JOURNAL OF APPROXIMATION THEORY **81**, 127–135 (1995)

On the Divergence of Lagrange Interpolation to $|x|$

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Communicated by Paul Nevai

Received November 23, 1993; accepted December 19, 1993

It is a classical result of Bernstein that the sequence of Lagrange interpolation polynomials to $|x|$ at equally spaced nodes in $[-1, 1]$ diverges everywhere, except at zero and the end-points. In the present paper we show that the case of equally spaced nodes is not an exceptional one in this sense. Namely, we prove that the divergence everywhere in $0 < |x| < 1$ of the Lagrange interpolation to $|x|$ takes place for a broad family of nodes, including in particular the Newman nodes, which are known to be very efficient for rational interpolation.

1. INTRODUCTION

Let $X = \{x_k^{(n)}\}$, $n = 0, 1, 2, ...,$ $0 \leqslant k \leqslant n$ be an infinite triangular matrix, where

$$-1 \leqslant x_n^{(n)} < x_{n-1}^{(n)} < \cdots < x_0^{(n)} \leqslant 1, \qquad n = 0, 1, 2, ...$$

and denote by $C[-1, 1]$ the Banach space of continuous functions on $[-1, 1]$ equipped with the uniform norm. To each $f \in C[-1, 1]$ there corresponds a unique interpolation polynomial $L_n(f; X; x)$ of degree at most n coinciding with $f(x)$ at the nodes of the nth row of X. The most important problem in interpolation theory is to characterize under what conditions on f and X the sequence $\{L_n(f; X; x)\}$, $n = 0, 1, 2, ...,$ converges to $f(x)$.

0021-9045/95 $6.00

The first negative result is due to Faber [3], who proved that for any matrix X there is a function $f \in C[-1, 1]$ such that

$$\lim_{n \to \infty} \|f - L_n(f; X)\| \neq 0.$$

A direct consequence of this result is the pointwise divergence of $\{L_n(f; X; x)\}$ at least at one point. The result of Faber was reinforced by Erdős and Vértesi [2], who showed that for any X there is a function $f \in C[-1, 1]$ such that the divergence of the interpolating process takes place *almost everywhere* on $[-1, 1]$.

It should be pointed out that the above mentioned negative results are valid for "bad," artificial functions, the construction of which is a difficult process. A striking contrast is an extremely simple example due to Bernstein [1], who proved that the sequence of interpolation polynomials to $|x|$ at equally spaced nodes in $[-1, 1]$ diverges *everywhere*, except at zero and the end-points. On the other hand, it was shown by Newman [5] that rational interpolation to $|x|$ is much more favorable. By choosing a special matrix of interpolation nodes (which will be referred to as the Newman nodes) defined by

$$N = (-1, -a, ..., -a^{n-1}, 0, a^{n-1}, ..., a, 1), \qquad n = 1, 2, ...,$$

where

$$a = a(n) = \exp(-1/\sqrt{n}),$$

Newman proved that the sequence of rational interpolants to $|x|$ at the N-nodes converges uniformly with exponential rate.

Motivated by the above mentioned results of Bernstein and Newman, in the present paper we consider the behavior of *polynomial* interpolation to $|x|$ at a family of nodes, which includes the N-nodes. We prove that the corresponding sequence of polynomial interpolants to $|x|$ diverges everywhere except at zero and the end-points. This result demonstrates that the divergence phenomenon for $|x|$ is rather general, and is linked to the nature of the polynomial interpolation process.

2. Results

Let $g(x) = |x|$ and let P be a family of nodal matrices of the form

$$P = P(a) = \{-1, -a, ..., -a^{n-1}, 0, a^{n-1}, ..., a, 1\}, \qquad n = 1, 2, ... \quad (1)$$

depending on the parameter $a = a(n)$, $0 < a < 1$ (here and in the sequel, when there is no reason for confusion, we omit the explicit dependence of

a on n). Note that for $a = \exp(-1/\sqrt{n})$, $P(a)$ coincides with the matrix N of Newman nodes. Our goal is to prove that under certain conditions on $a(n)$ the corresponding sequence of interpolating polynomials $L_{2n}(g; P; x)$ will diverge everywhere in $0 < |x| < 1$. More precisely, we prove the following

THEOREM 2.1. *Let P be the family of nodal matrices of the form* (1) *with*

$$a(n) = \exp(-1/r(n)), \tag{2}$$

where r(n) satisfies the conditions

$$\lim_{n \to \infty} r(n) = \infty; \qquad r(n+1) \geqslant r(n), \qquad n = 1, 2, ..., \tag{3}$$

$$\lim_{n \to \infty} [r(n+1) - r(n)] = 0, \tag{4}$$

$$\lim_{n \to \infty} \frac{r(n) \log[r(n)]}{n} = 0. \tag{5}$$

Then for any $x \in (-1, 1)$, $x \neq 0$

$$\limsup_{n \to \infty} L_{2n}(|x|; P; x) = \infty. \tag{6}$$

Remark 1. Condition (3) yields

$$a(n) \to 1, \qquad n \to \infty, \tag{3'}$$

while condition (4) implies, in view of the Schtolz theorem [4], that $r(n) = o(n)$, from which it follows that

$$[a(n)]^n \to 0, \qquad n \to \infty. \tag{4'}$$

The asymptotic relations (3′) and (4′) together guarantee that the set of nodal points (1) is dense in $[-1, 1]$.

It turns out that to prove the divergence we need some additional conditions to ensure that the rate of the convergence in (4′) is faster than that of (3′). As we will see in the process of the proof, condition (5) will be sufficient for this purpose. Note that although this condition is a bit more than we actually need, we find it convenient to formulate all the conditions for the divergence in terms of the sequence $\{r(n)\}$.

Remark 2. One can easily verify that for $r(n) = n^\alpha$, $0 < \alpha < 1$, conditions (3)–(5) are fulfilled. Therefore, the divergence property of $L_{2n}(|x|; P; x)$ takes place for a broad family of nodes, including in particular the Newman nodes.

Proof. We restrict ourselves to $x \in (-1, 0)$ (the case where $0 < x < 1$ is analogous) and introduce the function

$$f(x) = \begin{cases} 0, & -1 \leqslant x \leqslant 0, \\ x, & 0 \leqslant x \leqslant 1. \end{cases}$$

Since $g(x) = 2f(x) - x$, it suffices to prove divergence for the function $f(x)$. Newton's representation of the interpolating polynomial yields

$$\begin{aligned} L_{2n}(f; P; x) &= f[-1, -a, ..., -a^{n-1}, 0, a^{n-1}]\, Q_n(x) \\ &\quad + \sum_{k=2}^{n} f[-1, -a, ..., -a^{n-1}, 0, a^{n-1}, ..., a^{n-k}] \\ &\quad \times Q_n(x)(x - a^{n-1}) \cdots (x - a^{n-k+1}), \end{aligned} \tag{7}$$

where $f[\cdots]$ is the standard divided difference notation and

$$Q_n(x) = x \prod_{j=0}^{n-1} (x + a^j).$$

First we claim that

$$\begin{aligned} \operatorname{sgn}\{f[-1, -a, ..., -a^{n-1}, 0, a^{n-1}, ..., a^{n-k}]\} \\ = (-1)^{k-1}, \qquad k = 1, 2, ..., n. \end{aligned} \tag{8}$$

Indeed, it is clear that for the function $f(x)$ all the consecutive divided differences of the second order equal zero, except $f[-a^{n-1}, 0, a^{n-1}]$, which is positive. Therefore the vector, consisting of the consecutive divided differences of the third order (namely $\{f[-1, -a, -a^2, -a^3]$, $f[-a, -a^2, -a^3, -a^4], ..., f[a^3, a^2, a, 1]\}$) has the following sign pattern: $\{0, 0, ..., 0, +, -, 0, ..., 0, 0\}$. Continuing in the same manner we arrive at the conclusion that the consecutive divided differences of the order $(n+1)$ have alternating signs, namely

$$\begin{aligned} \operatorname{sgn}\{f[-a^{k-1}, -a^k, ..., -a^{n-1}, 0, a^{n-1}, ..., a^{n-k}]\} \\ = (-1)^{k-1}, \qquad k = 1, 2, ..., n. \end{aligned} \tag{9}$$

It remains to note that the sign alternation property of the divided differences in (9) guarantees the same sign pattern for the corresponding divided differences in (8).

Thus, all the summands on the right-hand side of (7) have the same sign and therefore the absolute value of $L_{2n}(f; P; x)$ is greater than the absolute value of the first summand, that is,

$$|L_{2n}(f; P; x)| > f[-1, -a, ..., -a^{n-1}, 0, a^{n-1}]\, |Q_n(x)|. \tag{10}$$

Further, by applying the well-known formula for the divided difference

$$f[x_0, x_1, ..., x_n] = \sum_{k=0}^{n} f(x_k) \Big/ \prod_{\substack{j=0 \\ j \neq k}}^{n} (x_k - x_j),$$

we find

$$\begin{aligned} f[-1, -a, ..., -a^{n-1}, 0, a^{n-1}] &= \frac{1}{2a^{n-1} \prod_{k=0}^{n-2} (a^{n-1} + a^k)} \\ &= \frac{1}{2a^{n(n-1)/2} \prod_{k=1}^{n-1} (1 + a^k)} \\ &> \frac{1}{2^n a^{n(n-1)/2}}. \end{aligned} \tag{11}$$

Now let us study the behavior of the polynomial factor $Q_n(x)$. To this end we fix $x \in (-1, 0)$ and assume that there is an index k, depending on n such that

$$-a^k \leqslant x < -a^{k+1}, \qquad k = 0, 1, ..., n-2. \tag{12}$$

Note that since $a^n \to 0$ it follows that $x < -a^{n-1}$ for n sufficiently large and therefore the case $-a^{n-1} \leqslant x < 0$ may be excluded from our consideration. It follows from (12) and (2) that

$$k \leqslant r(n)[-\log(-x)] < k + 1,$$

or, after the substitution $x = -\exp(-t)$, $t \in (0, \infty)$

$$k \leqslant tr(n) < k + 1.$$

Thus,

$$k = [tr(n)], \tag{13}$$

where $[\cdots]$ is the integer part notation.

Now we can estimate $|Q_n(x)|$ from below as

$$\begin{aligned} |Q_n(x)| &\geqslant (x + a^k)(-a^{k+1} - x) \left[a^{k+1} \prod_{j=k+2}^{n-1} (a^{k+1} - a^j) \prod_{j=0}^{k-1} (a^j - a^k) \right] \\ &= (x + a^k)(-a^{k+1} - x) \\ &\quad \times \left[a^{n(k+1) - k^2/2 - 5k/2 - 1} \prod_{j=1}^{k} (1 - a^j) \prod_{j=1}^{n-k-2} (1 - a^j) \right]. \end{aligned} \tag{14}$$

Taking into account that $1-a^j \geqslant 1-a$, $j=1, 2, ...,$ we have

$$\prod_{j=1}^{k} (1-a^j) \prod_{j=1}^{n-2-k} (1-a^j) > (1-a)^{n-2}. \tag{15}$$

Combining (10), (11), (14), and (15) yields

$$|L_{2n}(f; P; x)| \geqslant (x+a^k)(-a^{k+1}-x) \left\{\frac{(1-a)^{n-2}}{2^n a^{\mu(n,k)}}\right\}, \tag{16}$$

where

$$\mu(n,k) = \frac{n(n-1)}{2} - n(k+1) + \frac{k^2}{2} + \frac{5k}{2} + 1.$$

Inequality (16) may be rewritten as

$$|L_{2n}(f; P; x)| \geqslant \left[\frac{(1-a)^n}{2^n a^{\mu(n,k)-2k}}\right]\left[\frac{(x+a^k)(-a^{k+1}-x)}{(a^k-a^{k+1})^2}\right] \equiv B_n q_n(x). \tag{17}$$

To complete the proof of the divergence it will be sufficient to show that $B_n \to \infty$ as $n \to \infty$, while there is a subsequence of indices n, such that for some $\delta > 0$, $q_n(x) \geqslant \delta$. To this end we make use of the conditions (3)–(5).

Note first that conditions (3) and (5) yield

$$\lim_{n\to\infty} \frac{a^{cn}}{1-a} = 0, \tag{18}$$

where c is an arbitrary positive constant and $a = a(n) = \exp(-1/r(n))$. Indeed, it follows from (5) that

$$\lim_{n\to\infty} \left[\log r(n) - \frac{cn}{r(n)}\right] = -\infty,$$

which is equivalent to

$$\lim_{n\to\infty} \frac{r(n)}{e^{cn/r(n)}} = 0.$$

On the other hand

$$\lim_{n\to\infty} \frac{a^{cn}}{1-a} = \lim_{n\to\infty} \frac{e^{-cn/r(n)}}{1-e^{1/r(n)}} = \lim_{n\to\infty} \frac{r(n)}{e^{cn/r(n)}},$$

thus proving (18). In the following we will use (18) with $c = 1/8$.

Further (13) yields that $k=o(n)$ and therefore in the sequel we will assume that n is sufficiently large, so that

$$0 \leqslant k \leqslant \frac{n}{2}. \tag{19}$$

Now, since $\nu(n, k)=\mu(n, k)-2k$ is a monotone decreasing function of k in the interval (19) we have

$$\nu(n, k) \geqslant \frac{n^2}{8}-\frac{5n}{4}+1>\frac{n^2}{8}-\frac{5n}{4}=\frac{n(n-10)}{8}$$

and therefore

$$B_n>\left[\frac{1-a}{2a^{(n-10)/8}}\right]^n. \tag{20}$$

Combining (18) and (20) yields

$$\lim_{n \to \infty} B_n=\infty. \tag{21}$$

It remains to show that $q_n(x)$ which was defined by

$$q_n(x)=\frac{(x+a^k)(-a^{k+1}-x)}{(a^k-a^{k+1})^2}, \qquad a=a(n), \qquad k=k(n),$$

is bounded away from zero infinitely often. To this end it suffices to prove that

$$\frac{x+[a(n)]^{k(n)}}{[a(n)]^{k(n)}-[a(n)]^{k(n)+1}} \tag{22}$$

has a limiting point, different from zero.

The proof of this fact is based on the following lemma.

LEMMA 1. *Let $r(n)$, $n=1, 2, \ldots$, be a sequence, satisfying conditions* (3), (4) *of the Theorem. Let $t \in (0, \infty)$ be fixed and define the new sequence $b(n)$, $n=1, 2, \ldots$, by*

$$b(n)=\{tr(n)\}, \qquad n=1, 2, \ldots, \tag{23}$$

where $\{\cdots\}$ is the fractional part notation.

Then every point in (0, 1) *is an accumulation point of the sequence $b(n)$.*

Proof. For the proof it suffices to show that for any interval $[\alpha, \beta]$, $0<\alpha<\beta<1$, there exists an element of the sequence $b(n)$, belonging to

$[\alpha, \beta]$. To this end note first that condition (4) guarantees that there exists an n_1 such that

$$r(n+1) - r(n) < \frac{\beta - \alpha}{t}, \qquad \forall n > n_1. \tag{24}$$

Let $N = [tr(n_1)] + 1$ (here $[\cdots]$ is the integer part notation) and suppose that p is the first index such that

$$[tr(p)] = N,$$

while $q(q > p)$ is an index (which exists in view of (3)), such that

$$[tr(q)] = N + 1.$$

Suppose that $\{tr(k)\} \notin [\alpha, \beta]$, $k = p, p+1, ..., q$. Then there exists an index $j \in \{p, p+1, ..., q-1\}$ such that

$$tr(j) < N + \alpha,$$
$$tr(j+1) > N + \beta,$$

and therefore

$$r(j+1) - r(j) > \frac{\beta - \alpha}{t} \tag{25}$$

contradicting (24) and thus proving the lemma.

Returning to the proof of the theorem we can suppose in view of the lemma and taking into account (13) that there is a subsequence of indices $\{n_j\}$ such that

$$\lim_{j \to \infty} [tr(n_j) - k(n_j)] = L, \qquad L \in (0, 1).$$

Then since $x = -e^{-t}$ and $a = e^{-1/r(n)}$ we have

$$\begin{aligned}
&\lim_{j \to \infty} \frac{[a(n_j)]^{k(n_j)} + x}{[a(n_j)]^{k(n_j)} - [a(n_j)]^{k(n_j)+1}} \\
&= \lim_{j \to \infty} \frac{e^{-k(n_j)/r(n_j)} - e^{-t}}{e^{-k(n_j)/r(n_j)} - e^{-(k(n_j)+1)/r(n_j)}} \\
&= \lim_{j \to \infty} \frac{1 - [e^{-1/r(n_j)}]^{tr(n_j) - k(n_j)}}{1 - e^{-1/r(n_j)}} = L.
\end{aligned}$$

This completes the proof of the theorem.

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