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The scalar radius of the pion

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Abstract

The pion scalar radius is given by $\langle r_S^2\rangle = (6/\pi)\int_{4M_\pi^2}^\infty \mathrm{d}s\, \delta_S(s)/s^2$, with δ_S the phase of the scalar form factor. Below $\bar{K}K$ threshold, $\delta_S = \delta_\pi$, δ_π being the isoscalar, S-wave $\pi\pi$ phase shift. At high energy, $s > 2\ \mathrm{GeV}^2$, δ_S is given by perturbative QCD. In between I argued, in a previous letter, that one can interpolate $\delta_S \sim \delta_\pi$, because inelasticity is small, compared with the errors. This gives $\langle r_S^2\rangle = 0.75 \pm 0.07\ \mathrm{fm}^2$. Recently, Ananthanarayan, Caprini, Colangelo, Gasser and Leutwyler (ACCGL) have claimed that this is incorrect and one should have instead $\delta_S \simeq \delta_\pi - \pi$; then $\langle r_S^2\rangle = 0.61 \pm 0.04\ \mathrm{fm}^2$. Here I show that the ACCGL phase δ_S is pathological in that it is discontinuous for small inelasticity, does not coincide with what perturbative QCD suggests at high energy, and only occurs because these authors take a value for $\delta_\pi(4m_K^2)$ different from what experiment indicates. If one uses the value for $\delta_\pi(4m_K^2)$ favoured by experiment, the ensuing phase δ_S is continuous, agrees with perturbative QCD expectations, and satisfies $\delta_S \simeq \delta_\pi$, thus confirming the correctness of my previous estimate, $\langle r_S^2\rangle = 0.75 \pm 0.07\ \mathrm{fm}^2$.

1. Introduction

The quadratic scalar radius of the pion, $\langle r_S^2\rangle$, is defined via the scalar form factor, $F_{S,\pi}$:

$$F_{S,\pi}(t) \underset{t\to 0}{\simeq} F_{S,\pi}(0)\left\{1 + \frac{1}{6}\langle r_S^2\rangle t\right\}, \tag{1.1}$$

where

$$\langle\pi(p)\big|\big[m_u\bar{u}u(0) + m_d\bar{d}d(0)\big]\big|\pi(p')\rangle = (2\pi)^{-3}F_{S,\pi}(t); \tag{1.2}$$

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the (charged) pion states are normalized to $\langle\pi(p)|\pi(p')\rangle = 2p_0\delta(\mathbf{p}-\mathbf{p}')$, and $t=(p-p')^2$. To one loop in chiral perturbation theory (ch.p.t.), $\langle r_S^2\rangle$ is related to the important coupling constant $\bar{l}_4$ by

$$\langle r_S^2\rangle = \frac{3}{8\pi^2 f_\pi^2}\left\{\bar{l}_4 - \frac{13}{12}\right\}. \tag{1.3}$$

$f_\pi \simeq 93$ MeV is the decay constant of the pion.

An evaluation of $\langle r_S^2\rangle$ was given some time ago by Donoghue, Gasser and Leutwyler [1]; we will refer to this paper as DGL. These authors found (we quote the improved result from the second paper in Ref. [1])

$$\langle r_S^2\rangle_{\mathrm{DGL}} = 0.61 \pm 0.04\ \mathrm{fm}^2, \qquad \bar{l}_4 = 4.4 \pm 0.2. \tag{1.4}$$

The error comes from experimental errors and the estimated higher order corrections.

As noted in Ref. [2], one can obtain the scalar radius from the sum rule

$$\langle r_S^2\rangle = \frac{6}{\pi}\int_{4M_\pi^2}^{\infty} \mathrm{d}s\, \frac{\delta_S(s)}{s^2}, \tag{1.5}$$

where $\delta_S(s)$ is the phase of $F_{S,\pi}(s)$, and M_π is the charged pion mass. At low energy, $\delta_S(s) = \delta_\pi(s)$, where $\delta_\pi(s)$ is the phase shift for $\pi\pi$ scattering with isospin zero in the S wave. This equality holds with good accuracy up to the opening of the $\bar{K}K$ threshold, at $s = 4m_K^2$; for m_K we take the average kaon mass, $m_K = 496$ MeV. At high energy, $s > 2\ \mathrm{GeV}^2$, one can use the asymptotic estimate that perturbative QCD indicates for $\delta_S(s)$ (see below) and, between these two regions, what was considered in Ref. [2] a reasonable interpolation, viz., $\delta_S(s) \sim \delta_\pi(s)$. One then finds,

$$\langle r_S^2\rangle = 0.75 \pm 0.07\ \mathrm{fm}^2, \qquad \bar{l}_4 = 5.4 \pm 0.5. \tag{1.6}$$

This is about 2σ above the DGL value, Eq. (1.4).

The integral in (1.5) up to $s = 4m_K^2$ can be evaluated in a fairly unambiguous manner, and the contribution of the high energy region, $s > 2\ \mathrm{GeV}^2$, although evaluated with different methods, is found similar in Refs. [1–3]. The conflictive contribution is that of the intermediate region,

$$\int_{4m_K^2}^{2\ \mathrm{GeV}^2} \mathrm{d}s\, \frac{\delta_S(s)}{s^2}. \tag{1.7}$$

In fact, very recently Ananthanarayan, Caprini, Colangelo, Gasser and Leutwyler [3], that we will denote by ACCGL, have challenged the result of Ref. [2]. Their main objection is that the Fermi–Watson final state interaction theorem does *not* guarantee that $\delta_\pi(s)$ and $\delta_S(s)$ are equal, even if inelasticity is negligible; it only requires that they differ in an integral multiple of π:

$$\delta_S(s) \simeq \delta_\pi(s) + N\pi. \tag{1.8}$$

At $\pi\pi$ threshold, both δ_S and δ_π vanish, hence $N = 0$ here. Below $s = 4m_K^2$, continuity guarantees that the N in (1.8) still vanishes, as assumed in Ref. [2]. For $1.7\ \mathrm{GeV}^2 \lesssim s \lesssim 2\ \mathrm{GeV}^2$ inelasticity is also compatible with zero. However, since this is separated from the low energy region by the region $2m_K < s^{1/2} \lesssim 1.2$ GeV, where inelasticity is *not* negligible, one can have $N \neq 0$. Actually, ACCGL conclude that

$$\delta_S(s) \simeq \delta_\pi(s) - \pi, \qquad 1.1\ \mathrm{GeV} \simeq s^{1/2} \simeq 1.42\ \mathrm{GeV}. \tag{1.9}$$

According to ACCGL, this brings the value of $\langle r_S^2\rangle$ back to the DGL number in (1.4).

The remark of ACCGL leading to (1.8) is correct. Nevertheless, we will here show that their conclusion (1.9) is wrong. In fact, arguments of

(A) Continuity of $\delta_S(s)$ when the inelasticity goes to zero;
(B) The value experiment indicates for the quantity $\delta_\pi(4m_K^2)$;
(C) The value SU(3) ch.p.t. implies for the real part of the $\bar{K}K$ scattering length and $\delta_\pi(4m_K^2)$; and, finally,
(D) The matching with the phase expected from perturbative QCD at high virtuality,

all imply that the number N in (1.8) vanishes, therefore substantiating the claims of Ref. [2].

It should also be noted that the error analysis of DGL and ACCGL must be incomplete. With a correct error analysis, and even starting from their assumptions, DGL and ACCGL should have obtained a value for $\langle r_S^2\rangle$ compatible with that in Ref. [2], within errors. This is also discussed below.

2. Some definitions

Since we will only consider the S wave for isospin zero, we will omit isospin and angular momentum indices. We define a matrix for the partial wave amplitudes for the processes $\pi\pi \to \pi\pi, \pi\pi \to \bar{K}K (= \bar{K}K \to \pi\pi)$, and $\bar{K}K \to \bar{K}K$:

$$\mathbf{f} = \begin{pmatrix} f_{\pi\pi\to\pi\pi} & f_{\pi\pi\to\bar{K}K} \\ f_{\pi\pi\to\bar{K}K} & f_{\bar{K}K\to\bar{K}K} \end{pmatrix} = \begin{pmatrix} \frac{\eta e^{2i\delta_\pi}-1}{2i} & \frac{1}{2}\sqrt{1-\eta^2}e^{i(\delta_\pi+\delta_K)} \\ \frac{1}{2}\sqrt{1-\eta^2}e^{i(\delta_\pi+\delta_K)} & \frac{\eta e^{2i\delta_K}-1}{2i} \end{pmatrix}. \tag{2.1}$$

Below $\bar{K}K$ threshold, the elasticity parameter is $\eta(s) = 1$; above $\bar{K}K$ threshold[1] one has the bounds $0 \leqslant \eta \leqslant 1$. We will also use a K-matrix representation of $\mathbf{f}$:

$$\mathbf{f} = \left\{\mathbf{Q}^{-1/2}\mathbf{K}^{-1}\mathbf{Q}^{-1/2} - i\right\}^{-1}, \qquad \mathbf{Q} = \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}, \tag{2.2}$$

q_a are the momenta, $q_1 = \sqrt{s/4 - M_\pi^2}$, $q_2 = \sqrt{s/4 - m_K^2}$.

We may diagonalize $\mathbf{f}$ and find the *eigenphases*, $\delta^{(\pm)}$,

$$\begin{aligned} &\mathbf{f} = \mathbf{C}\{\mathbf{g}_D - i\}^{-1}\mathbf{C}^{\mathrm{T}}, \\ &\mathbf{g}_D = \begin{pmatrix} \cot\delta^{(+)} & 0 \\ 0 & \cot\delta^{(-)} \end{pmatrix}, \qquad \mathbf{C} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \end{aligned} \tag{2.3a}$$

We will define $\delta^{(+)}$ to be the eigenphase that matches δ_π: $\delta^{(+)}(4m_K^2) = \delta_\pi(4m_K^2)$. Then,

$$\begin{aligned} &\tan\delta^{(\pm)} = \frac{T \pm \sqrt{T^2 - 4\Delta}}{2}, \qquad \sin\theta = \left\{\frac{1}{2}\frac{T + \sqrt{T^2-4\Delta} - 2q_1K_{11}}{+\sqrt{T^2-4\Delta}}\right\}^{1/2}, \qquad T = q_1K_{11} + q_2K_{22}, \\ &\Delta = q_1q_2 \det\mathbf{K}. \end{aligned} \tag{2.3b}$$

This holds (near $\bar{K}K$ threshold) when $K_{11} > 0$. For $K_{11} < 0$, the $(\pm)$ signs should be exchanged in the right-hand side of the expression for $\tan\delta^{(\pm)}$, and the square roots in the expression for $\sin\theta$ get a minus sign. Near $\bar{K}K$ threshold, $\sin\theta$ and $\delta^{(-)}(s)$ vanish with q_2. If inelasticity were zero ($\eta = 1$) the channels would decouple and one would have $\mathbf{C} = 1$ and $\delta^{(+)} = \delta_\pi$, $\delta^{(-)} = \delta_K$.

The *phase* of the $\pi\pi \to \pi\pi$ amplitude will play an important role in the subsequent discussion. We will actually use the phase ϕ_π defined by

$$f_{\pi\pi\to\pi\pi} = \begin{cases} +|f_{\pi\pi\to\pi\pi}|e^{i\phi_\pi}, & 0 \leqslant \phi_\pi \leqslant \pi, \\ -|f_{\pi\pi\to\pi\pi}|e^{i\phi_\pi}, & \pi \leqslant \phi_\pi \leqslant 2\pi. \end{cases}$$

[1] In the present Letter we will neglect coupling of $\pi\pi$ to states other than $\bar{K}K$, for energies below 1.42 GeV.

This definition has to be adopted to agree with the standard definition of the phase (shift) δ for a purely elastic amplitude, given by $f = \sin\delta e^{i\delta}$, so that $f = \pm|f|e^{i\delta}$ with the $(\pm)$ signs as for ϕ_π above.

Using (2.1) one gets a simple expression for the tangent of ϕ_π:

$$\tan\phi_\pi = \left\{1 + \frac{1-\eta}{2\eta}\left(1 + \cot^2\delta_\pi\right)\right\}\tan\delta_\pi. \tag{2.4}$$

For ease of reference, we also give here the expressions of phase shift and inelasticity in terms of the K-matrix:

$$\tan\delta_\pi = \begin{cases} \frac{q_1|q_2|\det\mathbf{K} + q_1K_{11}}{1+|q_2|K_{22}}, & s \leqslant 4m_K^2, \\ \frac{1}{2q_1[K_{11}+q_2^2K_{22}\det\mathbf{K}]}\Big\{q_1^2K_{11}^2 - q_2^2K_{22}^2 + q_1^2q_2^2(\det\mathbf{K})^2 - 1 & \\ \quad + \sqrt{(q_1^2K_{11}^2 + q_2^2K_{22}^2 + q_1^2q_2^2(\det\mathbf{K})^2 + 1)^2 - 4q_1^2q_2^2K_{12}^4}\Big\}, & s \geqslant 4m_K^2; \end{cases} \tag{2.5a}$$

$$\eta = \sqrt{\frac{(1 + q_1q_2\det\mathbf{K})^2 + (q_1K_{11} - q_2K_{22})^2}{(1 - q_1q_2\det\mathbf{K})^2 + (q_1K_{11} + q_2K_{22})^2}}, \quad s \geqslant 4m_K^2. \tag{2.5b}$$

The connection with the scalar form factor of the pion comes about as follows. We form a vector $\mathbf{F}$ with $F_{S,\pi}$ and the form factor of the kaon, $F_{S,K}$, and define the vector $\mathbf{F}'$ by

$$\mathbf{F}' = \mathbf{C}^{\mathrm{T}}\mathbf{Q}^{1/2}\mathbf{F}, \qquad \mathbf{F} = \begin{pmatrix} F_{S,\pi} \\ F_{S,K} \end{pmatrix}. \tag{2.6a}$$

Then two-channel unitarity implies that

$$\begin{aligned} F_{S,\pi} &= q_1^{-1/2}\left\{(\cos\theta)\left|F_1'\right|e^{i\delta^{(+)}} + (\sin\theta)\left|F_2'\right|e^{i\delta^{(-)}}\right\}, \\ F_{S,K} &= q_2^{-1/2}\left\{(\cos\theta)\left|F_2'\right|e^{i\delta^{(-)}} - (\sin\theta)\left|F_1'\right|e^{i\delta^{(+)}}\right\}. \end{aligned} \tag{2.6b}$$

Near $\bar{K}K$ threshold or for small inelasticity, $\delta_S \simeq \delta^{(+)} \simeq \phi_\pi$.

3. The partial wave amplitudes from the experiment of Hyams et al.

We will here follow DGL and ACCGL and take the partial wave amplitudes as measured by Hyams et al. [4], although later we will also discuss other sets of $\pi\pi$ scattering data, as well as data [5] on $\pi\pi \to \bar{K}K$. Hyams et al. give three representations for their data: an energy-independent phase shift analysis that yields the values of the phase shift $\delta_\pi(s)$, and of the elasticity parameter $\eta(s)$, from $\pi\pi$ threshold to $s^{1/2} \simeq 1.9$ GeV; an energy-dependent parametrization of the K-matrix that interpolates these data in the whole range; and a second parametrization with a constant K-matrix that represents the data around $\bar{K}K$ threshold.

For the second, Hyams et al. write (Eq. (12a) and Table 1 in Ref. [4])

$$K_{ab}(s) = \alpha_a\alpha_b/(s_1 - s) + \beta_a\beta_b/(s_2 - s) + \gamma_{ab}, \tag{3.1}$$

where

$$\begin{aligned} &s_1^{1/2} = 0.11 \pm 0.15, \qquad s_2^{1/2} = 1.19 \pm 0.01, \\ &\alpha_1 = 2.28 \pm 0.08, \qquad \alpha_2 = 2.02 \pm 0.11, \qquad \beta_1 = -1.00 \pm 0.03, \qquad \beta_2 = 0.47 \pm 0.05, \\ &\gamma_{11} = 2.86 \pm 0.15, \qquad \gamma_{12} = 1.85 \pm 0.18, \qquad \gamma_{22} = 1.00 \pm 0.53. \end{aligned} \tag{3.2}$$

The numbers here are in the appropriate powers of GeV.

In the energy range around $\bar{K}K$ threshold, $0.9\ \text{GeV} \leqslant s^{1/2} \leqslant 1.1\ \text{GeV}$, Hyams et al. [4, p. 148] find that their data may be represented by a constant K-matrix with

$$K_{11} = 1.0 \pm 0.4\ \text{GeV}^{-1}, \qquad K_{12} = -4.4 \pm 0.3\ \text{GeV}^{-1}, \qquad K_{22} = -3.7 \pm 0.4\ \text{GeV}^{-1}. \tag{3.3}$$

The sign of K_{12} is undefined. We have chosen in (3.3) a sign opposite to that of Hyams et al. [4], to agree with what the same authors get from the energy-dependent K-matrix; see below, Eq. (3.4). This is somewhat different from what (3.2) gives at $\bar{K}K$ threshold: evaluating $K(s)$ with the central values in (3.2) one finds

$$K_{11}\left(4m_K^2\right) = -0.17\ \text{GeV}^{-1}, \qquad K_{12}\left(4m_K^2\right) = -4.0\ \text{GeV}^{-1}, \qquad K_{22}\left(4m_K^2\right) = -2.7\ \text{GeV}^{-1}. \tag{3.4}$$

Before starting with the actual analyses it is perhaps convenient to remark that what follows from *experiment* is the *energy-independent* set of phase shifts and elasticity parameters. The energy-dependent representations are model dependent. This is particularly true of (3.1), where one makes the choice of a specific functional form; the results vary somewhat if using other parametrizations.

4. The phase ϕ_π

We will here consider the value of the phase $\phi_\pi(s)$ that follows from the experimental analysis given above. Although $\phi_\pi(s)$ is different from the quantities $\delta_S(s)$ and $\delta^{(+)}(s)$, which are the ones that intervene in the evaluation of the scalar form factor, they follow the same pattern. This was noted by ACCGL, who discuss ϕ_π in detail to illustrate their conclusions on δ_S, and, indeed, it can be verified without too much trouble with the formulas of Section 2: explicitly for $\delta^{(+)}$ and to first order in q_2 or in ϵ for δ_S (the exact result for the last requires solving two coupled integral equations).

The advantage of ϕ_π is that it is given by the simple equation (2.4) in terms of the observable quantities δ_π, η. This will allow us to simplify the discussion enormously; in particular, it will let us use simple parametrizations of δ_π, η above $\bar{K}K$ threshold, which is the region where there is disagreement between the evaluation of Ref. [2] and DGL, ACCGL. This simplification is unnecessary in the sense that the results are almost identical to what one finds with the full K-matrix (that we will present later); but it permits us to exhibit, with great clarity, both the mechanisms at work and the issues involved.

To calculate ϕ_π around and above $\bar{K}K$ threshold we take

$$\delta_\pi(s) = \pi + d(s) + \frac{q_2}{s^{1/2}} c(s), \qquad \eta(s) = 1 - \epsilon(s) \tag{4.1}$$

and approximate, for $0.95\ \text{GeV} \lesssim s^{1/2} \lesssim 1.35\ \text{GeV}$,

$$d(s) = d_0 = \text{const}, \qquad c(s) = c_0 = \text{const}, \qquad \epsilon(s) = \left(\epsilon_1 \frac{q_2}{s^{1/2}} + \epsilon_2 \frac{q_2^2}{s}\right) \frac{M^2 - s}{s}, \quad M = 1.5\ \text{GeV}. \tag{4.2}$$

In the region immediately below $\bar{K}K$ threshold we replace q_2 by $|q_2|$ in (4.1).

The energy-independent set of data in Ref. [4] are well fitted with the numbers[2]

$$c_0 = 5 \pm 1, \qquad \epsilon_1 = 6.4 \pm 0.5, \qquad \epsilon_2 = -16.8 \pm 1.6, \tag{4.3}$$

and we will leave the value of d_0 (which is small) free for the moment. It will turn out that a key quantity in the analysis is the phase shift at $\bar{K}K$ threshold, $\delta_\pi(4m_K^2)$, and we want to be able to vary this.

[2] We have actually followed the fit of Ref. [6], which takes into account other data sets and is slightly below, both for δ_π and ϵ, from what Hyams et al. give, at the upper energy range.

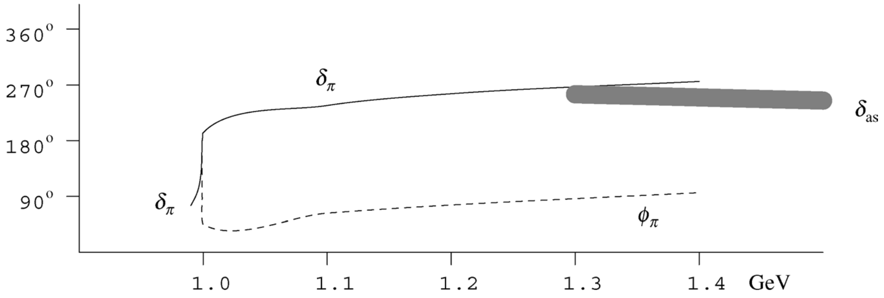

Fig. 1. The phases $\delta_\pi(s)$ (continuous line) and $\phi_\pi(s)$ (dashed line) corresponding to a Type I solution. Note that, below $\bar{K}K$ threshold $\delta_\pi(s) = \phi_\pi(s)$, hence $\phi_\pi(s)$ shows a very pronounced spike at $s = 4m_K^2$. The asymptotic phase (to be defined below) $\delta_{\mathrm{as.}}$ is represented by the thick gray line.

4.1. The phase ϕ_π of DGL, ACCGL

The authors of Refs. [1,3] take the K-matrix of Hyams et al. [4], with the central values as given in (3.2). What is important for us here is that this implies that the central value of $\delta_\pi(4m_K^2)$ is less than 180°:

$$\delta_\pi\left(4m_K^2\right) = 175^\circ. \tag{4.4a}$$

To reproduce this, we have to take d_0 in (4.2) negative and equal to

$$d_0 = -0.087. \tag{4.4b}$$

Care has to be exercised when crossing the energy s_0 at which $\delta_\pi(s)$ equals π, which, with (4.3) and (4.4b), occurs at $s_0^{1/2} = 992.6$ MeV,

$$\delta_\pi\left(s_0 = (992.6\ \mathrm{MeV})^2\right) = \pi,$$

and where (2.4) is singular. For the moment, we will tackle this by starting below s_0 and requiring continuity of *the phase* $\phi_\pi(s)$ across s_0. This we will call a solution of *Type I*, and is like what ACCGL find; indeed, the corresponding values of $\delta_\pi(s)$, $\phi_\pi(s)$, shown in Fig. 1, are practically identical to those in the Fig. 1 in ACCGL in the relevant region, around and above $\bar{K}K$ threshold. As can be seen in both figures, in the region $s^{1/2} \sim 1.35$ GeV, where inelasticity is negligible, $\delta_\pi(s)$ and $\phi_\pi(s)$ differ by π. $\delta_S(s)$ and $\delta^{(+)}(s)$ are very similar to $\phi_\pi(s)$ and thus also differ by π from $\delta_\pi(s)$.

This is the key remark of ACCGL: the phases $\delta_\pi(s)$ and $\phi_\pi(s)$, $\delta_S(s)$ are *not* equal above $s^{1/2} \sim 1.1$ GeV, but rather one has

$$\delta_S(s) \simeq \delta^{(+)}(s) \simeq \phi_\pi(s) \simeq \delta_\pi(s) - \pi, \quad s^{1/2} \gtrsim 1.1\ \mathrm{GeV}.$$

This accounts for the difference between the results of Refs. [1,3] (DGL, ACCGL) and my previous results [2] for the integral (1.7), hence for the different values of the scalar radius.

The situation, however, is not as simple as ACCGL seem to believe. First of all, the *inelasticity* given in Ref. [4] is much overestimated. After that paper was written, a number of experiments have appeared [5] in which the cross section $\pi\pi \to \bar{K}K$ was measured. Since there are no isospin-2 waves in $\pi\pi \to \bar{K}K$ scattering, and the $\pi\pi - \bar{K}K$ coupling is very weak for P, D waves, it follows that measurements of the differential cross section for $\pi\pi \to \bar{K}K$ give directly $1 - \eta^2$ with good accuracy. On the other hand, $\pi\pi$ scattering experiments like those of Hyams et al. [4] only measure the $\pi\pi \to \pi\pi$ cross section, so that η is obtained less precisely here: not only the $\pi\pi$ cross section depends on both δ_π, η, but other waves (notably S2, P and D0) interfere. Thus, these more recent, $\pi\pi \to \bar{K}K$ based, experimental values [5] for η are much more reliable than the older ones, in particular than those of Ref. [4].

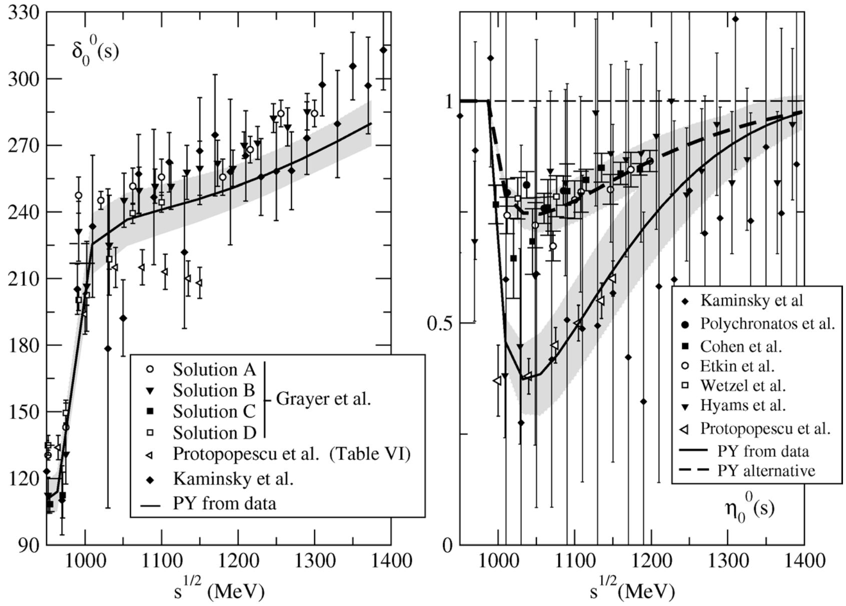

Fig. 2. Fit to the $I = 0$, S-wave inelasticity and phase shift between 950 and 1400 MeV, from Ref. [6] (so that the formula used for $\delta_0^{(0)} \equiv \delta_\pi$ is slightly different from (4.1)), and data from Refs. [4,5,8,11]. The shaded bands correspond to 1σ variation in the parameters of the fits. The fit to the phase shift corresponds to $d_0 = 0$. The difference between the determinations of η from $\pi\pi \to \pi\pi$ (*PY from data* in the figure) and from $\pi\pi \to \bar{K}K$ (*PY alternative*) is apparent here.

The value of the inelasticity the experiments in Ref. [5] give is about a *third* of what (4.3) indicates: η can be fitted with [6]

$$\epsilon_1 = 2.4 \pm 0.2, \qquad \epsilon_2 = -5.5 \pm 0.8. \tag{4.5}$$

The difference is shown graphically in Fig. 2.

If we now use (4.5) instead of (4.3) to calculate $\phi_\pi(s)$, keeping $\delta_\pi(s)$ fixed, a surprising result occurs: $\phi_\pi(s)$ does *not* become closer to $\delta_\pi(s)$ above the point s_0; on the contrary, it moves closer to $\delta_\pi - \pi$. In fact, one can decrease the inelasticity to zero, $\epsilon(s) \to 0$, *keeping* $\delta_\pi(s)$ *fixed*, and one finds that

$$\begin{aligned} &\phi_\pi(s) \to \delta_\pi(s), \qquad s^{1/2} < s_0^{1/2} = 992.6\ \text{GeV}, \\ &\phi_\pi(s) \to \delta_\pi(s) - \pi, \qquad s^{1/2} > s_0^{1/2} = 992.6\ \text{GeV}. \end{aligned} \tag{4.6}$$

That is to say: contrary to physical expectations, the limit of zero inelasticity does not coincide with inelasticity zero for, if we set $\epsilon(s) \equiv 0$, then $\delta_\pi(s)$ and $\phi_\pi(s)$ should be identical. This phenomenon was noticed by ACCGL who, however, failed to attach to it the due importance. As a matter of fact, the situation is even more complicated, as will be shown below: if we leave η fixed but vary d_0 in (4.2) across zero to a positive number, however small, the resulting ϕ_π is not continuous when d_0 crosses zero: it jumps by π.

What is the reason for this peculiar behaviour of ϕ_π? It is not difficult to identify: Eq. (2.4) does *not* determine ϕ_π, but only its tangent. Thus, ϕ_π is only fixed up to a multiple $N\pi$. N may be set to zero below the point s_0 where $\delta_\pi(s)$ crosses π, by requiring that $\phi_\pi(4m_K^2) = \delta_\pi(4m_K^2)$ and continuity above this. However, Eq. (2.4) shows that $\tan\phi_\pi(s)$ is *discontinuous* when s crosses s_0. Therefore, we may well add π to the $\phi_\pi(s)$ of the Type I solution found above, in the region $s > s_0$, since this does not change its tangent. We then find what we call a solution of

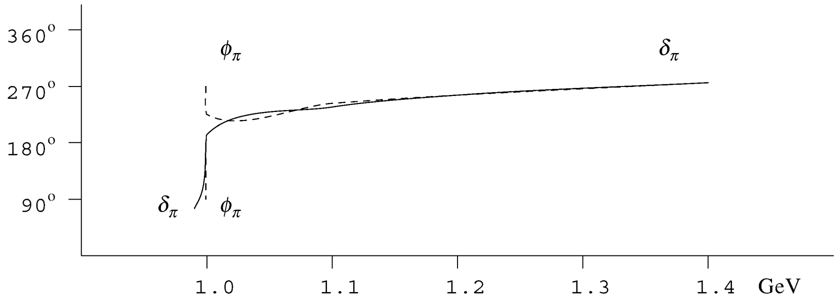

Fig. 3. The phases $\delta_\pi(s)$ (continuous line) and $\phi_\pi(s)$ (dashed line) corresponding to a solution of Type Id. The spike that appeared in Fig. 1 is now accompanied by a jump of $\phi_\pi(s)$, at $s = s_0$.

Type Id (*d* for discontinuous), depicted in Fig. 3. In this case, $\phi_\pi(s)$ is *not* continuous across s_0, but does tend[3] to $\delta_\pi(s)$, for all values of s, when the inelasticity tends to zero. It is also continuous (in the mean) for d_0 around zero. ACCGL appear to be unaware of the existence of solutions of Type Id.

It is not clear which of the two solutions, Type I or Type Id, should be considered correct: both types look awry. In fact, we will show that both Type I and Type Id are, with all probability, spureous solutions, artifacts due to the use of the parametrization (3.1), (3.2) over too wide a range, and with too little experimental information.

4.2. *The correct ϕ_π*

We next repeat the calculations of the previous section, but we will now assume that $\delta_\pi(4m_K^2)$ is *larger* than π, so that d_0 is *positive*. To get this it is sufficient to alter a little the parameters in (3.2). For example, if we move only one parameter by 1σ, just replacing in (3.2)

$$\alpha_1 \to 2.20 = 2.28 - 0.08, \tag{4.7}$$

then $\delta_\pi(4m_K^2)$ becomes 185°. Note that $\delta_\pi(s)$ is almost unchanged by this, as may be seen by comparing Figs. 1 and 4. The only important effect of the change in (4.7) is to push $\delta_\pi(4m_K^2)$ from a bit below to a bit above 180°; but then, this is a key point, as we will see.

A value for $\delta_\pi(4m_K^2)$ above 180° follows also for $s_1 = 0$, $\gamma_{11} = 3.0$ (as in Ref. [7]), which values are both less than 1σ off the central values in (3.2). In fact, a value $\delta_\pi(4m_K^2) > 180°$ can already be obtained with only a $\frac{1}{2}\sigma$ change,

$$\alpha_1 \to 2.24 = 2.28 - 0.04.$$

Thus, a value $\delta_\pi(4m_K^2) > 180°$ is perfectly compatible with the energy-dependent parametrization of Hyams et al., Eqs. (3.1), (3.2), when errors are taken into account.

We will use (4.7) for simplicity in the discussion and will thus repeat the calculations with

$$\delta_\pi\left(4m_K^2\right) = 185°, \qquad d_0 = +0.087. \tag{4.8}$$

In the present case, and as is obvious from (2.4), $\phi_\pi(s)$ is never singular and it stays *above* $\delta_\pi(s)$, up to the energy $s^{1/2} \sim 1.3$ GeV where $\delta_\pi(s)$ crosses $3\pi/2$, remaining close to it afterwards.[4]

This property is actually quite general, not tied to the specific approximations (4.2), (4.8), and depends only on the fact that $\delta_\pi(s)$ is an increasing function of s and that $\delta_\pi(4m_K^2) > \pi$. This is all we need for ϕ_π. To get the

[3] To be precise, one should remark that this limit applies *in the mean*; the isolated point s_0 remains singular. Convergence in the mean, however, is sufficient to ensure convergence of integrals involving ϕ_π.

[4] In fact, over the whole range, the difference between ϕ_π and δ_π is smaller than the experimental errors of the last: compare Figs. 2 and 4.

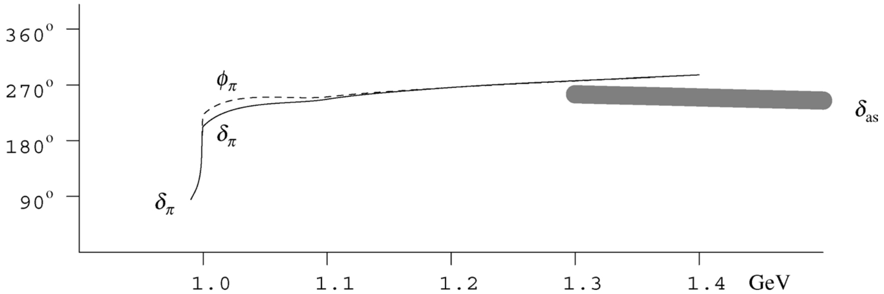

Fig. 4. The phases $\delta_\pi(s)$ (continuous line) and $\phi_\pi(s)$ (dashed line) corresponding to the solution of Type II. As in Fig. 1, the thick gray line is the asymptotic phase $\delta_{as.}$.

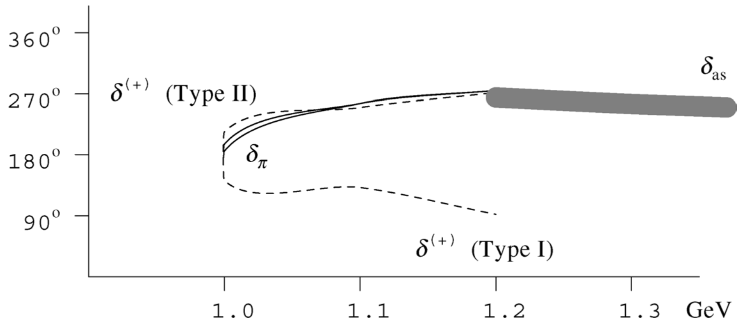

Fig. 5. The phases $\delta_\pi(s)$ (continuous lines) and $\delta^{(+)}(s)$ (dashed lines) evaluated in the K-matrix formalism, Eq. (3.1), with the central values of the parameters given in (3.2) (Type I) or (4.7), Type II. (The two lines for δ_π correspond also to (3.2), (4.7).) The asymptotic phase $\delta_{as.}$ (thick gray line) is also shown.

analogous property for $\delta^{(+)}$, δ_S we also require that $\det \mathbf{K}(4m_K^2) < 0$, something that is amply satisfied with the parameters of (3.2), (3.3) or, more generally, if, as implied by SU(3) ch.p.t., one has $\tan\delta_K < 0$ near $\bar{K}K$ threshold (see below).

A set of phases with these properties we will call a solution of *Type II*. In the specific case (4.8) we find the $\delta_\pi(s)$, $\phi_\pi(s)$ depicted in Fig. 4. Note that $\delta_\pi(s)$ and $\phi_\pi(s)$ are near each other all the time, as one expects physically since the inelasticity is small; this is particularly important in view of the results of Ref. [5]. Unlike what happened in solutions of Type I, or Type Id, the phase $\phi_\pi(s)$ is now a smooth function both of $\epsilon(s)$ and of s.

These results are not new. They were amply discussed more than thirty years ago, in connection with the eigenphases $\delta^{(\pm)}(s)$, by the present author in Ref. [7]. There it was noted that, by going from the values of the K-matrix parameters in (3.2) to values like those in (4.7), the eigenphase $\delta^{(+)}(s)$ changes from a fast decrease above the $\bar{K}K$ threshold, diverging from $\delta_\pi(s)$ by $\sim\pi$ (as does ϕ_π in a solution of Type I, see Fig. 1), to increasing above $\bar{K}K$ threshold with increasing s, staying close, but a bit above, $\delta_\pi(s)$ (again, as does ϕ_π in a solution of Type II, Fig. 4). The reader may compare our Figs. 1, 4 here with Fig. 2 in Ref. [7]. In Ref. [7] the M-matrix ($\mathbf{M} = \mathbf{K}^{-1}$) parametrization of experimental data of Protopopescu et al. [8] is also considered, and the same phenomenon is observed (Fig. 1 in Ref. [7]).

We give in Fig. 5 the eigenphases $\delta^{(+)}$ corresponding to Type I and Type II solutions. Here δ_π, as well as the eigenphase $\delta^{(+)}$, are evaluated with the K-matrix formalism, Eq. (3.1). For Type I we took the parameters (3.2); for Type II, those in (4.7). Our Fig. 5 here agrees with the corresponding parts of Figs. 1, 2 in Ref. [7].

5. The value of $\delta_\pi(4m_K^2)$

As is obvious from the previous discussion, a key quantity in this analysis is the $\pi\pi$ phase at $\bar{K}K$ threshold, $\delta_\pi(4m_K^2)$. If this is smaller than π, we have a situation of Type I; if, on the contrary, $\delta_\pi(4m_K^2) > \pi$, we have a solution of Type II and, in particular, we can approximate

$$\delta_S(s) \simeq \delta^{(+)}(s) \simeq \phi_\pi(s) \simeq \delta_\pi(s),$$

as was done in Ref. [2].

It should be clear that the parametrization (3.1), (3.2) is not a good guide to find the value of $\delta_\pi(4m_K^2)$. Not only $\delta_\pi(4m_K^2)$ crosses 180° when varying the parameters in (3.2) within their errors (as we have shown before) but, more to the point, (3.1) was devised to furnish an *approximate* representation of $\delta_\pi(s)$, $\eta(s)$ in the whole range $4M_\pi^2$ to 1.9^2 GeV^2. This may easily create local distortions; and, in fact, such distortions are expected. The inelasticity of Ref. [4] is overestimated, as proven by the more precise measurements of Ref. [5]: this will influence the phase δ_π above 1 GeV, hence, via the parametrization, around $\bar{K}K$ threshold. Such a distortion also occurs in the evaluation of Au et al. [9], who make a fit to η and δ_π, based on data of Ref. [4], over the whole energy range, which fit leads to a value of $\delta_\pi(4m_K^2)$ smaller than 180°: see Fig. 4 in Ref. [9]. We certainly need something more precise in the vicinity of the $\bar{K}K$ threshold, since $\delta_\pi(4m_K^2)$ is so near π.

For this we have several possibilities: the constant K-matrix fit around $\bar{K}K$ threshold of Hyams et al. [4]; the energy-independent analysis of this same reference; the results of other experiments; or certain theoretical arguments. As for the first, if we take the values K_{ab} in (3.3), obtained from a fit to data from 0.9 GeV to 1.1 GeV, we find

$$\delta_\pi\left(4m_K^2\right) = 205 \pm 8°, \tag{5.1}$$

3σ above 180°. A value above 180° is, of course, also found if interpolating the energy-independent analysis of Ref. [4]. The data of Protopopescu et al. [8] are not sufficiently precise to discriminate whether $\delta_\pi(4m_K^2)$ is below or above 180°: for some of the solutions in Ref. [8], $\delta_\pi(4m_K^2)$ is below, and for others above 180°, but in all cases, the errors cover the value 180°. However, a value clearly above 180° is found if extrapolating downward the experimental results of Ref. [10] (the phase shift is only measured for $s^{1/2} > 1$ GeV). This gives[5] $\delta_\pi(4m_K^2) = 203 \pm 7°$, including estimated systematic errors. A value $\delta_\pi(4m_K^2) > 180°$ is also found in all five solutions of Grayer et al. [11]: cf. Fig. 31 there. Finally, Kamiński et al. [11] find $\delta_\pi(4m_K^2) = 190 \pm 25°$. The experimental information thus clearly favours a value $\delta_\pi(4m_K^2) > 180°$, and hence a solution of Type II.

There are two other independent, theoretical arguments in favour of a solution of Type II. The first is based on chiral SU(3) calculations: unitarized SU(3) ch.p.t. produces central values of $\delta_\pi(4m_K^2)$ above 190° (with a value around 200° favoured; see, for example, Ref. [12]). Moreover, in Type II solutions, with the parameters in (4.7), one has a real part of the $\bar{K}K$ scattering length $a_r(\bar{K}K) \simeq -0.46M_\pi^{-1}$, in agreement with the unitarized current algebra (ch.p.t.) result that gives $a_r(\bar{K}K) \simeq -0.5M_\pi^{-1}$.

The second, more serious indication, is that the phase $\delta_S(s) \simeq \phi_\pi(s)$ for Type II solutions joins smoothly the result furnished by the perturbative QCD evaluation of $\delta_S(s)$, while a Type I solution $\delta_S(s)$ lies clearly below. We now turn to this.

[5] For the favoured solution in Ref. [10] which, incidentally, is the one with values of $\eta(s)$ more compatible with measurements based on $\pi\pi \to \bar{K}K$. For other solutions $\delta_\pi(4m_K^2)$ is even larger, except for one that yields a value near 180°.

6. The phase $\delta_S(t)$ at large t from QCD

Using the evaluations in Ref. [13] it is easy to get that, to leading order in the QCD coupling α_s, one has

$$F_{S,\pi}(t) = \frac{4\pi[m_u^2(\nu^2) + m_d^2(\nu^2)]C_F\alpha_s(\nu^2)}{-3t} I, \tag{6.1}$$

where, neglecting quark and pion masses,

$$I = \frac{1}{2}\left\{\int_0^1 \mathrm{d}\xi\, \frac{\Psi^*(\xi,\nu^2)}{1-\xi} \int_0^1 \mathrm{d}\eta\, \frac{\Psi(\eta,\nu^2)}{(1-\eta)^2} + \int_0^1 \mathrm{d}\xi\, \frac{\Psi^*(\xi,\nu^2)}{(1-\xi)^2} \int_0^1 \mathrm{d}\eta\, \frac{\Psi(\eta,\nu^2)}{1-\eta}\right\}. \tag{6.2}$$

Here ν^2 is the renormalization point and Ψ is the partonic wave function of the pion, defined by

$$(2\pi)^{3/2}\langle 0|\mathcal{S}{:}\bar{d}(0)\gamma_\lambda\gamma_5 D_{\mu_1}\cdots D_{\mu_n}u(0){:}\big|\pi(p)\big\rangle = \mathrm{i}^{n+1} p_\lambda p_{\mu_1}\cdots p_{\mu_n}\int_0^1 \mathrm{d}\xi\, \xi^n \Psi\big(\xi,\nu^2\big).$$

The D_μ are covariant derivatives, and $\mathcal{S}$ means symmetrization. The function Ψ is the same that appears in the evaluation of the vector form factor, and thus [13]

$$\Psi\big(\xi,\nu^2\big) \underset{\nu\to\infty}{\simeq} \xi(1-\xi)6\sqrt{2}f_\pi. \tag{6.3}$$

If we input (6.3) into (6.2) we get a divergent result. This divergence may be traced to the fact that we have neglected quark and pion masses, and may be cured by defining the form factor not for external momenta $p^2 = p'^2 = 0$, but with $p^2 = p'^2 = t_0$, t_0 being a fixed number; for example, we could take $t_0 = M_\pi^2$. Then we choose $\nu^2 = -t$ (for spacelike t) and find the asymptotic behaviour

$$F_{S,\pi}(t) \underset{t\to\infty}{\simeq} \frac{48\pi[m_u^2(-t) + m_d^2(-t)]C_F f_\pi^2\alpha_s(-t)\log(-t/t_0)}{-t} \to \frac{C[m_u^2(-t) + m_d^2(-t)]f_\pi^2}{-t} \tag{6.4}$$

with $C = 576\pi^2 C_F/(33-2n_f)$, and n_f is the number of quark flavours, that we take equal to three.

Unfortunately, the value of the constant C is changed when higher order corrections are included. These have the same structure as (6.4), with higher powers $[\alpha_s(-t)\log(-t/t_0)]^n$ which are not suppressed at large t. Therefore, the constant C gets contributions from all orders of perturbation theory with the result that the final value is unknown. However, it is very likely that the structure $[(\text{Constant}) \times \sum m_i^2(-t)/t]$ remains. This is sufficient to get a prediction for the asymptotic phase:

$$\delta_S(s) \underset{s\to\infty}{\simeq} \delta_{\text{as.}}(s) = \pi\left\{1 + \frac{2d_m}{\log(s/\Lambda^2)}\right\}, \quad d_m = \frac{12}{33-2n_f}. \tag{6.5}$$

Here Λ is the QCD parameter; in our calculations here we have allowed it to vary in the range $0.1\ \text{GeV}^2 \leqslant \Lambda^2 \leqslant 0.35\ \text{GeV}^2$. $\delta_{\text{as.}}(s)$ is the phase plotted in Figs. 1, 4, 5, where it is seen very clearly that it is consistent with Type II solutions, but not with the Type I solution of ACCGL.

7. Conclusions

There are other methods for finding directly $\bar{l}_4$, of which we only mention two. One can evaluate on the lattice the dependence of the quark condensate on the quark masses [14]; or one can fit $\bar{l}_4$ to the experimental $\pi\pi$ scattering lengths and effective range parameters obtained from experimental data [6], using ch.p.t. to one loop [15]. The results are summarized below, where we also repeat the results of Refs. [1–3]:

$$\bar{l}_4 = \begin{cases} 4.4 \pm 0.2 & \text{(Refs. [1,3])}, \\ 5.4 \pm 0.5 & \text{(Ref. [2])}, \\ 4.0 \pm 0.6 & \text{(lattice calculation, Ref. [14])}, \\ 7.2 \pm 0.7 & \text{(fitting } a_l^{(I)}, b_0^{(I)}, b_1, \text{ Ref. [15])}. \end{cases} \tag{7.1}$$

This is inconclusive; lattice calculations are known to suffer from large systematic errors, and the number following from the fit to experimental data is affected by higher order corrections, which the evaluation in Ref. [15] does not take into account. We have to fall back on our previous discussion, involving the phase of the scalar form factor.

In this case, and as we have shown, we have two types of solution: Type I, that occurs when $\delta_\pi(4m_K^2) < \pi$, and Type II, when $\delta_\pi(4m_K^2) > \pi$. The correctness of a solution of Type I, which is the one used in the evaluations of DGL, ACCGL is very unlikely: the experimental indications [4,10,11] favour values $\delta_\pi(4m_K^2) > \pi$. Moreover, in Type I solutions one has a discontinuous phase ϕ_π, when the inelasticity tends to zero. Type I solutions also exhibit a phase ϕ_π which is not continuous when $\delta_\pi(4m_K^2)$ moves around π. Finally, Type I solutions give a phase $\delta_S(s)$ rather different from what perturbative QCD suggests, Eq. (6.5), at large s. We think that Type I solutions are spureous, unphysical solutions, which appear only because one tries to fit, with too simple a formula, and without enough experimental information, the whole energy range from $\pi\pi$ threshold to 1.9 GeV, which distorts the results in the region of $\bar{K}K$ threshold. This last conjecture is confirmed by the evaluations of Moussallam [16]. This author uses, like DGL, ACCGL, fits that represent the quantities δ_π and η over the whole energy range; in particular, the fit of Au et al. [9]. Such parametrization gives $\delta_\pi(4m_K^2) \simeq 173°$, hence a Type I solution and thus, not surprisingly, Moussallam finds a value for $\langle r_S^2 \rangle$ similar to that of DGL.

Although this is not very important, because the very starting point of DGL, ACCGL (a Type I solution) is unlikely to be correct, one may question the methods of error analysis of these authors. As we discussed above, a value $\delta_\pi(4m_K^2) > 180°$ is obtained if replacing $\alpha_1 \to 2.28 - 0.04$, i.e., moving only $\frac{1}{2}\sigma$ off the central value in the fits of Hyams et al. [4, Eq. (3.2)]. Variation within errors of their parameters should have taken DGL, ACCGL to a Type II solution and, therefore, their error for $\langle r_S^2 \rangle$ should have comprised the value found with a Type II solution. With a complete error analysis DGL, ACCGL should have got[6] $\langle r_S^2 \rangle = 0.61^{+0.21}_{-0.04}\ \mathrm{fm}^2$.

For a Type II solution, on the other hand, the value of $\delta_\pi(4m_K^2) > \pi$ agrees with what experiment indicates; the phases $\phi_\pi(s)$, $\delta^{(+)}(s)$ and $\delta_S(s)$ are continuous both in s and when the inelasticity goes to zero; and the phase $\delta_S(s)$ agrees well with what perturbative QCD suggests at large s. We conclude that a situation of Type II is by far the more likely to be correct, thus confirming the validity of the approximations in Ref. [2]; in particular, the estimate

$$\langle r_S^2 \rangle = 0.75 \pm 0.07\ \mathrm{fm}^2. \tag{7.2}$$

A last question is whether one can improve on the evaluation in Ref. [2]. This is very unlikely, for the contribution of the region $4m_K^2 \leqslant s \leqslant 2$ GeV, Eq. (1.7). First of all, the incompatibility of the central values for η in analyses based on $\pi\pi \to \pi\pi$ scattering [4,10,11] with what one finds in $\pi\pi \to \bar{K}K$ experiments [5], implies that the phase δ_π obtained from $\pi\pi \to \pi\pi$ scattering must be biased. And, secondly, to find the eigenphases $\delta^{(\pm)}$ and mixing angle θ which are necessary to disentangle the form factors $F_{S,\pi}$, $F_{S,K}$ (cf. Eq. (2.6)), one requires, as discussed in detail in Ref. [7], experimental measurements of the *three* reactions $\pi\pi \to \pi\pi, \pi\pi \to \bar{K}K, \bar{K}K \to \bar{K}K$. Failing this, we are only left with approximate evaluations, like those in Ref. [2].

[6] Note that the converse is not true, in the sense that we do *not* have to enlarge the errors to encompass the DGL number: while it is true that the *parametrization* (3.1), (3.2) is compatible with both a solution of Type I and one of Type II, we have shown in Section 5 that the *experimental* data point clearly to $\delta_\pi(4m_K^2) > 180°$, hence a solution of Type II, that SU(3) ch.p.t. calculations also indicate a solution of Type II and, finally, in Section 6, we have argued that only a solution of Type II is compatible with the asymptotic behaviour indicated by perturbative QCD.

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