(t, x) + V(x, D)u(t, x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\
u(0, x) &= u_0(x), \quad x \in \mathbb{R}^n.
\end{align*}
\]  

(1.4)

As a consequence of Theorem 4.4, certain \(L^p - L^q\) estimates of solution for Eq. (1.4) can be obtained by employing Strauß's fractional powers (see Theorem 4.7). When \(V(x, D)\) is a suitable integrable function \(V(x)\), similar arguments for Eq. (1.4) have also been discussed in [5,30,34,41,43], etc., by semigroup methods. Nevertheless, none of them can deal with Eq. (1.4) with a differential perturbed operator \(V(x, D)\).

Finally, with respect to the classical Schrödinger equation, i.e. Eq. (1.4) with \(P(D) = -\Delta\) and \(V(x, D) = iV(x)\) where \(V(x)\) is a suitable real potential, the study of the \(L^p - L^q\)-estimates for Schrödinger group \(e^{-it(\Delta - V)}\) has received great attentions in recent years motivated by nonlinear problems (see Bourgain [9, pp. 17–27], also refer to Schlag [35] for a recent survey on this subject). However, for higher-order Schrödinger group \(e^{it(P(D) + V)}\), a similar analysis of the \(L^p - L^q\)-estimates seems undeveloped to the best of the authors’ knowledge. Clearly, it would be very interesting to further investigate this problem. On the other hand, several other topics related to higher-order Schrödinger operator \(P(D) + V(x, D)\) have been studied for many years by many people, for instance, see Hörmander [25, Chapters XIV, XXX] for scattering theory, Schechter [36] for spectral theory on \(L^p(\mathbb{R}^n)\), etc.

The paper is organized as follows.

Section 2 is to establish the \(L^p - L^q\) estimates of \(D^d e^{itP(D)}\) \((t \neq 0)\) by using the pointwise estimates and interpolation theorems, and also deduce local Strichartz inequalities.

Section 3 is to give the proof of Theorem 2.1, that is, we show the pointwise estimates for \(\mathcal{F}^{-1} (\alpha(\cdot) e^{i\xi P})(x)\). Here our proof depends on the flexible frequency decomposition method from [27,41], but needs more delicate analysis in order to obtain our decay results. In addition, it should be pointed out that our method is different from the polar coordinate transform method used in [5,2,16,17,29], since the polar coordinate transform method fails generally to degenerate cases.

Finally, combining with the \(L^p - L^q\) estimates, in Section 4 we show that higher-order Schrödinger operator \(iP(D) + V(x, D)\) in \(L^p(\mathbb{R}^n)\) generates the fractionally integrated group by perturbed method, from which we deduce some \(L^p\) estimates for the solution of Eq. (1.4).

2. The \(L^p - L^q\) estimates of the solution with regularity

Throughout this paper, let \(n \geq 1\) and \(m\) be an even positive integer \(\geq 2\). Suppose \(P : \mathbb{R}^n \to \mathbb{R}\) is always a real elliptic polynomial of order \(m\) with \(P_m(\xi) > 0\) for \(\xi \neq 0\) and satisfies the condition (H\(_b\)) for some \(b \in (0, 1)\). Thus there exist absolute constants \(c_i > 0\) \((i = 1, 2)\) and \(L > 0\) such that for any \(|\xi| \geq L\), we have

\[
\begin{align*}
\min_k \{\lambda_k(\xi)\} &\geq c_1 |\xi|^{(m-2)b} , \\
c_1 |\xi|^{m-1} &\leq |\nabla P(\xi)| \leq c_2 |\xi|^{m-1} ,
\end{align*}
\]  

(2.1)

and

\[
\begin{align*}
\|\partial^\alpha P(\xi)\| &\leq c_2 |\xi|^{m-|\alpha|} , \quad \forall |\alpha| \leq n + 1.
\end{align*}
\]  

(2.3)

Indeed, the inequality (2.1) is from the condition (H\(_b\)), (2.2) is due to the ellipticity of \(P\) and (2.3) actually is true for any polynomial of order \(m\). Obviously, these conditions are mainly imposed on the high frequency of \(P\) which is the most difficult part in estimating the oscillatory integral \(I(t, x)\) defined below (see Lemmas 3.2 and 3.3 in Section 3), whereas for the lower frequency part corresponding to the region: \(|\xi| < L\), the pointwise estimate easily follows by Fourier transform (see Lemma 3.1). In the following, constants \(c_1, c_2\) and \(L\) above are regarded as some fixed absolute constants, just like as \(m, n\), etc.

Note that any second-order elliptic polynomial \(P\) can be written as \((\xi, A\xi) + B\xi + C\), where \(A\) is a positively defined matrix, \(B \in \mathbb{R}^n\) and \(C \in \mathbb{R}\). Hence it is certainly nondegenerate, and naturally assumed that \(b = 1\) in the condition (2.1) as \(m = 2\). In addition, when \(m \geq 4\), we observe that any homogeneous elliptic polynomial \(P\) satisfying the condition (H\(_b\)) for some \(b \in (0, 1)\) must be nondegenerate, i.e. such \(P\) also satisfies the condition (H\(_2\)). Indeed, using the homogeneity of \(P\) and (2.1), it follows that the Hessian’s determinant \(HP(\xi)\) of \(P\) also is a homogeneous polynomial of order \(m(m-2)\) and satisfies

\[
HP(\xi) = \sum_1^n \lambda_k(\xi) \geq c_1 |\xi|^{bn(m-2)} \quad \text{for} \quad |\xi| \geq L.
\]

Hence, it implies that \(HP(\xi) \geq c|\xi|^{n(m-2)}\) for any \(\xi \in \mathbb{R}^n\) where \(c = c_1 L^{(1-b)n(m-2)}\), and \(P\) naturally is nondegenerate by (1.2).
By above remarks, we know that if a degenerate elliptic polynomial \( P \) satisfies the \((H_b)\) with some \( b < 1 \), then the \( P \) must be nonhomogeneous, and its lower-order part \((P - P_m)\) of \( P \) has an essential effect for the condition \((2.1)\). At this point, the condition \((H_b)\) relative to nondegeneracy seems to share certain similarity with the hypoellipticity relative to ellipticity (see [25, Chapter II, p. 61]). In particular, it was well known that any homogeneous hypoelliptic polynomial with constant coefficients must be elliptic.

Let \( d \in \mathbb{R} \), we denote by \( S^d(\mathbb{R}^n) \) the set of all \( \alpha(\xi) \in C^\infty(\mathbb{R}^n) \) such that for all \( \alpha \in \mathbb{R}^n \), the derivative \( \partial^\alpha \alpha(\xi) \) has the following bound

\[
|\partial^\alpha \alpha(\xi)| \leq C_{\alpha} (1 + |\xi|)^{d-|\alpha|} \quad \forall \xi \in \mathbb{R}^n.
\]

Furthermore, we also define \( \mu_s \) and \( v_s \) by

\[
\mu_s = \frac{n(m-2) - 2s}{2(m-1)}, \quad v_s = \frac{n + 2s}{2(m-1)} \quad \text{for} \ s \in \mathbb{R}.
\]  

(2.4)

Clearly, \( \mu_s + v_s = \frac{n}{2} \).

Suppose \( \alpha(\xi) \in S^d(\mathbb{R}^n) \), then we consider the following oscillatory integral:

\[
I(t, x) := \mathcal{F}^{-1}(a(t) e^{itP})(x) = \int_{\mathbb{R}^n} e^{itP(\xi) + it\langle x, \xi \rangle} \alpha(\xi) \, d\xi,
\]  

(2.5)

which has the pointwise estimate as follows:

**Theorem 2.1.** Let \( P \) satisfy the condition \((H_b)\) for some \( b \in (0, 1) \). If \(-\frac{n}{2} \leq d \leq \frac{n}{2}(m-2)(2b-1)\) and \( s_d = d + (1-b)n(m-2) \), then we have

\[
|I(t, x)| \leq \begin{cases} C (1 + |t|^{-1}|x|)^{-\mu_d}, & |t| \geq 1, \\ C |t|^{-\frac{n+m}{2} (1 + |t|^{-1/m}|x|)^{-\mu_d}}, & 0 < |t| < 1, \end{cases}
\]

(2.6)

where \( C \) is some absolute constant independent of \((t, x)\)-variables. In particular, the following uniform estimate

\[
|I(t, x)| \leq C (1 + |t|^{-\frac{n+m}{2}}), \quad t \neq 0,
\]

holds.

The proof of Theorem 2.1 is lengthy and given in the next section. It is easy to obtain from (2.4) that

\[
\mu_d = \frac{n(m-2)(2b-1) - 2d}{2(m-1)},
\]

which obviously displays how the parameters \( b \) and \( d \) affect the decay index of the oscillatory integral (2.5). Actually, the \( \mu_d \) increases with rate \( n(m-2)/(m-1) \) as \( b \) converges to 1 (i.e. the decrease of nondegeneracy on \( P \)), and decreases with speed \( 1/(m-1) \) as \( d \) increases (i.e. the increase of regularity). Hence, in this sense the degenerate index \( b \) has the larger impact on the pointwise estimate than the index \( d \) of regularity. Moreover, since

\[-\frac{n}{2} \leq d \leq \frac{n}{2}(m-2)(2b-1),\]

thus it implies that \( \mu_d \geq 0 \), and we need assume that \( b \in [\frac{m-3}{2(m-2)}, 1] \) for \( m \geq 4 \). Of course, we also remember that \( b = 1 \) when \( m = 2 \).

Finally, note that

\[
[0, 1] \subset \left[ \frac{m-3}{2(m-2)}, 1 \right] \quad \text{for} \ m \geq 4,
\]

and \( \frac{1}{2}n(m-2)(2b-1) \geq 0 \) as \( b \in [\frac{1}{2}, 1] \). Therefore, we can take \( \alpha(\xi) = 1 \) in (2.5) (correspondingly \( d = 0 \)). Thus from Theorem 2.1 we have the following consequence immediately.

**Theorem 2.2.** If \( P \) satisfies the condition \((H_b)\) for some \( b \in [\frac{1}{2}, 1] \), and \( s_0 = (1-b)n(m-2) \), then we have

\[
|\mathcal{F}^{-1}(e^{itP})(x)| \leq \begin{cases} C (1 + |t|^{-1}|x|)^{-\mu_0}, & |t| \geq 1, \\ C |t|^{-\frac{n+m}{2} (1 + |t|^{-1/m}|x|)^{-\mu_0}}, & 0 < |t| < 1, \end{cases}
\]

(2.7)

where \( C \) is some positive constant independent of \((t, x)\)-variables. In particular, the following uniform estimate

\[
|\mathcal{F}^{-1}(e^{itP})(x)| \leq C (1 + |t|^{-\frac{n+m}{2}}), \quad t \neq 0,
\]

holds.
Remark 2.3. (i) Obviously, Theorem 2.2 improves the corresponding result of [41, Theorem 2.2], where under the same assumptions as in Theorem 2.2, the only uniform estimate (1.3) was obtained by using a similar (less refined) process comparing with the one here (see the proof of Theorem 2.1 in the next section). With respect to the estimate (2.7), we comment on that when $b = 1$, it is sharp at least for local time (see Remark 2.3(ii) below). But, when $b < 1$, the optimality of (2.7) is unknown up to now. It seems to be hard, and at least involves the resolution of singularity on $P$. For example, see [7] for a detailed study of $\mathcal{F}^{-1}(e^{itQ})(x)$, where $Q$ is any three-order polynomials of two variables.

(ii) If $b = 1$ (i.e. $P$ is nondegenerate), then it follows from (2.7) that

$$
|\mathcal{F}^{-1}(e^{itP})(x)| \leq \begin{cases} 
C(1 + |t|^{-1/m})^{\frac{n(m - 2)}{2(m - 1)}}, & |t| \geq 1, \\
C|t|^\frac{n}{m} (1 + |t|^{-1/m}) - \frac{n-2}{m+n-2}, & 0 < |t| < 1,
\end{cases}
$$

which are exactly identical with the results in [29] based on a different method originated in [5]. If $P$ is homogeneous and nondegenerate, then by scaling the estimates (2.8), it can be unified into the following sharp form in $(t, x)$-variable:

$$
|\mathcal{F}^{-1}(e^{itP})(x)| \leq C|t|^\frac{n}{m} (1 + |t|^{-1/m}) - \frac{n-2}{m+n-2}, \quad t \neq 0.
$$

In particular, we remark that the index $\frac{n(m-2)}{2(m-1)}$ is optimal by testing the special case $e^{it|x|^m}$. In fact, from Proposition 5.1(ii) of Miyachi [31, p. 289], we can obtain that

$$
|\mathcal{F}^{-1}(e^{it|x|^m})(x)| \in C^\infty(\mathbb{R}^d)
$$

and

$$
F^{-1}(e^{it|x|^m})(x) = A|x|^{-\frac{n(m-2)}{m-1}} e^{it|x|^{m-1}} + O(|x|^{-\frac{n(m-2)}{m-1}}) \quad \text{as } |x| \to \infty,
$$

where $A, B$ are two absolute constants. Clearly, this implies that there exists a positive constant $C'$ such that

$$
|\mathcal{F}^{-1}(e^{it|x|^m})(x)| \geq C'(1 + |x|)^{-\frac{n(m-2)}{m-1}}, \quad x \in \mathbb{R}^d,
$$

which shows that the decay index $\frac{n(m-2)}{2(m-1)}$ cannot be improved even for the simple nondegenerate polynomial $|x|^m$ ($m$ is even integer emphasized as above). Note that in this case the corresponding level surface $\Sigma$ is exactly the spherical surface $S^{n-1}$ of $\mathbb{R}^d$.

(iii) If $b = 1$ and $a(\xi) \in S^d(\mathbb{R}^d)$, then $\mu_{sg} = \frac{n(m-2)-2d}{2(m-1)}$ in (2.6), which also is a sharp decay index in $x$ variable. In particular, when $0 \leq d \leq n(m-2)/2$, from (2.6) the uniform estimate

$$
|\mathcal{F}^{-1}(a(\xi)e^{itP})(x)| \leq C(1 + |t|^{-\frac{d}{m-1}}), \quad t \neq 0,
$$

holds. As we know, the estimate with some different $a(\xi)$ can be found in several papers. For instance, if taking $a(\xi) = |HP(\xi)|^{1/2} \in S^{n(m-2)/2}(\mathbb{R}^d)$ where $HP(\xi)$ is the determinant of the Hessian matrix of $P$, then we have

$$
|\mathcal{F}^{-1}(e^{itP})(x)| \leq C(1 + |t|^{-\frac{n}{2}}), \quad t \neq 0,
$$

which was directly implied in [27, Lemmas 2.7, 3.4] where a general phase function $\phi(\xi)$ was considered for some similar nondegenerate conditions as $P$. As $P$ is a general one-dimensional phase function of order $m \geq 2$ (may be odd integer), such uniform estimate also appeared in [11,10]. Moreover, some specific nondegenerate fourth-order cases of multi-dimension also were studied in [6].

(iv) Let $b, d$ be as in Theorem 2.1. Given any $T > 0$, by a simple calculation from (2.6) we can deduce the following local pointwise estimate:

$$
|I(t, x)| \leq C_T |t|^{-\frac{n+kd}{m-1}} (1 + |t|^{-1/m})^{-\mu_{sg}}, \quad 0 < |t| < T,
$$

where $s_d = d + (1 - b)n(m - 2)$ and $C_T$ increases polynomially as $T$ becomes large (except for homogeneous cases). If $b = 1$ and $-n/2 < d \leq n(m - 2)/2$, then $s_d = d$ and the estimate can be recognized by [16, Theorem 2.1] for one-dimensional cases and [17, Corollary 1.2] for higher-dimensional cases, where any global estimate about time $t$ like (2.6) was not obtained.

(v) Let $a(z) = |z|^2$ where $z = d + iy \in C$ and $|\xi| = (1 + |\xi|^2)^{1/2}$. Then $a(z) \in S^d(\mathbb{R}^d)$ for each $y \in \mathbb{R}$. Furthermore, if $-\frac{n}{2} \leq d \leq \frac{1}{2}n(m-2)(2b-1)$, and keeping track of $y$ in the proof of Theorem 2.1, then the estimate (2.6) with such $a(z)$ can be improved to

$$
|J_z(t, x)| \leq \begin{cases} 
C(1 + |y|^2(1 + |t|^{-1/m}))(1 + |t|^{-1/m})^{-\mu_{sg}}, & |t| \geq 1, \\
C(1 + |y|^2e^{itP})^{-\frac{n+kd}{m-1}} (1 + |t|^{-1/m})(1 + |t|^{-1/m})^{-\mu_{sg}}, & 0 < |t| < 1,
\end{cases}
$$

where $J_z(t, x) := \mathcal{F}^{-1}(a(z)e^{itP})(x)$ and $C$ is a constant independent of $t, x, y$.

Now, we turn to establish the $L^p - L^q$ estimates of $|D|^{d/2}e^{itP/D}$ ($t \neq 0$) where $|D|^d = (1 - \Delta)^{d/2}$ denotes the Bessel potential of order $-d$. In the sequel, we assume that $b \in [\frac{1}{2}, 1]$ and $0 \leq d \leq \frac{1}{2}n(m-2)(2b-1)$ in view of our application in Section 4. Of course, there also exist some similar arguments as Theorem 2.4 for other pairs of $(b, d)$ by the same method. In this paper we omit the presentation of these parallel results for the sake of simplicity.
To state our main results (i.e. Theorem 2.4), we define

\[
\delta = \delta(b, d) = \begin{cases} 
\infty, & b = \frac{1}{2}, \\
\frac{2n(m-1)}{n(m-2)(2b-1)-2d}, & b \in (\frac{1}{2}, 1].
\end{cases}
\]
\[
\tau = \tau(b, d) = \begin{cases} 
2, & b = \frac{1}{2}, \\
\frac{2n(m-1)(2b-1)}{n(m-2)(2b-1)-2d}, & b \in (\frac{1}{2}, 1].
\end{cases}
\]

(2.10)

(2.11)

and denote by \( \Box_{b,d} \) a closed quadrangle of the plane by the following four vertex (also see Fig. 1 below):

\[
A = \left( \frac{1}{\tau}, \frac{1}{\tau} \right), \quad B = \left( 1, \frac{1}{\delta} \right), \quad C = (1, 0), \quad D = \left( \frac{1}{\delta}, 0 \right).
\]

where \( \delta' \) and \( \tau' \) satisfy \( \frac{1}{\delta'} + \frac{1}{\tau'} = 1 \) and \( \frac{1}{\delta'} + \frac{1}{\tau'} = 1 \), respectively.

Moreover, we denote by \( L^{p,q}(\mathbb{R}^n) \) the Lorentz space \( (L^p, \infty)(\mathbb{R}^n) = L^{\infty}(\mathbb{R}^n) \) (see [21] for the definition of \( L^{p,q}(\mathbb{R}^n) \)). Also denote by \( \| \cdot \|_{L^p} \) the norm in \( L^p(\mathbb{R}, \mathbb{R}^n) \) (the space of all bounded linear operators from \( L^p(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \)).

For any \( u_0 \in S(\mathbb{R}^n) \) (the Schwartz function space), we can write

\[
\langle D \rangle^d e^{i\xi P(D)} u_0 = J_d(t, \cdot) * u_0,
\]

(2.12)

where \( J_d(t, x) = \mathcal{F}^{-1}(\cdot)^d e^{i\xi P}(x) \).

**Theorem 2.4.** If \( P \) satisfies the condition (H2) for some \( b \in (\frac{1}{2}, 1] \) and \( 0 \leq d \leq \frac{1}{2}(m - 2)(2b - 1) \), then

\[
\| (D)^d e^{i\xi P(D)} \|_{L^p \to L^q} \leq \begin{cases} 
C |t|^\frac{n}{p} \frac{1}{|\xi|}, & |t| \geq 1, \\
C |t|^\frac{n}{p} (\frac{1}{|\xi|} - \frac{m}{p}), & 0 < |t| < 1,
\end{cases}
\]

(2.13)

where \( \frac{1}{p}, \frac{1}{q} \in \Box_{b,d}, s_d = d + (1 - b)n(m - 2) \) and

\[
L^p_d - L^n_d := \begin{cases} 
L^1 - L^{\infty}, & \text{if } \left( \frac{1}{p}, \frac{1}{q} \right) = (1, \frac{1}{2}), \\
L^{\infty} - L^{1}, & \text{if } \left( \frac{1}{p}, \frac{1}{q} \right) = (\frac{1}{n}, 0), \\
L^p - L^q, & \text{otherwise}.
\end{cases}
\]

Proof. Let us start with \( b \in (\frac{1}{2}, 1] \). Since \( \xi \in S(\mathbb{R}^n) \) by Theorem 2.1, for each \( t \neq 0 \) we obtain \( J_d(t, \cdot) \in L^p(\mathbb{R}^n) \) for \( q > \delta \) and \( J_d(t, \cdot) \in L^{\infty}(\mathbb{R}^n) \). Therefore, from (2.12) and the Young (or weak Young) inequality (see [21, p. 22]), it follows that

\[
\| (D)^d e^{i\xi P(D)} \|_{L^p \to L^q} \leq \begin{cases} 
C |t|^\frac{n}{p} \frac{1}{|\xi|}, & |t| \geq 1, \\
C |t|^\frac{n}{p} (\frac{1}{|\xi|} - \frac{m}{p}), & 0 < |t| < 1,
\end{cases}
\]

(2.14)

which prove the points \( (1, \frac{1}{2}) \) in the side \( \mathcal{B} \).

To show the case \( \left( \frac{1}{p}, \frac{1}{q} \right) = (1, \frac{1}{2}) \) (i.e. the end point A of Fig. 1), we introduce an analytic family of operators as follows:

\[
W_{\sigma + iy}(t) u_0 := \langle F^{-1}(\cdot)^{\sigma + iy} P(D) \rangle * u_0 = J_{\sigma + iy}(t, \cdot) * u_0,
\]

where \( 0 \leq \sigma \leq \frac{1}{2}(n - 2)(2b - 1) \) and \( y \in \mathbb{R} \).

When \( \sigma = \frac{1}{2}(n - 2)(2b - 1) \), it follows from Remark 2.3(v) and the Young inequality that

\[
\| W_{\sigma + iy}(t) \|_{L^1 \to L^\infty} \leq \begin{cases} 
C (1 + |y|^p)^p, & |t| \geq 1, \\
C (1 + |y|^p |t|^\frac{n}{2}), & 0 < |t| < 1.
\end{cases}
\]

On the other hand, if \( \sigma = 0 \), then the Plancherel's theorem gives \( \| W_{iy}(t) \|_{L^2 \to L^2} \leq 1 \). Thus by the Stein analytic interpolation theorem (see [21, p. 38]), for any \( d \in [0, \frac{1}{2}(n - 2)(2b - 1)] \) we have

\[
\| (D)^d e^{i\xi P(D)} \|_{L^1 \to L^\infty} \leq \begin{cases} 
C, & |t| \geq 1, \\
C |t|^{\frac{n}{p} \left( \frac{1}{2} - \frac{1}{p} \right)}, & 0 < |t| < 1.
\end{cases}
\]

(2.15)

Note that

\[
d \leq \frac{1}{2}(n - 2)(2b - 1) \iff \frac{n}{2} \left( \frac{1}{\tau} - \frac{1}{\tau'} \right) \geq \frac{n}{m} \left( \frac{1}{\tau} - \frac{1}{\tau'} \right) - \frac{sd}{m}.
\]
However, if $P$ is also homogeneous, then by scaling it follows from (2.14) with (2.16), by the Marcinkiewicz interpolation theorem (see [21, p. 56]), we obtain (2.13) for all points in the triangle $\triangle_{ABC}$. Next, by duality the desired conclusion for $\triangle_{ABC}$ follows immediately from the triangle $\triangle_{ABC}$. Thus we complete the proof in the case $b \in (\frac{1}{2}, 1]$.

Finally, if $b = \frac{1}{2}$, then $d = 0$ and

$$\mathbb{D}_{1/2.0} = \left\{ \left( \frac{1}{p}, \frac{1}{p'} \right) : 1 \leq p \leq 2 \right\}. \quad (2.17)$$

Clearly, the case $(\frac{1}{p}, \frac{1}{q}) = (1, 0)$ follows simply from the Young inequality and Theorem 2.2, and the case $(\frac{1}{p}, \frac{1}{q}) = (\frac{1}{2}, \frac{1}{2})$ from the Plancherel’s theorem. Therefore, the Riesz–Thorn theorem yields the desired estimate. Thus the whole proof of Theorem 2.4 is concluded. $\square$

**Remark 2.5.** In the above proof, if taking interpolation between the estimates (2.14) and (2.15), then we can obtain a slight better estimate than (2.13):

$$\| (D)^d e^{itP} \|_{L^p \to L^p'} \leq \begin{cases} C |t|^{\frac{1}{2} - \frac{1}{p}}, & |t| \geq 1, \\ C |t|^{\frac{n}{m} - \frac{1}{2} - \frac{1}{p} + \frac{1}{m} \cdot \frac{d}{m - 2 - d}}, & 0 < |t| < 1. \end{cases} \quad (2.18)$$

where $(\frac{1}{p}, \frac{1}{q}) \in \mathbb{D}_{b,d}$ and $\tau$ is defined in (2.11). Note that if $b \in [\frac{1}{2}, 1]$ and $0 \leq d \leq \frac{1}{2} n (m - 2) (2b - 1)$, then $\frac{n}{m} - \frac{d}{m - 2 - d} \geq 0$, and (2.17) leads to (2.13) again. If taking $d = 0$ (corresponding $\tau = 2$), then $(\frac{1}{p}, \frac{1}{p'}) = \mathbb{D}_{b,0}$ for $1 \leq p \leq 2$, and from (2.17) we have

$$\| e^{itP} \|_{L^p \to L^p'} \leq C (1 + |t|^{\frac{n}{m} \left( \frac{1}{2} - \frac{1}{p} \right)}), \quad t \neq 0. \quad (2.19)$$

In particular, when $(\frac{1}{p}, \frac{1}{q}) = (\frac{1}{2}, \frac{1}{2})$, (2.18) exactly yields that $\| e^{itP} \|_{L^2 \to L^2} \leq C$ which cannot be obtained from (2.13). However, if $b = 1$, i.e. $P$ is nondegenerate, then from (2.13) or (2.17), we always obtain the following the same corollary.

**Corollary 2.6.** If $P$ is nondegenerate and $0 \leq d \leq \frac{1}{2} n (m - 1)$, then we have

$$\| (D)^d e^{itP} \|_{L^p \to L^p'} \leq \begin{cases} C |t|^{\frac{1}{2} - \frac{1}{p}}, & |t| \geq 1, \\ C |t|^{\frac{n}{m} \left( \frac{1}{2} - \frac{1}{p} \right) - \frac{d}{m}}, & 0 < |t| < 1. \end{cases} \quad (2.19)$$

In particular, if $P$ is also homogeneous, then by scaling it follows from (2.19) that

$$\| (D)^d e^{itP} \|_{L^p \to L^p'} \leq C |t|^{\frac{n}{m} \left( \frac{1}{2} - \frac{1}{p} \right) - \frac{d}{m}}, \quad t \neq 0, \quad (2.20)$$

where $(\frac{1}{p}, \frac{1}{q}) \in \mathbb{D}_{1,d}$ and $|D|^d := (-\Delta)^{d/2}$ denotes the Riesz potential of order $-d$. 

![Fig. 1. The region ABCD of all points $(\frac{1}{p}, \frac{1}{q})$ for $(D)^d e^{itP} (t \neq 0)$ when $d > 0$.](image-url)
Proof. Clearly, (2.19) follows from (2.13) (or (2.17)) by letting \( b = 1 \). As for (2.20), note that when \( t = 1 \) and \((\frac{1}{p}, \frac{1}{q}) \in \square_{1,d}\), from (2.19) we have

\[
\| \langle D \rangle^d e^{itP(D)} \|_{L^p_t L^q_x} \leq C.
\]

Since the operator \( |D|^d \langle D \rangle^{-d} \) is bounded in \( L^r(\mathbb{R}^n) \) for \( 1 \leq r \leq \infty \) (see [39, pp. 133–134]), so it follows that

\[
\| |D|^d e^{itP(D)} \|_{L^p_t L^q_x} \leq C, \quad \| \langle D \rangle^d e^{itP(D)} \|_{L^p_t L^q_x} \leq C,
\]

where \((\frac{1}{p}, \frac{1}{q}) \in \square_{1,d}, r = 1\) for the case \((1, \frac{1}{2})\) and \( r = q \) for other cases. Thus by the homogeneity of \( P \) and \( |\xi|^d \), the desired conclusion (2.20) follows by a scaling argument. \( \Box \)

Remark 2.7. (i) We remark that the estimate (2.13) (or (2.17)) represents a kind of global smoothing effect for the solution for Eq. (1.1) (see also [27] for an original description on the effect). More precisely, let \( m \geq 4, b \in (1/2, 1) \) and \( 0 < d \leq \frac{1}{2} n(m-2)(2b-1) \). If \((\frac{1}{p}, \frac{1}{q}) \in \square_{b,d} \setminus \{(1, \frac{1}{2}), (\frac{1}{2}, 0)\}\), and initial value \( u_0 \in L^p(\mathbb{R}^n) \), then from Theorem 2.4 it follows that the solution of Eq. (1.1), i.e. \( u(t, \cdot) = e^{itP(D)} u_0 \), gains \( d \)-order derivatives in \( L^q(\mathbb{R}^n) \) for each \( t \neq 0 \), equivalently, \( u(t, \cdot) \in W^d,q(\mathbb{R}^n) \) (Sobolev space). On the other hand, in contrast to global regularizing effect, there exist abundant works on local smoothing effect for general dispersive type equations under rather weak assumptions (see [26,14,15,27,3,13] and references therein).

(ii) Let \( b \in [1/2, 1] \) and \( 0 \leq d \leq \frac{1}{2} n(m-2)(2b-1) \). For any \( T > 0 \), by some simple calculations the estimates (2.13) and (2.17) can be transformed into the following local forms:

\[
\| \langle D \rangle^d e^{itP(D)} \|_{L^p_t L^q_x} \leq C_T |t|^{-\frac{n}{m} (1 - \frac{1}{p}) - \frac{d}{q}}, \quad 0 < |t| \leq T,
\]

and

\[
\| \langle D \rangle^d e^{itP(D)} \|_{L^p_t L^q_x} \leq C_T |t|^{-\frac{n}{m} (1 - \frac{1}{p}) - \frac{d}{q} + \left(\frac{d}{q} - \frac{d}{p} - \frac{d}{m} - \frac{d}{n}\right) \frac{1}{2}}, \quad 0 < |t| \leq T,
\]

where \((\frac{1}{p}, \frac{1}{q}) \in \square_{b,d}, s_d = d + (1 - b)n(m-2) \) and \( C_T \) polynomially depends on \( T \) except for homogeneous cases. In particular, if \( b = 1 \) and \( 0 \leq d \leq n(m-2)/2 \), then (2.21) and (2.22) are completely identical, that is

\[
\| \langle D \rangle^d e^{itP(D)} \|_{L^p_t L^q_x} \leq C_T |t|^{-\frac{n}{m} (1 - \frac{1}{p}) - \frac{d}{q}}, \quad 0 < |t| \leq T,
\]

which has been obtained by [16], [17, Theorem 3.2(1)] from a local decay pointwise estimate of Remark 2.3(iv) above.

Because of the finite of \( T \), the estimates (2.21) and (2.22) cannot be used in Section 4. However, such local \( L^p - L^q \) estimates can deduce some important local Strichartz inequalities, which would play a key role in well-posed analysis of nonlinear dispersive equations (e.g. see [28,12,9,40] for more backgrounds and contents of these related topics). For instance, if \( b \in [1/2, 1] \) and \( d = 0 \), then for any \( 1 \leq p \leq 2 \), from (2.22) one has

\[
\| e^{itP(D)} u_0 \|_{L^p_t L^q_x} \leq C_T |t|^{-\frac{n}{m} \left(\frac{1}{p} - \frac{1}{q}\right)}, \quad 0 < |t| \leq T.
\]

Using the estimate (2.23), the following theorem is actually implied by a famous result of [28, Theorem 1.2], and also can be proved by standard arguments (\( TT^* \) method and Hardy–Littlewood–Sobolev inequality) apart from some endpoint cases.

Let \( \sigma = \frac{n+s_0}{m} \) where \( s_0 = (1 - b)n(m-2) \). A pair \((q, r)\) is \( \sigma \)-admissible if \( q, r \geq 2, (q, r, \sigma) \neq (2, \infty, 1) \) and

\[
\frac{1}{q} + \frac{\sigma}{r} = \frac{\sigma}{2}.
\]

Theorem 2.8. If \( P \) satisfies the condition (H_b) for some \( b \in [1/2, 1] \), and \( I = [0, T] \), then the estimates

\[
\| e^{itP(D)} u_0 \|_{L^q_t(L^p_x(\mathbb{R}^n))} \leq C_T \| u_0 \|_{L^p_x(\mathbb{R}^n)}, \tag{2.24}
\]

\[
\| \int e^{-isP(D)} g(s, \cdot) \, ds \|_{L^q_t(L^p_x(\mathbb{R}^n))} \leq C_T \| g \|_{L^q_t(L^p_x(\mathbb{R}^n))}, \tag{2.25}
\]

and

\[
\| \int_0^t e^{i(t-s)P(D)} g(s, \cdot) \, ds \|_{L^q_t(L^p_x(\mathbb{R}^n))} \leq C_T \| g \|_{L^q_t(L^p_x(\mathbb{R}^n))}, \tag{2.26}
\]

hold for all \( \sigma \)-admissible pairs \((q, r), (\bar{q}, \bar{r})\).
Proof. Let \( W(t) g := \chi(t) e^{itP(D)} g \). Then by the estimate (2.23), it is easy to check that \( W(t) \) satisfies the assumptions of [28, Theorem 1.2]. Thus the inequalities (2.24)–(2.26) follow from the corresponding estimates (5)–(7) in [28, Theorem 1.2] immediately. \( \Box \)

3. The proof of Theorem 2.1

In the sequel, we denote by the letters \( C \) (or \( C_i \), etc.) generic constants which may depend on many admissible constants, such as \( m, n, b, P, L \), etc., but must be independent of \( \xi, t \) and \( x \). Also denote by \( A \sim B \) the equivalence \( C_1 A \leq B \leq C_2 A \) with some generic constants \( C_i \) (i = 1, 2).

Proof of Theorem 2.1. Let \( (t, x) \in (R \setminus \{0\}) \times R^d \). We decompose \( R^d = \bigcup_{j=1}^3 \Omega_j \) where

\[
\Omega_1 = \{ \xi \in R^d \mid |\xi| < 4L \},
\]

\[
\Omega_2 = \{ \xi \in R^d \mid |\xi| > L, \ \left| \nabla P(\xi) + \frac{x}{t} \right| > \frac{1}{2} |x| \},
\]

\[
\Omega_3 = \{ \xi \in R^d \mid |\xi| > L, \ \left| \nabla P(\xi) + \frac{x}{t} \right| < \frac{1}{2} |x| \},
\]

and choose a partition of unity \( \{ \psi_j(\xi) \}_{j=1}^3 \) subordinate to this covering \( \{ \Omega_j \}_{j=1}^3 \) (cf. [27,41]):

\[
\psi_1(\xi) = \varphi(\xi/3L),
\]

\[
\psi_2(\xi) = (1 - \varphi(\xi/3L))(1 - \varphi(\nabla P(\xi) + \frac{x}{t} / \frac{3}{8} |x|)),
\]

\[
\psi_3(\xi) = (1 - \varphi(\xi/3L))\varphi(\nabla P(\xi) + \frac{x}{t} / \frac{3}{8} |x|),
\]

where \( \varphi \in C_0(R^d) \) such that

\[
\varphi(\xi) = \begin{cases} 
0, & |\xi| > 1, \\
1, & |\xi| \leq \frac{1}{2}.
\end{cases}
\]  

(3.1)

Clearly, \( \sum_{j=1}^3 \psi_j(\xi) = 1 \) for each \( \xi \in R^d \). Furthermore, by a direct calculation for each \( \xi \in R^d \), we also have

\[
|\partial^\alpha \psi_1(\xi)| \leq C_\alpha \left(1 + |\xi|\right)^{-|\alpha|}, \quad \forall \alpha \in \mathbb{N}_0^d,
\]

and

\[
|\partial^\alpha \psi_2(\xi)| \leq C'_\alpha |\xi|^{-|\alpha|}, \quad \forall \alpha \in \mathbb{N}_0^d \text{ and } j = 2, 3,
\]

where \( C_\alpha \) and \( C'_\alpha \) are some admissible constants independent of \( (t, x) \)-variables. Next, we define the following three integrals:

\[
I_j^0(t, x) = \int_{R^d} e^{itP(\xi) + \frac{x}{t} \cdot \xi} a(\xi) \varphi(\xi) \psi_j(\xi) d\xi, \quad j = 1, 2, 3 \text{ and } 0 < \varepsilon \leq 1.
\]  

(3.2)

Since

\[
\sum_{j=1}^3 I_j^0(t, x) = \mathcal{F}^{-1}(a(\cdot) \varphi(\xi) e^{itP}(\cdot)) (x) := I^0(t, x),
\]

and as \( \varepsilon \to 0 \), \( I^0(t, x) \) converges uniformly to \( I(t, x) \) in \( x \)-variable at each compact subsets of \( R^d \) for each \( t \neq 0 \). Thus to estimate \( I(t, x) \), it suffices to estimate the three integrals \( \{ I_j^0(t, x) \}_{j=1}^3 \) uniformly in \( \varepsilon \in (0, 1) \), which are given in the following three lemmas, respectively. Hence combining (3.4), (3.7) and (3.14) below, the desired conclusion (2.6) immediately follows. \( \Box \)

Lemma 3.1. Let \( \varepsilon \in (0, 1) \). If \( s \in R \) and \( s \leq \frac{n(m-2)}{2} \), then

\[
|I_j^0(t, x)| \leq C_s \left(1 + |t|\right)^{\mu_j} \left(1 + |x|\right)^{-\mu_j},
\]

where \( \mu_j = \frac{n(m-2)-2s}{2(m-1)} \) is defined in (2.4). Furthermore, for any \( s \in \left[-\frac{n}{2}, \frac{n(m-2)-2s}{2(m-1)}\right] \), we also have
Moreover, if \( \psi \in C_0^\infty(\mathbb{R}^n) \) and \( \text{supp} \psi_1 \subset \Omega_1 \), so \( \exp(a(\xi)) \psi(\xi) \psi_1(\xi) \) for each \( \xi \in (0, 1) \). Note that \( |\partial^\alpha(\psi(\xi))| \leq C_\alpha (1 + |\xi|^{-|\alpha|}) \) uniformly in \( \xi \in (0, 1) \) for any \( \alpha \in \mathbb{N}_0^n \) (cf. [25, III, p. 66]), then integrations by part give that

\[
[1] \frac{t^s}{n} (1 + |t|^{-1} |x|^{-\mu_s}, \quad |t| \geq 1, \\
C |t|^{\frac{n+3}{m}(1 + |t|^{-1/2} |x|^{-\mu_s}, \quad 0 < |t| < 1,
\]

where \( C_\alpha (1 + |\xi|^{-|\alpha|}) \) is an admissible constant independent of \( \varepsilon \).

**Proof.** Since \( \psi_1(\xi) \in C_0^\infty(\mathbb{R}^n) \) and \( \text{supp} \psi_1 \subset \Omega_1 \), so \( \exp(a(\xi)) \psi(\xi) \psi_1(\xi) \) for each \( \xi \in (0, 1) \). Note that \( |\partial^\alpha(\psi(\xi))| \leq C_\alpha (1 + |\xi|^{-|\alpha|}) \) uniformly in \( \xi \in (0, 1) \) for any \( \alpha \in \mathbb{N}_0^n \) (cf. [25, III, p. 66]), then integrations by part give that

\[
[1] \frac{t^s}{n} (1 + |t|^{-1} |x|^{-\mu_s}, \quad |t| \geq 1, \\
C |t|^{\frac{n+3}{m}(1 + |t|^{-1/2} |x|^{-\mu_s}, \quad 0 < |t| < 1,
\]

where \( C_\alpha (1 + |\xi|^{-|\alpha|}) \) is an admissible constant independent of \( \varepsilon \).

**Proof.** Since \( \psi_1(\xi) \in C_0^\infty(\mathbb{R}^n) \) and \( \text{supp} \psi_1 \subset \Omega_1 \), so \( \exp(a(\xi)) \psi(\xi) \psi_1(\xi) \) for each \( \xi \in (0, 1) \). Note that \( |\partial^\alpha(\psi(\xi))| \leq C_\alpha (1 + |\xi|^{-|\alpha|}) \) uniformly in \( \xi \in (0, 1) \) for any \( \alpha \in \mathbb{N}_0^n \) (cf. [25, III, p. 66]), then integrations by part give that

\[
[1] \frac{t^s}{n} (1 + |t|^{-1} |x|^{-\mu_s}, \quad |t| \geq 1, \\
C |t|^{\frac{n+3}{m}(1 + |t|^{-1/2} |x|^{-\mu_s}, \quad 0 < |t| < 1,
\]

where \( C_\alpha (1 + |\xi|^{-|\alpha|}) \) is an admissible constant independent of \( \varepsilon \).

**Lemma 3.2.** Let \( \varepsilon \in (0, 1] \). If \( \frac{n}{2} \leq d \leq \frac{n(m-2)}{2} \), then

\[
[1] \frac{t^s}{n} (1 + |t|^{-1} |x|^{-\mu_s}, \quad |t| \neq 0.
\]

Moreover, if \( \frac{n}{2} \leq d \leq \frac{n(m-2)}{2} \), then for any \( s \in [d, \frac{n(m-2)}{2}] \), we also have

\[
[1] \frac{t^s}{n} (1 + |t|^{-1} |x|^{-\mu_s}, \quad |t| \neq 0.
\]

where \( \mu_s = \frac{n(m-2)(m-1)}{2m} \) is defined in (2.4) and \( C_\alpha (1 + |\xi|^{-|\alpha|}) \) is an admissible constant independent of \( \varepsilon \).

**Proof.** We first prove (3.5). Since \( \text{supp} \phi_2 \subset \Omega_2 \), we consider the decomposition \( \Omega_2 = \bigcup_{j=1}^n U_j \) where

\[
U_j = \{ \xi \in \Omega_2 : |\partial_j P(\xi) + \frac{x_j}{\tau} | \geq \frac{1}{2\sqrt{n}} |\nabla P(\xi) + \frac{x}{\tau} | \}
\]

and choose the following partition of unity of \( \Omega_2 \) subordinate to this covering: \( \phi_{2j} = \phi_2 \theta_j / \sum_{l=1}^n \theta_l \ (j = 1, \ldots, n) \) where

\[
\theta_j(\xi) = \psi \left( 2\sqrt{n} \left| \partial_j P(\xi) + \frac{x_j}{\tau} \right| / \left| \nabla P(\xi) + \frac{x}{\tau} \right| \right),
\]

and \( \psi \in C_\infty(\mathbb{R}) \) with

\[
\psi(s) = \begin{cases} 
1, & |s| \geq 2, \\
0, & |s| \leq 1.
\end{cases}
\]
Since
\[
\left| \nabla P(\xi) + \frac{X}{t} \right| \geq C|\nabla P(\xi)| \sim |\xi|^{m-1} \quad \text{for} \ \xi \in \Omega_2,
\]
we can check that \(|\partial^\alpha \phi_2j| \leq C_{\alpha}|\xi|^{-|\alpha|}\) for any \(\alpha \in \mathbb{N}_0^j\) and \(\xi \in \Omega_2\). Now to estimate \(I^\varepsilon_{2}\), it suffices to consider the following integral:
\[
I^\varepsilon_{21} = \int_{U_1} e^{i[P_2(\xi) + (x, \xi)]} a(\xi) \varphi(\xi) \phi_{21}(\xi) \, d\xi.
\]
Again to the \(I^\varepsilon_{21}\), we divide two cases to discuss in the sequel.

**Case (i).** \(|t|^{-1/m}|x| \geq 1\).

Let \(r = |t|^{-\frac{\nu}{2\alpha}}|x|^{-\frac{d}{2\alpha}}\) where \(\mu_d\) and \(\nu_d\) are defined in (2.4), and write
\[
U_1 = V_1 \cap V_2 := \{\xi \in U_1: |\xi| < r \} \cap \{\xi \in U_1: |\xi| > r/4\}.
\]
To the covering \(\{V_j\}_{j=1}^2\), we can choose the partition \(\{\phi_{21j}\}_{j=1}^2\) satisfying \(|\partial^\alpha \phi_{21j}| \leq C_{\alpha}|\xi|^{-|\alpha|}\) for \(\xi \in V_j\), where \(C_{\alpha}\) is independent of \(r\). Furthermore, we split the \(I^\varepsilon_{21}\) into \(I^\varepsilon_{211}\) and \(I^\varepsilon_{212}\) associated with \(\phi_{211}\) and \(\phi_{212}\), respectively. Obviously,
\[
|I^\varepsilon_{211}| \leq \int_{V_1} |a(\xi)\varphi(\xi)\phi_{211}(\xi)| \, d\xi \leq Cr^{n+d} = C|t|^{-\nu_d}|x|^{-\mu_d}.
\]  

(3.9)

where \(C\) is independent of \(\varepsilon\). Next consider \(I^\varepsilon_{212}\). Define \(D_+f = \check{\theta}(gf)\) for \(f \in C^\infty(\mathbb{R}^n)\) where \(g = (it\check{\theta}P + i\chi_1)^{-1}\), then
\[
D_+f = \sum_{\alpha} a_\alpha(\partial_1^{\alpha_1}g) \cdots (\partial_1^{\alpha_j}g) (\partial_1^{\alpha_{j+1}}f), \quad \alpha \in \mathbb{N}_0^j,
\]  

(3.10)

where the sum runs over all \(\alpha = (\alpha_1, \ldots, \alpha_{j+1}) \in \mathbb{N}_0^{j+1}\) such that \(|\alpha| = j\) and \(0 \leq \alpha_1 \leq \cdots \leq \alpha_j\).

By Leibniz’s rule, one has
\[
|\partial_1^{\alpha}g| \leq C|x|^{-\alpha-j} \quad \text{and} \quad |\partial_1^{\alpha}g| \leq C|t|^{-1}\xi|^{-m-\alpha-j}, \quad \xi \in U_1.
\]  

(3.11)

Thus for any \(\theta \in [0, 1]\),
\[
|\partial_1^{\alpha}g| \leq C|t|^{-\theta}|x|^{-(1-\theta)}|\xi|^{-(\alpha-j)(m+1)}, \quad \xi \in U_1.
\]  

(3.11)

Therefore, combining (3.10) with (3.11), we have
\[
|D_+^\alpha(a(\xi)\varphi(\xi)\phi_{212}(\xi))| \leq C|t|^{-\nu_d}|x|^{-(n+\alpha-j)(m+1)}, \quad \xi \in V_2.
\]

where \(C\) is an admissible constant independent of \(\varepsilon\). Hence \(n\)-times integrations by parts lead to
\[
|I^\varepsilon_{212}| = \left| \int_{\mathbb{R}^n} e^{i[P_2(\xi) + (x, \xi)]} D_+^\alpha(a(\xi)\varphi(\xi)\phi_{212}(\xi)) \, d\xi \right| \leq C|t|^{-\nu_d}|x|^{-(n+\alpha-j)(m+1)} \int_{|\xi| > r} |\xi|^{d-n-\alpha(n-1)} \, d\xi.
\]

Note that \(-n/2 \leq d \leq n(m - 2)/2\) implies that \(\frac{2\nu_d}{n} \in [0, 1]\), hence by taking \(\theta = \frac{2\nu_d}{n}\), it follows that
\[
|I^\varepsilon_{212}| \leq C|t|^{-2\nu_d}|x|^{-2\mu_d}|r|^{-(n+\alpha-j)} = C|t|^{-\nu_d}|x|^{-\mu_d}.
\]  

(3.12)

where \(r = |t|^{-\frac{\nu}{2\alpha}}|x|^{-\frac{d}{2\alpha}}\).

Thus when \(|t|^{-1/m}|x| \geq 1\), from (3.9) and (3.12) it follows that
\[
|I^\varepsilon_{21}| \leq C|t|^{-\nu_d}|x|^{-\mu_d} \leq C|t|^{-\frac{d}{2\alpha}} (1 + |t|^{-1/m}|x|)^{-\mu_d}.
\]

So true for \(I^\varepsilon_{2}\), therefore (3.6) is concluded in Case (i).

**Case (ii).** \(|t|^{-1/m}|x| < 1\).
To deal with Case (ii), we choose \( r = |t|^{-1/m} \) and \( \theta = 1 \), then a slight change in the proof of Case (i) can lead to
\[
|I_{21}^3| = |I_{211}^3 + I_{212}^3| \leq C \left( |t|^{-\frac{4n}{m}} + |t|^{-n} \int_{|\xi| = \frac{1}{2} |t|^{-1/m}} |\xi|^{-\mu \theta} d\xi \right).
\] (3.13)
Note that \(-\frac{n}{2} \leq d \leq \frac{1}{2}(m-2)(2b-1)\), then for any \( s \in [sd, \frac{n(m-2)}{2}]\), we have
\[
|I_{21}^3(t, x)| \leq \left\{ \begin{array}{ll}
C(1 + |t|^{-1/|x|})^{-\mu s}, & |t| \geq 1, \\
C|t|^{-\frac{4n}{m}} (1 + |t|^{-1/|x|})^{-\mu s}, & 0 < |t| < 1,
\end{array} \right.
\] (3.14)
where \( sd = d + (1-b)(m-2), \mu_s = \frac{n(m-2)-2s}{2(m-1)} \) is defined in (2.4) and \( C \) is an admissible constant independent of \( \varepsilon \).

Proof. Clearly, it suffices to just consider \( s = sd \). Since
\[
|\nabla P(\xi)| \sim |\xi|^{-m-1} \sim \frac{|x|}{t} \quad \text{for} \quad \xi \in \Omega_3,
\]
there exist constants \( c_3, c_4 > 0 \) such that
\[
\Omega_3 \subset \{ \xi \in \mathbb{R}^n : |\xi| > L, \quad 2c_3 \lambda < |\xi| < c_4 \lambda \},
\]
where \( \lambda = |x|/t^{1/(m-1)} \).
If \( L \geq c_4 \lambda \), then clearly \( I_{21}^3 = 0 \).
If \( c_3 \lambda \leq L < c_4 \lambda \), then \( \Omega_3 \subset \{ \xi \in \mathbb{R}^n : L < |\xi| < c_4 \lambda^{-1} L \} \). For this case, \( I_{21}^3 \) is the same type as \( I_1 \). Hence (3.14) follows from Lemma 3.1.
If \( L < c_3 \lambda \), then consider further truncated cone decomposition of \( \Omega_3 \). To this end, choose a finite set \( \{ \xi_k \} \subset S^{n-1} \) (the unit sphere in \( \mathbb{R}^n \)) such that for every \( \xi \in S^{n-1} \)
\[
|\xi - \xi_k| \geq \frac{1}{4} \quad (k \neq k') \quad \text{and} \quad \min |\xi - \xi_k| < \frac{1}{4}.
\]
Notice that the set \( \{ \xi_k \} \) contains at most \( C4^n \) elements where \( C = C(n) \). Corresponding to \( \{ \xi_k \} \), write \( \Omega_3 = \bigcup_k \Omega_3^k \) where
\[
\Omega_3^k = \{ \xi \in \Omega_3 : \left| \frac{\xi}{|\xi|} - \xi_k \right| \leq \frac{1}{2} \},
\]
and choose the following partition of unity subordinate to the covering: \( X_3^k = \varphi_3^k(\xi) = (\sum_j \xi_j)^{-1} \) where \( \xi_k(\xi) = \varphi(2(|\xi|/|\xi_k| - \xi_k)) \) and \( \varphi \in C^\infty_0(\mathbb{R}^n) \) is defined in (3.1). By the Leibniz rule, we can get that \( |\partial^\alpha X_3^k| \leq C_\alpha |\xi|^{-|\alpha|} \) for \( \xi \in \Omega_3^k \). Then \( I_{21}^3 = \sum_k I_{21}^{3,k} \) where
\[
I_{21}^{3,k} = \int_{\mathbb{R}^n} \exp \left[ i(tP(\xi') + (x, \xi)) \right] \varphi(\xi) X_3^k(\xi) d\xi.
\]
To estimate \( I_{21}^3 \), in view of the finite of the set \( \{ \xi_k \} \) it suffices to consider \( I_{21}^{3,k} \). For this, we first prove the following inequality:
\[
|\nabla P(\xi) - \nabla P(\xi')| \geq c_4 \lambda b(m-2) |\xi - \xi'| \quad \text{for} \quad \xi, \xi' \in \Omega_3^k.
\] (3.15)
where the constant \( c = c_4 \lambda b(m-2) \) and \( c_4 \) is the absolute constant in the condition (2.1).
In fact, let
\[
f(\gamma) = (\omega, \nabla P(\varepsilon + \gamma (\xi - \xi'))),
\]
where \( \gamma \in [0, 1] \), \( \xi \neq \xi' \) and \( \omega = \frac{\xi - \xi'}{|\xi - \xi'|} \). By the intermediate value theorem one has

\[
|\nabla P(\xi) - \nabla P(\xi')| \geq \int_{0}^{1} f'(\gamma) d\gamma = \left( \int_{0}^{1} (\omega, HP(\xi + \gamma(\xi - \xi'))(\xi - \xi') \omega) d\gamma \right) |\xi - \xi'| \\
\geq |(\omega, HP(\bar{\xi}) \omega)| |\xi - \xi'| \geq \left( \min_{k} \lambda_{k}(\bar{\xi}) \right) |\xi - \xi'|.
\]

Since \( \bar{\xi} \) is located in the convex hull of \( \Omega_{3}^{\lambda} \), we can show that \( |\bar{\xi}| \geq c_{3} \lambda \geq L \). It follows by the assumption (2.1) on \( P \) that

\[
\min_{k} \lambda_{k}(\bar{\xi}) \geq c_{1}|\bar{\xi}|^{b(m-2)} \geq c\lambda^{b(m-2)},
\]
as desired.

Now return to \( I_{3}^{\epsilon, \kappa} \). Pick up \( \xi_{0} \in \Omega_{3}^{\lambda} \) such that

\[
|\nabla P(\xi_{0}) + \frac{X}{l}| \leq \frac{C}{4} \lambda^{b(m-2)} r,
\]
where \( r = |t|^{-\frac{1}{2}} \lambda(1-2b)(m-2)/2 \). The case that \( \xi_{0} \) does not exist can be easily treated in the end (see also [27, p. 53]). Corresponding to the sets:

\[
V_{1}^{\epsilon} := \{ \xi \in \Omega_{3}^{\lambda} \mid |\xi - \xi_{0}| < r \} \quad \text{and} \quad V_{2}^{\epsilon} := \{ \xi \in \Omega_{3}^{\lambda} \mid |\xi - \xi_{0}| > \frac{1}{4} r \},
\]
we split \( I_{3}^{\epsilon, \kappa} \) into two integrals \( I_{31}^{\epsilon, \kappa} \) and \( I_{32}^{\epsilon, \kappa} \).

For \( I_{31}^{\epsilon, \kappa} \), since \( |\xi| \sim \lambda \) for \( \xi \in \Omega_{3} \), we easily obtain that

\[
|I_{31}^{\epsilon, \kappa}| \leq C \int_{V_{1}} |\xi|^{d} \leq C' \lambda^{d-m} \leq C'' |t|^{-\gamma_{d}} |x|^{-\mu_{d}}.
\]  

(3.16)

For \( I_{32}^{\epsilon, \kappa} \), write \( V_{2} = \bigcup_{j=1}^{n} W_{j}^{\epsilon} \) where \( W_{j}^{\epsilon} \) is defined as \( U_{j} \) in the proof of Lemma 3.2, and split \( I_{32}^{\epsilon, \kappa} \) into \( n \) new integrals. To estimate \( I_{32}^{\epsilon, \kappa} \), it suffices to estimate one of them, for example,

\[
|I_{321}^{\epsilon, \kappa}| = \left| \int_{\mathbb{R}^{n}} e^{(tP(\xi) + (x, \xi))} D_{n}^{\mu}(\phi^{\epsilon}(\xi) \eta(\xi)) d\xi \right|,
\]

where

\[
\phi^{\epsilon}(\xi) = a(\xi) \varphi(\xi \epsilon) \chi_{\lambda}^{\epsilon}(\xi) (1 - \varphi(r^{-1}(\xi - \xi_{0}))), \quad \eta(\xi) = \theta_{1}(\xi) / \sum_{i=1}^{n} \theta_{i},
\]

\( \varphi \) and \( \theta_{i} \) are defined in (3.1) and (3.8), respectively.

Since \( |\xi| \sim \lambda \) for \( \xi \in W_{j}^{\epsilon} \) \( (\subset \Omega_{3}) \), then

\[
|\xi - \xi_{0}| \leq |\xi| + |\xi_{0}| \leq C|\xi|.
\]

By the Leibniz’s formula

\[
|\partial_{X}^{d} \phi^{\epsilon}(\xi)| \leq C|\xi|^{d} |\xi - \xi_{0}|^{-j}, \quad \xi \in W_{j}^{\epsilon},
\]

where \( C \) is independent of \( \epsilon \), and

\[
|\partial_{X}^{d} \eta(\xi)| \leq C|\xi|^{j(m-2)} |\partial_{1} \eta| + x/t|^{-j}, \quad \xi \in W_{j}^{\epsilon}.
\]

Hence using the formula (3.10) again, we obtain

\[
|D_{n}^{\mu}(\phi^{\epsilon}(\eta))| \leq C|t|^{-n} \sum_{j+\ell=n} |\xi|^{j(m-2)} |\partial_{1}^{\ell} (\phi^{\epsilon}(\eta))| |\partial_{1} \eta| + x/t|^{-n}\end{align}.

\[
\leq C|t|^{-n} \sum_{j=0}^{n} \lambda^{j(m-2)} |\xi - \xi_{0}|^{j} |\partial_{1} \eta| + x/t|^{-2n-j} \leq C'' |t|^{-n} \sum_{j=0}^{n} \lambda^{j(m-2)} |\xi - \xi_{0}|^{j} |\partial_{1} \eta| + x/t|^{-2n-j}, \quad \xi \in W_{j}^{\epsilon}, \ |\xi| \sim \lambda.
\]

where constants \( C, C' \) and \( C'' \) are all admissible and independent of \( \epsilon \).
Notice that if $\xi \in V_2^k$, then
\[
\left| \nabla P(\xi) + \frac{X}{t} \right| \geq \frac{1}{2} \left| \nabla P(\xi) - \nabla P(\xi_0) \right| \geq \frac{C}{2} \lambda^{b(m-2)} |\xi - \xi_0|.
\] (3.17)
which can be concluded by the following two inequalities:
\[
\left| \nabla P(\xi_0) + \frac{X}{t} \right| \leq \frac{C}{4} \lambda^{b(m-2)} r
\]
and
\[
\left| \nabla P(\xi) - \nabla P(\xi_0) \right| \geq C \lambda^{b(m-2)} |\xi - \xi_0| \geq \frac{C}{2} \lambda^{b(m-2)} r, \quad \xi \in V_2^k \quad (\text{i.e. by (3.15)}).
\]
Consequently, by (3.17) we have
\[
|D_n^0 (\phi^e n_1)(\xi)| \leq C |t|^{-n} \lambda^{d+1-2b} |\xi - \xi_0|^{-2n} \sum_{j=0}^n \lambda^{-j} (1-2b) (m-2), \quad \xi \in W_1^k.
\]
Since for $\xi \in W_1^k$, $\lambda \sim |\xi| > L$, so it follows that
\[
|f_{22}^k| \leq C |t|^{-1} |x|^{-\mu_d}.
\] (3.18)
Now combining (3.16) with (3.18), we obtain
\[
|f_3^j| \leq \sum_k |f_{3,12}^k| \leq C |t|^{-1} |x|^{-\mu_d}.
\] (3.19)

Note that $\lambda = |x|^{(1/m-1)} > L / c_3$ (i.e. $|t|^{-1} |x| > (L / c_3)^{m-1}$), thus if $|t| > 1$, then from (3.19) it immediately follows that
\[
|f_3^j| \leq C |t|^{-1} |x|^{-\mu_d} \leq C (1 + |t|^{-1} |x|)^{-\mu_d} = C (1 + |t|^{-1} |x|)^{-\mu_d}.
\]
If $0 < |t| < 1$, then when $|t|^{-1/m} |x| > 1$, (3.19) gives again
\[
|f_3^j| \leq C |t|^{-1} |x|^{-\mu_d} \leq C |t|^{-n} \left( |t|^{-1} \right)^{\mu_d} \leq C |t|^{-n} \left( 1 + |t|^{-1} |x| \right)^{-\mu_d},
\]
and when $|t|^{-1/m} |x| \leq 1$ (i.e. $|x| \leq |t|^{1/m}$), it directly follows that from the definition (3.2) of $f_3^j$
\[
|f_3^j| \leq C |t|^{n} \left( |t|^{-1} \right)^{\mu_d} \leq C |t|^{-n} \left( 1 + |t|^{-1} |x| \right)^{-\mu_d}.
\]
Thus we have proved the desired estimate (3.14) from above discussions.

Finally, let us consider the case that there is not $\xi_0$ in $\Omega_2^k$ such that
\[
\left| \nabla P(\xi_0) + \frac{X}{t} \right| \leq \frac{C}{4} \lambda^{b(m-2)} r.
\]
Clearly, it implies
\[
\inf_{\xi \in \Omega_2^k} \left| \nabla P(\xi) + \frac{X}{t} \right| \geq \frac{C}{4} \lambda^{b(m-2)} r > 0.
\]
Now choose $\xi_0 \in \Omega_2^k$ such that
\[
\left| \nabla P(\xi_0) + \frac{X}{t} \right| \leq 2 \inf_{\xi \in \Omega_2^k} \left| \nabla P(\xi) + \frac{X}{t} \right|.
\]
Then
\[
\left| \nabla P(\xi) + \frac{X}{t} \right| \geq \frac{1}{3} |\nabla P(\xi) - \nabla P(\xi_0)|,
\]
and the above proof of $f_{22}^k$ also works for the case in the same way. Thus the whole proof of Lemma 3.3 is completed. $\square$
Remark 3.4. Obviously, all the conditions (2.1)-(2.3) are essentially used in Lemma 3.3 which deals with the most difficult part of $I(t,x)$ with critical points, i.e. these points satisfying equations $\nabla P(\xi) + \xi = 0$ (may be not isolated in degenerate cases). Especially, the assumption (2.1) is used to deduce the key inequalities (3.15) in above proof.

4. Applications for higher-order Schrödinger operators

As an application of Theorem 2.4, in this section we will show that higher-order Schrödinger operator $iP(D) + V(x,D)$ generates a fractionally integrated group on $L^p(\mathbb{R}^n)$. For this, let us start with giving the definition of a fractionally integrated group.

Let $\alpha \geq 0$. A densely defined linear operator $A$ on a Banach space $X$ is called the generator of an $\alpha$-times integrated semigroup if there exists an exponentially bounded, strongly continuous family $T(t)$ $(t \geq 0)$ of bounded linear operators on $X$ such that

$$(\lambda - A)^{-1}x = \lambda^{\alpha} \int_{0}^{\infty} e^{-\lambda t}T(t)x \, dt \quad \text{for large } \lambda \text{ and } x \in X.$$  \hspace{1cm} (4.1)

If $A$ and $-A$ both are generators of $\alpha$-times integrated semigroups, $A$ is called the generator of an $\alpha$-times integrated group.

In view of our application, we need the following perturbation results of the fractionally integrated semigroup (see [22, Theorem 5.1] and [30, Theorems 3.1, 3.3]). In particular, the second assertion in the following lemma is the special case of Theorem 3.3(a) in [30], where the same conclusion can hold on a class of Banach spaces of Fourier type $s \in [1, 2]$.

Lemma 4.1. Let $(A, D(A))$ be the generator of an $\alpha$-times integrated semigroup on $X$ and let $(B, D(B))$ be a linear operator on $X$ such that $D(A) \subseteq D(B)$ and there exist constants $M, \omega \geq 0$ such that $\|B(\lambda - A)^{-1}\| \leq M < 1$ for Re $\lambda > \omega$. Then $(A + B, D(A))$ generates a $\beta$-times integrated semigroup on $X$, where $\beta > \alpha + 1$. Moreover, if $X = L^p(\mathbb{R}^n)$ $(1 < p < \infty)$, then in the case we can take $\beta > \alpha + \max\{\frac{1}{p}, \frac{1}{q}\}$.

In the sequel, we consider the higher-order Schrödinger operator of the form:

$$iP(D) + V(x, D) := iP(D) + \sum_{j=1}^{k} h_{j}(x)Q_{j}(D),$$  \hspace{1cm} (4.2)

where $P(\xi)$ is a real elliptic polynomial of order $m$ in $\mathbb{R}^n$ where $n \geq 2$. For each $1 \leq j \leq k$, $h_{j}(x)$ is an (or complex value) measurable function on $\mathbb{R}^n$ and $Q_{j}(\xi)$ belongs to symbol class $S^{m_{j}}(\mathbb{R}^n)$ (see Section 2) for $m_{j} \geq 0$. Of course, $Q_{j}(D)$ can be a partial differential operator of order $m_{j}$ if $m_{j}$ is a positive integer.

When $V(x, D) \equiv 0$, it was well known that the elliptic operator $iP(D)$ (even any general elliptic operator) generates an $\alpha$-times integrated semigroup $T(t)$ on $L^p(\mathbb{R}^n)$ for $\alpha > \frac{n}{2} - \frac{1}{p}$ and $1 < p < \infty$ (see [23,42]). Hence, when $V(x, D) \neq 0$, in order to show that $iP(D) + V(x, D)$ in $L^p(\mathbb{R}^n)$ generates a fractionally integrated semigroup, we shall use $V(x, D)$ to perturb $iP(D)$ in view of Lemma 4.1. For this, we need establish the $L^p-L^q$ estimates of the operator $(D)^{\delta}(\lambda - iP(D))^{-1}$ where Re $\lambda \neq 0$ and

$$(D)^{\delta}(\lambda - iP(D))^{-1}u_0 = \mathcal{F}^{-1}\big(\xi^{d}(\lambda - iP(\xi))^{-1}u_0\big), \quad u_0 \in \mathcal{S}(\mathbb{R}^n).$$  \hspace{1cm} (4.3)

Now set

$$\Delta_{b,d} = \square_{b,d} \cap \{(1/p, 1/q); \quad 1/p - 1/q < (m - s_d)/n\}$$  \hspace{1cm} (4.4)

and

$$\ell_{b} = \sup\{d; \quad \Delta_{b,d} \neq \emptyset, \quad 0 \leq d \leq \frac{1}{2}n(m - 2)(2b - 1)\}.$$  \hspace{1cm} (4.5)

In particular, if $b = 1$, then we have $\ell_{1} = m - 2$.

Theorem 4.2. If $P$ satisfies the condition $(H_{b})$ for some $b \in [\frac{1}{2}, 1], 0 \leq d < \ell_{b}$ and $s_d = d + (1 - b)n(m - 2)$, then for any Re $\lambda \neq 0$ we have

$$\|(D)^{\delta}(\lambda - iP(D))^{-1}\|_{L^p-L^q} \leq C|\text{Re } \lambda|^{-1}\left(|\text{Re } \lambda|^{-\frac{n}{2} - \frac{1}{p}} + |\text{Re } \lambda|^{-\frac{n}{2} - \frac{1}{q}} + \frac{\delta}{m}\right),$$  \hspace{1cm} (4.6)

where $(\frac{1}{p}, \frac{1}{q}) \in \Delta_{b,d} \setminus \{(1, \frac{1}{2}), (\frac{1}{p}, 0)\}$, and $\delta = \delta(b, d)$ is defined in (2.10).
Proof. When \( \text{Re}\lambda > 0 \), by (4.3) one has
\[
\langle D \rangle^{\beta} (\lambda - iP(D))^{-1} u_0 = \int_{0}^{\infty} e^{-\lambda t} \langle D \rangle^{\beta} e^{itP(D)} u_0 \, dt \quad \text{for} \quad u_0 \in \mathcal{S}(\mathbb{R}^n).
\]
Therefore, for each \((\frac{1}{p}, \frac{1}{q}) \in \Delta_{b,d} \setminus \{(1, \frac{1}{2}), (\frac{1}{q}, 0)\} \), it follows from Theorem 2.4 that
\[
\| \langle D \rangle^{\beta} (\lambda - iP(D))^{-1} \|_{L^{p'}-L^p} \leq C \int_{0}^{\infty} e^{-\lambda t} (\lambda^{\frac{1}{q}} + t^{\frac{1}{p'}}) \, dt \leq C |\text{Re}\lambda|^{-\frac{1}{q}} + |\text{Re}\lambda|^{\frac{1}{p'}},
\]
where \(\lambda < 0\), we notice that \(\lambda < 0\), we notice that
\[
\langle D \rangle^{\beta} (\lambda - iP(D))^{-1} u_0 = \int_{0}^{\infty} e^{\lambda t} \langle D \rangle^{\beta} e^{-itP(D)} u_0 \, dt \quad \text{for} \quad u_0 \in \mathcal{S}(\mathbb{R}^n),
\]
and the desired result also similarly holds. \(\square\)

Corresponding to Corollary 2.6, one also gets the following consequence immediately.

**Corollary 4.3.** If \( P \) is nondegenerate and \( 0 \leq d < \ell_1 = m - 2 \), then for any \( \text{Re}\lambda \neq 0 \), we have
\[
\| \langle D \rangle^{\beta} (\lambda - iP(D))^{-1} \|_{L^{p'}-L^p} \leq C |\text{Re}\lambda|^{-\frac{1}{q}} + |\text{Re}\lambda|^{\frac{1}{p'}},
\]
where \((\frac{1}{p}, \frac{1}{q}) \in \Delta_{b,d} \setminus \{(1, \frac{1}{2}), (\frac{1}{q}, 0)\} \).

Now we are in a position to state the main results in this section. For \( 1 \leq p \leq \tau \), denote by \( \Lambda(p, b, d) \) the following subset of \([1, \infty)\):
\[
\Lambda(p, b, d) = \left\{ r : \frac{1}{r} = \frac{1}{p} - \frac{1}{q}, \quad \left( \frac{1}{p}, \frac{1}{q} \right) \in \Delta_{b,d} \right\},
\]
where \(\tau = \tau(b, d)\) and \(\Delta_{b,d}\) are defined in (2.11) and (4.4), respectively.

**Theorem 4.4.** Let \( P \) satisfy the condition (Hb) for some \( b \in \left[ \frac{n}{2}, 1 \right] \), and \( 0 \leq d < \ell_b \), where \( \ell_b \) is defined in (4.5) and \( V(x, D) \) is defined in (4.2).

(a) If \( h_j \in L^{q_j}(\mathbb{R}^n) \) with \( r_j \in \Lambda(p, b, d) \) for some \( 1 < p \leq \tau \), and \( \text{deg}(Q_j(D)) = m_j \leq d \) \((j = 1, 2, \ldots, k)\), then the operator
\[
L(x, D) = iP(D) + V(x, D)
\]
generates a \(\beta\)-times integrated group on \(L^p(\mathbb{R}^n)\) for any \(\beta > n\left(\frac{1}{2} - \frac{1}{p} \right) + \frac{1}{p}\).

(b) If \( h_j \in L^{q_j}(\mathbb{R}^n) \) with \( r_j \in \Lambda(p', b, d) \) for some \( \tau' < p < \infty \), and \( \text{deg}(Q_j(D)) = m_j \leq d \) \((j = 1, 2, \ldots, k)\), then the dual operator \( L^{\tau}(x, D) \) generates a \(\beta\)-times integrated group on \(L^p(\mathbb{R}^n)\) for any \(\beta > n\left(\frac{1}{2} - \frac{1}{p} \right) + \frac{1}{p}\).

**Proof.** Since \( iP(D) + V(x, D) \) and \(-iP(D) + V(x, D))\) satisfy the same assumptions, it suffices to show that \( iP(D) + V(x, D) \) generates a \(\beta\)-times integrated semigroup on \(L^p(\mathbb{R}^n)\).

We first consider the case \( 1 < p \leq \tau \). Let \(\frac{1}{q_j} = \frac{1}{p} - \frac{1}{r_j} \) \((j = 1, 2, \ldots, k)\). Then \( r_j \in \Lambda(p, b, d) \) implies \((1/p, 1/q_j) \in \Delta_{b,d}\) by (4.8). So we obtain by Theorem 4.2 and Hölder’s inequality that
\[
\| V(x, D)(\lambda - iP(D))^{-1} \|_{L^{p'}-L^p} \leq C \sum_{j=1}^{k} \bigg( \int_{0}^{\infty} e^{-\lambda t} \| V_j(D) \|_{L^{r_j}-L^{r_j}} (\lambda - iP(D))^{-1} \|_{L^{p}-L^{q_j}} \bigg) \]
\[
\leq C \sum_{j=1}^{k} (|\text{Re}\lambda|^{-\frac{1}{q_j} - \frac{1}{p} - 1} + |\text{Re}\lambda|^{\frac{1}{r_j} + \frac{1}{p} - 1} \bigg),
\]
where the boundedness of the operator $Q_j(D)(D)^{-d}$ in $L^p(\mathbb{R}^n)$ is from Mihlin multiplier theorem (cf. [21, p. 362]). Since by (4.4)
$$\frac{n}{mr_j} + \frac{5d}{m} = 1 < 0, \quad j = 1, 2, \ldots, k,$$
thus there exists $\omega > 1$ such that
$$\|V(x, D)(\lambda - iP(D))^{-1}\|_{L^p \to L^p} \leq 1/2 \quad \text{for } \Re \lambda > \omega.$$
Consequently, by Lemma 4.1 and the fact that $iP(D)$ generates an $\alpha$-times integrated semigroup on $L^p(\mathbb{R}^n)$ for $\alpha \geq n[\frac{1}{2} - \frac{1}{p}]$, it follows that $iP(D) + V(x, D)$ generates a $\beta$-times integrated semigroup on $L^p(\mathbb{R}^n)$ for $\beta > n[\frac{1}{2} - \frac{1}{p}] + \frac{1}{p}$.

Next, let us consider the case $\tau' \leq p < \infty$. From the proof of (a), it is easy to see that $\hat{L}(x, D) = -iP(D) + \mathcal{V}(x, D)$ is densely defined on $L^p(\mathbb{R}^n)$. Thus, its dual operator $\hat{L}^*(x, D)$ exists and is also densely defined on $L^p(\mathbb{R}^n)$. Since $s_j \in \Lambda_{p'}$ and
$$n \left| \frac{1}{2} - \frac{1}{p} \right| = n \left| \frac{1}{2} - \frac{1}{p} \right|,$$
for all $d > m - 2$. If function $h_\alpha \in L^{1+r}(\mathbb{R}^n)$ for all $|\alpha| \leq d$ where $r_\alpha \in \Lambda(p, d)$, then the operator
$$L(x, D) = iP(D) + \sum_{|\alpha| \leq d} h_\alpha(x)D^\alpha$$
generates a $\beta$-times integrated group on $L^p(\mathbb{R}^n)$ for any $\beta > n[\frac{1}{2} - \frac{1}{p}] + \frac{1}{p}$.

**Remark 4.6.** Comparing with the conclusions of Theorem 3.3 and Corollary 3.5 in [41], Theorem 4.4 and Corollary 4.5 in this paper have some improvements in two ways. Firstly, here the perturbed operator $V(x, D)$ can be a differential operator, not just a potential function $V(x)$. Secondly, with respect to the integrated semigroup’s times, here we have
$$\beta > n[1/2 - 1/p] \max \{1/p, 1'/p\},$$
rather than $\beta > n[1/2 - 1/p] + 1$ in [41]. Clearly, this becomes better because for $1 < p < \infty$,
$$n[1/2 - 1/p] + 1 > n[1/2 - 1/p] \max \{1/p, 1'/p\}.$$

Finally, in order to obtain $L^p-L^p$ estimates of the solution for Eq. (1.4), we use the definition of fractional powers by van Neerven and Straub [33]. Let $\alpha_0 > 0$. If $A$ is the generator of an $\alpha$-times integrated group for every $\alpha > \alpha_0$, then the fractional powers $(\omega \pm A)^{\alpha}$ are well defined for large $\omega \in \mathbb{R}$ and their domains all contain the dense subspace $D(A^{[\alpha]+1})$. The following result is a consequence of Theorem 1.1 in [33] and Theorem 4.4(a) above.

**Theorem 4.7.** Suppose $P, V, p$ and $\beta$ satisfy the assumptions of Theorem 4.4(a). Then there exist constants $C, \omega > 0$ such that for every data
$$u_0 \in \mathcal{D}(\omega + L(x, D)^{\beta}) \cap \mathcal{D}(\omega - L(x, D)^{\beta}),$$
Eq. (1.4) has a unique solution $u \in C(\mathbb{R}, L^p(\mathbb{R}^n))$ and
$$\|u(t)\|_{L^p} \leq C\omega|t|\|\omega \pm L(x, D)^{\beta}u_0\|_{L^p}, \quad t \in \mathbb{R},$$
where we choose $+ (\text{resp. } -)$ if $t \geq 0 (\text{resp. } < 0)$.

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References