On the explicit structure of $K_2(\mathbb{F}_p G)$ for $G$ a finite abelian $p$-group

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A B S T R A C T

Let $\mathbb{F}_p$ be a finite field with $p$ a prime number and $G$ a finite abelian $p$-group. We give the explicit structure of $K_2(\mathbb{F}_p G)$; in particular $K_2(\mathbb{F}_p G)$ is not an elementary abelian $p$-group when the $p^2$-rank of $G$ is greater than 1.

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1. Introduction

Let $\mathbb{F}_p$ be a finite field of characteristic $p$ and $G$ a finite abelian group; the $p'$-rank of $G$ is defined to be $\dim_{\mathbb{F}_p}G^{p^{-1}}/G^p$. For elementary abelian $p$-groups $G$, Dennis et al. [1] calculated $K_2(\mathbb{F}_p G)$ and then got lower bounds for the order of $K_2(G)$ and $Wh_2(G)$. For $p = 2$ and with the 4-rank of $G \leq 1$, Magurn [3] proved that $K_2(\mathbb{F}_p G)$ is an elementary abelian 2-group and gave Steinberg symbols as generators. We [2] extended Magurn’s results to odd prime numbers $p$ and calculated $K_2(\mathbb{F}_p G)$ for when the $p^2$-rank of $G$ is less than 1. Now a natural question arises:

**Question.** If the $p^2$-rank of $G \geq 2$, what is the structure of $K_2(\mathbb{F}_p G)$?

If $G$ is a finite abelian group and $C_{p^r}$ a cyclic group of order $p^r$, by (3.1) in [2] we have the following decomposition formula:

$$ K_2(\mathbb{F}_p G \times C_{p^r}) \cong K_2(\mathbb{F}_p G) \oplus K_2(\mathbb{F}_p G^{p^{-1}}/G^p)(t) ). $$

(1.1)

For $G$ a finite abelian $p$-group, the order of the finite $p$-group $K_0(\mathbb{F}_p G)$ was given by Oliver [4], so the order of the finite $p$-group $K_2(\mathbb{F}_p H[1]/(t^{p^r})(t))$ for any finite abelian $p$-group $H$ can be calculated by using the above decomposition formula. After finding a sufficient number of Dennis–Stein symbols for generating $K_2(\mathbb{F}_p H[1]/(t^{p^r})(t))$, we get the explicit structure of $K_2(\mathbb{F}_p H[1]/(t^{p^r})(t))$. And by repeated use of (1.1), the explicit structure of $K_2(\mathbb{F}_p G)$ for arbitrary finite abelian $p$-group $G$ is given.

The main results of this paper are the following two theorems.

**Theorem 1.1.** Let $\mathbb{F}_p$ be a finite field with $p$ a prime number. Let $G = C_{p^r_1} \times \cdots \times C_{p^r_n} = \langle \sigma_1 \rangle \times \cdots \times \langle \sigma_n \rangle$ be a finite abelian $p$-group and $\alpha_1, \ldots, \alpha_n \geq m$; then

$$ K_2(\mathbb{F}_p G[1]/(t^{p^m})(t)) \cong \bigoplus_{i=1}^{m-1} C_{p^i}^{\left| \mathbb{F}_p \right|^i \left( (mp^i+1-(n-1)p^{i+1}-1 \right) \left( (mp^i-n-1)1^2-(n-1)p^{i+1}{(p^i-1)} \right) } . $$

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**Theorem 2.1.** Let $\mathbb{F}_p$ be a finite field with $p$ prime and $G$ a finite abelian $p$-group of exponent $p^e$. Let $r_i$ denote the $p^i$-rank of $G$. Then
\[
K_2(\mathbb{F}_p G) = C_{p^{e-1}}^{(r_1-1)(|G^{p-1}| - r_1 + 1 - |G|)} \bigoplus_{i=1}^{e-1} C_{p^i}^{(r_1-1)(|G^{p-1}| - r_1 + 1 - (r_i+1)(|G^{p-1}| - |G|))}.
\]

**Theorem 2.1** extends the corresponding results in [1–3], showing that $K_2(\mathbb{F}_p G)$ is no longer an elementary abelian $p$-group when the $p^2$-rank of $G$ is greater than 1.

**2. Preliminaries**

Let $R$ be a commutative ring with unit and $I \subseteq \text{rad}(R)$. By (1.4) of [3], the relative $K_2$-group $K_2(R, I)$ is generated by Dennis–Stein symbols $(a, b)$ with $a$ or $b$ in $I$, satisfying the following relations:

\begin{align*}
(DS1) & \quad (a, b) = -(b, a) \quad \text{if } a \in I; \\
(DS2) & \quad (a, b) + (a, c) = (a, b + c - abc) \quad \text{if } a \in I \text{ or } b, c \in I; \\
(DS3) & \quad (a, bc) = (ab, c) + (ac, b) \quad \text{if } a \in I.
\end{align*}

The relations (2.1) and (2.2) in the following lemma are frequently used in the computation of Dennis–Stein symbols in this paper.

**Lemma 2.1.** (See [5, Lemma 1.5].) Let $R$ and $I$ be as above. If $R$ is an $\mathbb{F}_p$-algebra, then
\[
(a, b) = \langle ab^p, b^p \rangle, \quad (a, b, c) = \langle (a+b+c-abc), abc \rangle, \quad (a, b, c, d) = \langle (a+b+c-d), (a+b+c), (a+b+c), (a+b+c) \rangle.
\]

For an arbitrary ring $R$ and positive integer $m$,
\[
(a, b^m) = m \langle ab^{m-1}, b \rangle.
\]

Let $K$ be an unramified extension of the $p$-adic field $\mathbb{Q}_p$ with $[K : \mathbb{Q}_p] = f$ and $A$ the valuation ring of $K$; then $A/(p)$ is just the finite field $F_q$ with $q = p^f$, so Proposition 6.3 and Theorem 6.6 in [4] give the precise order of $K_2(\mathbb{F}_q G)$, which is essential for determining the structure of $K_2(\mathbb{F}_q G)$ through direct calculation of symbols.

**Theorem 2.2.** (See [4, Proposition 6.3].) Let $F_q$ be a finite field with $q = p^f$ and $G$ a finite abelian $p$-group. If $\exp(G) = p^e$ and $r_i = p^k - rk(G)(1 \leq i \leq e)$, then
\[
\text{ord}_p[K_2(F_q G)] = f ((r_1 - 1)|G| - (r_1 - r_2)|G^p| - \cdots - (r_{e-1} - r_e)|G^{p-1}| - (r_e - 1)).
\]

**3. The main results**

Let $R$ be a commutative ring and $I$ a radical ideal of $R$. If $a, b \in I$ or $c \in I$, then by (DS2) we have the following equations in $K_2(R, I)$:
\[
(a + b, c) = (a, c) + (b, c) + ((1 - (a + b - abc)c)^{-1}abc, c).
\]

Let $b, c \in I$ or $a \in I$. As (3.1) we have
\[
(a, b + c) = (a, b) + (a, c) + (a, (1 - (b + c - abc)a)^{-1}abc).
\]

We first need three lemmas to prove **Theorem 3.4** below, which gives a relatively small number of generators of $K_2(\mathbb{F}_p G[1]/(t^{p^m}), (t))$.

**Lemma 3.1.** Let $a_i, \ldots, a_{mpm-1}$ be arbitrary elements of $R, I > 1$; then in $K_2(R[t]/(t^{p^n}), (t))$, $\langle a_it^l + \cdots + a_{mp-1}t^{p^m-1}, t \rangle$ is a sum of elements in $\{ \langle at^l, t \rangle | l \leq i \leq mp - 1, a \in R \}$.

**Proof.** If $l = pm - 1$, the lemma is obviously true. Let $l = k$; by (3.1) we have
\[
\langle a_{it} + \cdots + a_{mp-1}t^{p^m-1}, t \rangle = (a_{it}, t) + (a_{i+1}t^k + \cdots + a_{mp-1}t^{p^m-1}, t) + (\theta, t),
\]
where $\theta \in (t^{2k+2})$. Since $k + 1, 2k + 2 > k$, by the induction hypothesis the lemma is true for $l = k$ and the lemma is proved. \(\square\)

**Lemma 3.2.** Let $a_i, b_1, \ldots, b_{mp-1}$ be arbitrary elements of $R, 1 \leq i \leq pm - 1$; then in $K_2(R[t]/(t^{p^n}), (t))$, $\langle at^l, b_1t + \cdots + b_{mp-1}t^{p^m-1} \rangle$ is a sum of elements of $\{ \langle at^l, t \rangle, \langle at^l, b_i \rangle | i \geq l, j > l, a, b \in R \}$.

**Proof.** By (3.2) we have
\[
\langle at^l, b_1t + \cdots + b_{mp-1}t^{p^m-1} \rangle = (a_{it}, b_1t) + (a_{it}, b_2t^2 + \cdots + b_{mp-1}t^{p^m-1}) + (a_{it}, \theta).
\]
where \( \theta \in (t^{l+1}) \) and by (DS3)
\[
\langle at^1, bt \rangle = \langle a_t b_1 t^1, t \rangle + \langle at^{l+1}, b_1 \rangle.
\]
Now an easy induction yields the lemma. \( \square \)

**Lemma 3.3.** Let \( a_1, \ldots, a_{p-1} \) and \( b \) be arbitrary elements of \( R \); then \( \langle at^1 + \cdots + a_{p-1}t^{p-1}, b \rangle \) is a sum of elements of \( \{\langle at^i, b \rangle | a \in R, i \geq 1 \} \) in \( K_2(\mathbb{F}_p[t]/(t^p))^m \), \( (t) \).

**Proof.** By (3.1) we have
\[
\langle at^1 + \cdots + a_{p-1}t^{p-1}, b \rangle = \langle at^1, b \rangle + \langle a_{p-1}t^{p-1} + \cdots + a_1t^1, b \rangle + \langle \theta, b \rangle,
\]
where \( \theta \in (t^{2l+1}) \); now an easy induction yields the lemma. \( \square \)

**Theorem 3.4.** Let \( \mathbb{F}_p \) be a finite field with \( p \) elements and \( G = \langle \sigma_1 \rangle \times \cdots \times \langle \sigma_n \rangle \) be a finite abelian \( p \)-group. Then \( K_2(\mathbb{F}_pG[t]/(t^p)), (t) \) can be generated by
\[
S = \{\langle gt^k, t \rangle, (gt^k, \sigma_i) | g \in G, 1 \leq k < p^m, 1 \leq i \leq n \}.
\]

**Proof.** By Proposition 1.7 in [5], \( K_2(\mathbb{F}_pG[t]/(t^p)), (t) \) is generated by elements \( \langle at^1, t \rangle \) and \( \langle at^i, b \rangle \) with \( a, b \in \mathbb{F}_pG \) and \( 1 \leq i \leq p^m - 1 \), and we define a filtration on \( K_2(\mathbb{F}_pG[t]/(t^p)), (t) \) using these elements. Let \( F_0 = 0 \) and:

1. when \( 1 \leq k \leq p^m - 1 \),
   \[ F_k = \text{the subgroup generated by } F_{k-1} \text{ and symbols of the type } \langle at^{p^m-k}, t \rangle; \]
2. when \( p^m \leq k \leq 2p^m - 2 \),
   \[ F_k = \text{the subgroup generated by } F_{k-1} \text{ and symbols of the type } \langle at^{2p^m-k-1}, b \rangle. \]

Then \( F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{p^m-1} \subseteq F_{2p^m-2} \subseteq \cdots \subseteq F_{2p^m-2} = K_2(\mathbb{F}_pG[t]/(t^p)), (t) \). To prove the theorem, it suffices to prove that the image of \( S \) in \( F_{2p^m-1} \) under the natural map \( F_k \rightarrow F_{k}/F_{k-1} \) is a set of generators of \( F_k/F_{k-1} \).

1. For \( 1 \leq k \leq p^m - 1 \), \( F_k/F_{k-1} \) is generated by \( \langle at^{p^m-k}, t \rangle \) with \( a \in \mathbb{F}_pG \). Let \( a_1, a_2 \) be arbitrary elements of \( \mathbb{F}_pG \); then by (3.1)
   \[
   \langle at^{p^m-k}, t \rangle + \langle a_2t^{p^m-k}, t \rangle = \langle a_1 + a_2, t^{p^m-k} \rangle - \langle \theta, t \rangle,
   \]
   where \( \theta \in (t^{2p^m-2k+1}) \) and \( 2p^m - 2k + 1 > p^m - k \). By Lemma 3.1, \( (\theta, t) \in F_{k-1} \). Since each element \( a \) of \( \mathbb{F}_pG \) is an \( \mathbb{F}_p \)-linear combination of \( g, \sigma_i \) in \( G \), the image of \( \{\langle gt^{p^m-k}, t \rangle | g \in G \} \) generates \( F_k/F_{k-1} \) for \( 1 \leq k \leq p^m - 1 \).
2. For \( p^m \leq k \leq 2p^m - 2 \), \( F_k/F_{k-1} \) is generated by \( \langle at^{2p^m-k-1}, b \rangle \) with \( a, b \in \mathbb{F}_pG \). Let \( b_1, b_2 \) be arbitrary elements of \( \mathbb{F}_pG \); by (3.2),
   \[
   \langle at^{2p^m-k-1}, b_1 \rangle + \langle at^{2p^m-k-1}, b_2 \rangle = \langle at^{2p^m-k-1}, b_1 + b_2 \rangle - \langle at^{2p^m-k-1}, \theta \rangle,
   \]
   where \( \theta \in (t^{2p^m-k-1}) \). By Lemma 3.2, we have \( \langle at^{2p^m-k-1}, \theta \rangle \in F_{k-1} \). Then \( \langle at^{2p^m-k-1}, b \rangle \) can be generated by \( \langle at^{2p^m-k-1}, \sigma_1^{h_1}, \cdots, \sigma_n^{h_n} \rangle \). By (DS3) for \( a' \in \mathbb{F}_pG \),
   \[
   \langle a', \sigma_1^{h_1}, \cdots, \sigma_n^{h_n} \rangle = \sum_{i=1}^n h_i \langle a' \sigma_1^{h_1}, \cdots, \sigma_i^{h_i-1}, \sigma_1^{h_1}, \cdots, \sigma_n^{h_n} \rangle.
   \]
So \( \langle at^{2p^m-k-1}, b \rangle \) is generated by symbols in \( \{\langle at^{2p^m-k-1}, \sigma_i | a \in \mathbb{F}_pG, 1 \leq i \leq n \} \). Let \( a_1, a_2 \) be arbitrary elements of \( \mathbb{F}_pG \); by (3.1),
\[
\langle at^{2p^m-k-1}, \sigma_i \rangle + \langle a_2t^{2p^m-k-1}, \sigma_i \rangle = \langle (a_1 + a_2)t^{2p^m-k-1}, \sigma_i \rangle + \langle \theta, \sigma_i \rangle,
\]
where \( \theta \in (t^{2p^m-k-1}) \). By Lemma 3.3, \( (\theta, \sigma_i) \in F_{k-1} \); then \( \langle at^{2p^m-k-1}, \sigma_i \rangle \) is generated by symbols \( \{\langle gt^{2p^m-k-1}, \sigma_i | g \in G \} \). Now the theorem is proved. \( \square \)

The next lemma further reduces the number of Dennis–Stein symbols needed to generate \( K_2(\mathbb{F}_pG[t]/(t^p)), (t) \).

**Lemma 3.5.** Let \( \mathbb{F}_p, G \) and \( S \) be as in **Theorem 3.4**. Let
\[
T_1 = \{\langle gt^k, t \rangle | g \in G, 1 \leq k < p^m \};
\]
\[
T_2 = \{\langle \sigma_1^{l_1} \cdots \sigma_i^{l_i}g^p t^k, \sigma_i \rangle | 0 \leq l_1, \ldots, l_i \leq l - 1, 0 \leq k \leq p - 2, g \in G, 2 \leq k < p^m, 1 \leq i \leq n \};
\]
\[
T_3 = \{\langle \sigma_1^{l_1} \cdots \sigma_i^{l_i}g^p t^k, \sigma_i \rangle | 1 \leq i \leq n, g \in G, 2 \leq k < p^m, k \equiv 0 \mod p \}.
\]
Set \( T = T_1 \cup T_2 \cup T_3 \). Then \( K_2(\mathbb{F}_pG[t]/(t^p)), (t) \) can be generated by Dennis–Stein symbols in \( S \setminus T \).
Proof. By Theorem 3.4, we need only to show that each symbol in $T$ is a sum of symbols in $S \setminus T$. Now we consider the symbols in $T_1, T_2$ and $T_3$ separately.

1) Symbols in $T_1$.

Let $g = \sigma_1^{l_1} \cdots \sigma_n^{l_n}$. If $k + 1 \equiv 0 \mod p$,

\[
\langle g^pt^k, t \rangle = \langle g^p, t^{k+1} \rangle + \langle t^k, g^pt \rangle
\]

\[
= -(t^{k+1}, g^p) - (g^pt^k, t)
\]

\[
= -p(g^p t^{k+1}, g) - k(g^p t^k, t),
\]

and by (2.2) we have

\[
(1 + k)(g^p t^k, t) = -p(g^p t^{k+1}, g) = -p \sum_{i=1}^n l_i(g^p \sigma_i^{-1} t^{k+1}, \sigma_i).
\]

Since $k + 1 \equiv 0 \mod p, \langle g^p \sigma_i^{-1} t^{k+1}, \sigma_i \rangle \in S \setminus T$. Since $K_2(F_p G[t]/(t^m), (t))$ is a finite abelian $p$-group, $\langle g^p t^k, t \rangle$ is a sum of symbols in $S \setminus T$ when $k + 1 \equiv 0 \mod p$.

If $k + 1 \equiv 0 \mod p$, let $k = lp - 1, 1 \leq l \leq p^{m-1}$; then $\langle g^p t^k, t \rangle = \langle g^p t^{lp-1}, t \rangle$. If $l = 1$,

\[
\langle g^p t^{lp-1}, t \rangle = p(g, t) = -p(t, g) = -p \sum_{i=1}^n l_i(g \sigma_i^{-1} t, \sigma_i).
\]

Obviously $\langle g \sigma_i^{-1} t, \sigma_i \rangle \in S \setminus T$, and $\langle g^p t^{lp-1}, t \rangle$ is a sum of symbols in $S \setminus T$.

If $l > 1$ and $g \notin G^p$, $\langle g^p t^{lp-1}, t \rangle = p(g t^{l-1}, t)$.

Also $\langle g t^{l-1}, t \rangle \in S \setminus T$.

If $l > 1$ and $g \in G^p$, let $g = g' \theta$ for some $g' \in G$; then

\[
\langle g^p t^{lp-1}, t \rangle = p(g^p t^{l-1}, t). \tag{3.3}
\]

Since $\langle g^p t^{l-1}, t \rangle \in T_1$, we can repeat the discussion above to show that either $\langle g^p t^{l-1}, t \rangle$ is a sum of symbols in $S \setminus T$ or $\langle g^p t^{lp-1}, t \rangle = p(g^p t^{l-1}, t)$ by (3.3) when $g' \in G^p$ and $l = \theta p - 1 > 1$. But $l' - 1 < l < l - l p - 1$, so after a finite number of steps we can show that $\langle g^p t^{lp-1}, t \rangle$ is a sum of symbols in $S \setminus T$ when $g' \in G^p$ and $l > 1$.

2) Symbols in $T_2$.

Let $g = \sigma_1^{l_1} \cdots \sigma_n^{l_n}$; then

\[
\langle \sigma_1^{l_1} \cdots \sigma_i^{l_i} g^p t^k, \sigma_i \rangle = -(\sigma_1, \sigma_1^{l_1} \cdots \sigma_i^{l_i} g^p t^k)
\]

\[
= -(\sigma_1^{l_1} \cdots \sigma_i^{l_i+1} g^p, t^k) - (\sigma_i t^k, \sigma_1^{l_1} \cdots \sigma_i^{l_i} g^p)
\]

\[
= -(\sigma_1^{l_1} \cdots \sigma_i^{l_i+1} g^p t^{k-1}, t) - p(\sigma_1^{l_1} \cdots \sigma_i^{l_i+1} g^{p-1} t^k, g) - \langle \sigma_1 g^p t^k, \sigma_1^{l_1} \cdots \sigma_i^{l_i} \rangle.
\]

For the last two terms in the above equation,

\[
\langle \sigma_1^{l_1} \cdots \sigma_i^{l_i+1} g^{p-1} t^k, g \rangle = \sum_{j<i} l'_j \langle \sigma_1^{l_1} \cdots \sigma_j^{l_j} \sigma_i^{l_i+1} g^p t^k, \sigma_j \rangle + \sum_{j>i} l'_{j-1} \langle \sigma_1^{l_1} \cdots \sigma_{j-1}^{l_{j-1}} \sigma_j^{l_j+1} g^p t^k, \sigma_j \rangle
\]

\[
\langle \sigma_1 g^p t^k, \sigma_1^{l_1} \cdots \sigma_i^{l_i} \rangle = \sum_{j<i} l_j \langle \sigma_1^{l_1} \cdots \sigma_j^{l_j} \sigma_i^{l_i+1} g^p t^k, \sigma_j \rangle + l_i \langle \sigma_1^{l_1} \cdots \sigma_i^{l_i+1} g^p t^k, \sigma_i \rangle,
\]

so we have

\[
(1 + p'_i + l_i)\langle \sigma_1^{l_1} \cdots \sigma_i^{l_i} g^p t^k, \sigma_i \rangle = -(\sigma_1^{l_1} \cdots \sigma_i^{l_i+1} g^p t^{k-1}, t) - \sum_{j<i} l_j \langle \sigma_1^{l_1} \cdots \sigma_j^{l_j-1} \sigma_i^{l_i+1} g^p t^k, \sigma_j \rangle
\]

\[
- p \left( \sum_{j<i} l'_j \langle \sigma_1^{l_1} \cdots \sigma_j^{l_j-1} \cdots \sigma_i^{l_i+1} g^p t^k, \sigma_j \rangle + \sum_{j>i} l'_{j-1} \langle \sigma_1^{l_1} \cdots \sigma_{j-1}^{l_{j-1}} \sigma_i^{l_i+1} g^p t^k, \sigma_j \rangle \right).
\]

Since $1 \leq l_1 + 1 \leq p - 1$, and so $\sigma_i^{l_i+1} \notin G^p, \langle \sigma_1^{l_1} \cdots \sigma_i^{l_i+1} g^{p+1} t^{k-1}, t \rangle, \langle \sigma_1^{l_1} \cdots \sigma_i^{l_i+1} \cdots \sigma_j^{l_j+1} g^p t^k, \sigma_j \rangle (j < i)$ and $\langle \sigma_1^{l_1} \cdots \sigma_i^{l_i+1} \cdots \sigma_j^{l_j+1} g^p t^k, \sigma_j \rangle (j > i)$ in the above equation are in $S \setminus T$. And because $(1 + p'_i + l_i) \equiv 0 \mod p, K_2(F_p G[t]/(t^p), (t))$ is a finite abelian $p$-group, so $\langle \sigma_1^{l_1} \cdots \sigma_i^{l_i} g^p t^k, \sigma_i \rangle \in T_2$ is a sum of symbols in $S \setminus T$.

3) Symbols in $T_3$.

Let $k = lp, 1 \leq l \leq p^{m-1} - 1$; then

\[
\langle \sigma_1^{l-1} g^{lp}, \sigma_1 \rangle = p(g t, \sigma_1).
\]
If \( \langle g^l, \sigma_i \rangle \in T_2 \), by the discussion in (2), \( \langle g^l, \sigma_i \rangle \) is a sum of symbols in \( S \setminus T \); if \( \langle g^l, \sigma_i \rangle \in T_3 \), then \( \langle g^l, \sigma_i \rangle = p \langle g^l t^j, \sigma_i \rangle \) and \( l' < l < lp \). But if \( l = 1 \), \( \langle g^l, \sigma_i \rangle \in S \setminus T \). So after a finite number of steps, we can show that \( \langle \sigma_i^{n-1} g^l t^j, \sigma_i \rangle \) is a sum of Dennis–Stein symbols in \( S \setminus T \). Now the lemma is proved.

Now we are ready to prove Theorems 1.1 and 1.2.

**Proof of Theorem 1.1.** Let \( N \) denote nonnegative integers and

\[ V = \{ (l_1, \ldots, l_n) \in \mathbb{N}^n \mid 0 < l_j < p^{n_j}, 1 \leq j \leq n \}. \]

For each \( k, 1 \leq k < n \), we shall define a partition \( N_{k,1}, \ldots, N_{k,m-k} \) of \( V \). We shall show below that if \( (l_1, \ldots, l_n) \in N_{k,m-k} \), the exponent of \( \langle s^1 \ldots s^n t^k x \rangle \) is \( \leq p^{m-k} \); if \( (l_1, \ldots, l_n) \in N_{k,1} \), \( 1 \leq i < m - s \), the exponent of \( \langle s^1 \ldots s^n t^k x \rangle - \langle s^1 \ldots s^n t^k x \rangle \) is \( \leq p' \). The meanings of \( s, x \) and \( (l_1', \ldots, l_n') \) are explained below.

If \( p^s \leq k < p^{s+1} \), \( 0 \leq s \leq m - 1 \), set

\[ N_{k,i} = \{ (l_1, \ldots, l_n) \in \mathbb{N}^n \mid 0 < l_j < p^{n_j-i+1}, 1 \leq j \leq n \}, \quad 1 \leq i \leq m - s. \]

Then \( N_{k,m-s} = \{ (l_1, \ldots, l_n) \in \mathbb{N}^n \mid (l_1', \ldots, l_n') \in N_{k,i+1} \}, 1 \leq i < m - s \); so \( N_{k,1}, \ldots, N_{k,m-k} \) is a partition of \( V \). If \( i < m - s \) and \( (l_1, \ldots, l_n) \in N_{k,i} \), then there exists only one \( (l_1', \ldots, l_n') \in N_{k,i+1} \) such that \( l_j \equiv l_j' \mod p^{n_j-i} \).

For convenience, let \( x \) stand for \( t \) or \( \sigma_j \), \( 1 \leq j \leq n \). If \( (l_1, \ldots, l_n) \in N_{k,m-k} \), since \( k - p^{m-s} \geq p_m \),

\[ p^{m-s} \langle \sigma_{i}^{1} \cdots \sigma_{n}^{l} t \rangle, x \rangle = \langle (\sigma_{i}^{1} \cdots \sigma_{n}^{l} p^{m-s-1} t^{k} p^{m-s})^{k}, x \rangle = 0, \]

so the order of \( \langle \sigma_{i}^{1} \cdots \sigma_{n}^{l} t \rangle, x \rangle \) is \( \leq p^{n-s} \). If \( (l_1, \ldots, l_n) \in N_{k,i}, i < m - s \),

\[ p' \langle \sigma_{i}^{1} \cdots \sigma_{n}^{l} t \rangle, x \rangle - \langle \sigma_{i}^{1} \cdots \sigma_{n}^{l} t \rangle, x \rangle = 0, \]

so the order of \( \langle \sigma_{i}^{1} \cdots \sigma_{n}^{l} t \rangle, x \rangle - \langle \sigma_{i}^{1} \cdots \sigma_{n}^{l} t \rangle, x \rangle \leq p' \). For \( i < m - s \) and \( (l_1, \ldots, l_n) \in N_{k,i} \), if \( \sigma_{i}^{1} \cdots \sigma_{n}^{l} t \rangle, x \rangle \in S \setminus T \), then \( \langle \sigma_{i}^{1} \cdots \sigma_{n}^{l} t \rangle, x \rangle \in S \setminus T \), then if we replace \( \sigma_{i}^{1} \cdots \sigma_{n}^{l} t \rangle, x \rangle \in S \setminus T \) by \( \langle \sigma_{i}^{1} \cdots \sigma_{n}^{l} t \rangle, x \rangle - \langle \sigma_{i}^{1} \cdots \sigma_{n}^{l} t \rangle, x \rangle \), the new set is still a generating set of \( K_2(\mathbb{F}_p[G]/(t_{m}^{\alpha})), (t) \) and the number of these generators does not change. We still use \( S \setminus T \) to denote it.

Now we begin to count the number of Dennis–Stein symbols in the new generating set \( S \setminus T \). For \( p^s \leq k < p^{s+1}, 0 \leq s \leq m - 1 \), let \( \mu_{k,s} \) denote the number of \( \langle \sigma_{i}^{1} \cdots \sigma_{n}^{l} t \rangle, x \rangle \in S \setminus T \) with \( (l_1, \ldots, l_n) \in N_{k,s} \). Let \( \mu_{k,1} \) denote the number of \( \langle \sigma_{i}^{1} \cdots \sigma_{n}^{l} t \rangle, x \rangle - \langle \sigma_{i}^{1} \cdots \sigma_{n}^{l} t \rangle, x \rangle \in S \setminus T \) with \( (l_1, \ldots, l_n) \in N_{k,1} \), \( i < m - s \). Using the definitions of \( T_i \) in Lemma 3.5 we can get the value of \( \mu_{k,1} \). If \( x = t \) and \( p' \leq k < p^{s+1} \),

\[ \mu_{k,1} = |G|p^{-n(m-s)}(p^n - 1), \quad \mu_{k,s} = |G|p^{-n(n)1}(1 - p^{-n}), \quad i < m - s. \]

Now let \( x = \sigma_j, 1 \leq j \leq n \). If \( k = 1 \),

\[ \mu_{k,1} = |G|p^{-n(m)}p^1, \quad \mu_{k,s} = |G|p^{-n(n)1}(1 - p^{-n}), \quad i < m. \]

When \( p^s \leq k < p^{s+1}, 2 \leq k \) and \( k \equiv 0 \mod p \),

\[ \mu_{k,1} = |G|p^{-n(m-s)}(p^n - p^l + p^l-1); \]

\[ \mu_{k,s} = |G|p^{-n(n)1}(p^n - p^l + p^l-1), \quad i \leq m - s; \]

When \( p^s \leq k < p^{s+1}, 2 \leq k \) and \( k \equiv 0 \mod p \),

\[ \mu_{k,1} = |G|p^{-n(m-s)}(p^n - p^l + p^l-1); \]

\[ \mu_{k,s} = |G|p^{-n(n)1}(p^n - p^l + p^l-1), \quad i \leq m - s. \]

Let

\[ \beta_m = \sum_{k=1}^{p^{m}} \left( \mu_{k,1} + \sum_{j=1}^{n} \mu_{k,j} \right), \quad \beta_i = \sum_{k=1}^{p^{m-i-1}} \left( \mu_{k,1} + \sum_{j=1}^{n} \mu_{k,j} \right). \]

So the order of \( K_2(\mathbb{F}_p[G]/(t_{m}^{\alpha})), (t) \) is \( \leq \prod_{i=1}^{m} p^{\beta_i} = p^{G(p^{-n(m-s)} - (n-1))} \). Since \( K_2(\mathbb{F}_p[G \times C_p]) \cong K_2(\mathbb{F}_p G \times K_2(\mathbb{F}_p[G]/(t_{m}^{\alpha})), (t)) \), by Theorem 2.2, we know that

\[ |K_2(\mathbb{F}_p[G]/(t_{m}^{\alpha})), (t)| = |K_2(\mathbb{F}_p[G \times C_p])|/|K_2(\mathbb{F}_p G)| = p^{G(p^{-n(m-s)} - (n-1))}. \]
So
\[ K_2(\mathbb{F}_p G[t]/(t^{b^m})), (t)) = \bigoplus_{i=1}^{m} C_{p^i}^{\beta_i}, \]
and the theorem is proved. \(\square\)

**Proof of Theorem 1.2.** We use induction on the \(p\)-rank of \(G\). As noted in [3], \(K_2(\mathbb{F}_p G) = 1\) if \(G\) is finite cyclic. Let \(C_{p^l}\) be a direct summand of least order in the decomposition of \(G\) as a direct sum of cyclic \(p\)-groups and \(H \leq G\) such that \(G = H \times C_{p^l}\), \(l \leq e\); so if \(G\) is not cyclic, \(G\) and \(H\) have the same exponent \(p^e\). Let \(r_i'\) and \(r_i\) denote the \(p^i\)-ranks of \(G\) and \(H\) respectively. By (3.1) in [2],
\[ K_2(\mathbb{F}_p G) \cong K_2(\mathbb{F}_p H) \oplus K_2(\mathbb{F}_p H[t]/(t^{p^i})), (t)) \]

Let
\[ K_2(\mathbb{F}_p H) = \bigoplus_{i=1}^{e} C_{p^i}, K_2(\mathbb{F}_p H[t]/(t^{p^i})), (t)) = \bigoplus_{i=1}^{l} C_{p^i}. \]

By the induction hypothesis, all the \(\alpha_i\) are known.

We will consider \(e > l\) and \(e = l\) separately. First suppose that \(e > l\). For \(l < i \leq e\), since \(r_i = r_i'\) and \(|H^{p^i}| = |G^{p^i}|\), we have
\[ \alpha_i = (r_i' - 1)(|G^{p^{i-1}}| - 1), \quad \alpha_i = (r_i' - 1)(|G^{p^{i-1}}| - |G^{p^i}|) - (r_{i+1}' - 1)(|G^{p^i}| - |G^{p^{i+1}}|), \quad l < i < e. \]

For \(i = 1\), obviously \(r_i' - 1 = r_1\) and \(|G^{p^{-1}}| = |H^{p^{-1}}|p^{p^{-1}}\). By Theorem 1.1 and the induction hypothesis,
\[ \alpha_1 + \beta_1 = (r_1 - 1)|H^{p^{-1}}| - |H^{p^1}| - (r_1 - 1)(H^{p^1} | - |H^{p^1}) + |H^{p^1}|(r_1 p^{p^{-1}} - (r_1 - 1)p^{p^{-1}} - 1) \]
\[ = |H^{p^1}|(r_1 - 1)(p^{p^{-1}} - 1 - r_{1+1}' - 1)(|G^{p^1}| - |G^{p^1})|) + |H^{p^1}|(r_1 p^{p^{-1}} - (r_1 - 1)p^{p^{-1}} - 1) \]
\[ = (r_1 - 1)(|G^{p^{-1}}| - |G^{p^1}|) - (r_{1+1}' - 1)(|G^{p^1}| - |G^{p^1})|). \]

For \(1 \leq i \leq l\), all the \(r_i\) are equal, as are the \(r_i'\). Let \(r' = r_i'\) and \(r = r_i\); then \(r' - 1 = r\). Suppose \(1 \leq i < l\); by the induction hypothesis,
\[ \alpha_i = (r - 1)(|H^{p^{-1}}| - |H^{p^1}) - (r - 1)(|H^{p^1} | - |H^{p^{i+1}}) \]
\[ = (r - 1)|H^{p^{i+1}}| (p^{p^{-1}} - 1^2). \]

By Theorem 1.1 and the fact \(|H^{p^{i+1}}| = |H^{p^1}|p^{p^{-1}}\), we have
\[ \beta_i = |H^{p^1}|(r p^{p^{-1}} p^{p^{-1}} - 1 - 1^2) - (r - 1)p^{p^{-1}}(p^{p^{-1}} - 1^2) \]
\[ = |H^{p^{i+1}}| (p^{p^{-1}} - 1 - 1^2) - (r - 1)(p^{p^{-1}} - 1^2). \]
\[ = r|H^{p^{i+1}}|p^{p^{-1}}(p^{p^{-1}} - 1^2) - \alpha_i. \]

Since
\[ (r' - 1)(|G^{p^{-1}}| - |G^{p^1}|) - (r' - 1)(|G^{p^1}| - |G^{p^{i+1}}|) \]
\[ = r(|H^{p^{-1}}|p^{p^{-1}} - |H^{p^1}| p^{p^{-1}} - r(|H^{p^1}|p^{p^{-1}} - |H^{p^{i+1}}|p^{p^{-1}}) \]
\[ = r|H^{p^{i+1}}| (p^{p^{-1}} - 1 - 1^2) - |H^{p^{i+1}}| (p^{p^{-1}} - 1^2) \]
\[ = r|H^{p^{i+1}}|p^{p^{-1}}(p^{p^{-1}} - 1^2), \]
we have
\[ \alpha_i + \beta_i = (r_i' - 1)(|G^{p^{-1}}| - |G^{p^1}|) - (r_{i+1}' - 1)(|G^{p^1}| - |G^{p^{i+1}}|), \quad 1 \leq i < l. \]

Now we have proved that the formula for \(K_2(\mathbb{F}_p G)\) is correct when \(e > l\).

Next we consider the case \(e = l\), that is, \(G\) is a homogeneous abelian \(p\)-group. Let \(r'\) and \(r\) denote the \(p^i\)-ranks of \(G\) and \(H\) respectively; for \(i \leq e, r' - 1 = r\). By the induction hypothesis and Theorem 1.1,
\[ \alpha_e = (r - 1)(|H^{p^{-1}}| - 1) = (r - 1)(p^{p^{-1}} - 1), \quad \beta_e = rp^{p+1} - (r - 1)p^{p-1}. \]

So
\[ \alpha_e + \beta_e = r(p^{p+1} - 1) = (r - 1)(|G^{p^{-1}}| - 1). \]
If $1 \leq i < l$,
\[
\alpha_i + \beta_i = (r - 1)(|H_{p^{i-1}} - 2|H_{p^i} + |H_{p^{i+1}}|) + |H_{p^l}|(r^{p^{e-i-1}}(p^{i+1} - 1)^2 - (r - 1)p^{e-i}(p^e - 1)^2) \\
= (r - 1)|H_{p^{i+1}}|(p^{e-i} - 2p^e + 1) + |H_{p^{i+1}}|(r^{p^{e-i-1}}(p^{i+1} - 1)^2 - (r - 1)(p^e - 1)^2) \\
= r|H_{p^{i+1}}|p^{e-(i+1)}(p^{2r+2} - 2p^{e+1} + 1) \\
= r|G_{p^{i+1}}|(p^{2r} - 2p^e + 1) \\
= r(|G_{p^{i+1}}| - 2|G_{p^e}| + |G_{p^{i+1}}|).
\]
So the formula for $K_2(F_pG)$ holds if $e = l$. Now the theorem is proved. \(\square\)

**Examples 3.6.** Let us now determine the explicit structure of $K_2(F_2[C_4 \times C_4])$. By (1.1) we know that $K_2(F_2[C_4 \times C_4]) \cong K_2(F_2C_4[t]/(t^4), (t))$. By Theorem 3.4, $K_2(F_2C_4[t]/(t^4), (t))$ is generated by Dennis–Stein symbols in the following six matrices, denoted by $S$.

$k = 1, \begin{pmatrix}
\langle t, t \rangle & \langle \sigma t, t \rangle \\
\langle \sigma^2 t, t \rangle & \langle \sigma^3 t, t \rangle
\end{pmatrix}, \begin{pmatrix}
\langle t, \sigma \rangle & \langle \sigma t, \sigma \rangle \\
\langle \sigma^2 t, \sigma \rangle & \langle \sigma^3 t, \sigma \rangle
\end{pmatrix}$.

$k = 2, \begin{pmatrix}
\langle t^2, t \rangle & \langle \sigma t^2, t \rangle \\
\langle \sigma^2 t^2, t \rangle & \langle \sigma^3 t^2, t \rangle
\end{pmatrix}, \begin{pmatrix}
\langle t^2, \sigma \rangle & \langle \sigma t^2, \sigma \rangle \\
\langle \sigma^2 t^2, \sigma \rangle & \langle \sigma^3 t^2, \sigma \rangle
\end{pmatrix}$.

$k = 3, \begin{pmatrix}
\langle t^3, t \rangle & \langle \sigma t^3, t \rangle \\
\langle \sigma^2 t^3, t \rangle & \langle \sigma^3 t^3, t \rangle
\end{pmatrix}, \begin{pmatrix}
\langle t^3, \sigma \rangle & \langle \sigma t^3, \sigma \rangle \\
\langle \sigma^2 t^3, \sigma \rangle & \langle \sigma^3 t^3, \sigma \rangle
\end{pmatrix}$.

As in Lemma 3.5, let

$T_1 = \{\langle t, t \rangle, \langle \sigma^2 t, t \rangle, \langle t^2, t \rangle, \langle \sigma^2 t^2, t \rangle, \langle t^3, t \rangle, \langle \sigma^2 t^3, t \rangle\};$

$T_2 = \{\langle t^2, \sigma \rangle, \langle \sigma^2 t^2, \sigma \rangle, \langle t^3, \sigma \rangle, \langle \sigma^2 t^3, \sigma \rangle\};$

$T_3 = \{\langle \sigma t^2, \sigma \rangle, \langle \sigma^3 t^2, \sigma \rangle\}$.

Set $T = T_1 \cup T_2 \cup T_3$; then $K_2(F_2C_4[t]/(t^4), (t))$ is generated by symbols in $S \setminus T$, and the symbols in $T$ are given a hat in the above matrices to distinguish them from the other symbols. Using the notation from the proof of Theorem 1.1,

$V = \{0, 1, 2, 3\}$.

If $k = 1, s = 0$, so the partition of $V$ is $N_{1,2} = \{0, 1\}, N_{1,1} = \{2, 3\}$. If $k = 2, 3, s = 1$, so the partition of $V$ is $N_{2,1} = N_{3,1} = \{0, 1, 2, 3\}$. As in the proof of Theorem 1.1, the symbols of order $2^2$ in $S \setminus T$ are

$\langle \sigma t, t \rangle, \langle t, \sigma \rangle, \langle \sigma t, \sigma \rangle$.

The Dennis–Stein symbols of order 2 in $S \setminus T$ are

$\langle \sigma^3 t, t \rangle - \langle \sigma t, t \rangle, \langle \sigma^2 t, \sigma \rangle - \langle t, \sigma \rangle, \langle \sigma^3 t, \sigma \rangle - \langle \sigma t, \sigma \rangle,$

$\langle \sigma t^2, t \rangle, \langle \sigma^3 t^2, t \rangle, \langle t^3, t \rangle, \langle \sigma^3 t^3, t \rangle, \langle \sigma t^3, \sigma \rangle, \langle \sigma^3 t^3, \sigma \rangle$.

so we have $K_2(F_2[C_4 \times C_4]) \cong K_2(F_2C_4[t]/(t^4), (t)) = C_4^1 \oplus C_2^2$.

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**References**


