ELSEVIER Stochastic Processes and their Applications 59 (1995) 125–142

stochastic processes and their applications

A Chung type law of the iterated logarithm for subsequences of a Wiener process ★

Qi-Man Shao *

Department of Mathematics, Hangzhou University, Hangzhou, Zhejiang, P.R. China

Received 13 October 1994

Abstract

Let $\{W(t), t \geqslant 0\}$ be a standard Wiener process and $\{t_n, n \geqslant 1\}$ be an increasing sequence of positive numbers with $t_n \to \infty$. We consider the limit inf for the maximum of a subsequence $|W(t_i)|$. It is proved in this paper that the Chung law of the iterated logarithm holds, i.e., $\liminf_{n\to\infty}(t_n/\log\log t_n)^{-1/2} \max_{i\leqslant n} |W(t_i)| = \pi/\sqrt{8}$ *a.s.* if $t_n - t_{n-1} = o(t_n/\log\log t_n)$ and that the assumption $t_n - t_{n-1} = o(t_n/\log\log t_n)$ cannot be weakened to $t_n - t_{n-1} = O(t_n/\log\log t_n)$.

Keywords: Law of the iterated logarithm; Limit inferior; Subsequence; Wiener process

1. Introduction and main results

Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables with $EX_n = 0$ and $EX_n^2 < \infty$ for each $n \geqslant 1$. Put

$$S_0 = 0, \quad S_n = \sum_{i=1}^{n} X_i, \quad t_n = \sum_{i=1}^{n} EX_i^2, \ n = 1, 2, \ldots$$

Assume

$$t_n \to \infty \quad \text{as } n \to \infty. \tag{1.1}$$

$\{X_n, n \geqslant 1\}$ is said to satisfy the Chung law of the iterated logarithm (Chung LIL) if

$$\liminf_{n\to\infty}(t_n/\log\log t_n)^{-1/2} \max_{1\leqslant i\leqslant n} |S_i| = \pi/\sqrt{8} \quad \text{a.s.} \tag{1.2}$$

Here, and in the sequel, $\log x = \ln \max(x, e)$ and ln is the natural logarithm.

It is well-known (Jain and Pruitt, 1975) that the Chung LIL (1.2) holds for $\{X_n, n \geqslant 1\}$ i.i.d. random variables with mean zero and finite variance. The assumptions $EX_1 = 0$ and $EX_1^2 < \infty$ are also necessary (cf. Csáki, 1978). For independent, not necessarily

★ Supported in part by the Fok Yingtung Education Foundation of China and by Charles Phelps Taft Postdoctoral Fellowship of the University of Cincinnati

* Corresponding author; Present address: Department of Mathematics, National University of Singapore, Lower Kent Ridge Road, Singapore 0511, Singapore. Fax: 65 7795 452. E-mail: matsqm@math.nus.sg.

SSDI 0304-4149(95)00025-9

identically distributed random variables one can refer to Martikainen (1986) and Shao (1992). We restate the Chung LIL of Shao (1992) here for easy reference.

Theorem A. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables with $EX_n = 0$ and $EX_n^2 < \infty$ for every $n \geqslant 1$. Set $\sigma_{k,n} = \sum_{i=k+1}^{k+n} EX_i^2$, $k \geqslant 0$, $n \geqslant 1$. Assume that for any $\varepsilon > 0$*

$$\lim_{n\to\infty} \max_{k+j \leqslant n: \sigma_{k,j} \geqslant \varepsilon t_n / \log\log t_n} \sum_{i=k+1}^{k+j} EX_i^2 I\{|X_i| \geqslant \varepsilon \sigma_{k,j}^{1/2}\} / \sigma_{k,j} = 0. \tag{1.3}$$

Then (1.2) holds.

It is easy to see that if $\{X_n^2/EX_n^2, n \geqslant 1\}$ is uniformly integrable and

$$\max_{1 \leqslant i \leqslant n} EX_i^2 = \mathrm{o}(t_n / \log\log t_n) \quad \text{as} \quad n \to \infty, \tag{1.4}$$

then (1.3) is satisfied. Obviously, $\{X_n^2/EX_n^2, n \geqslant 1\}$ is uniformly integrable for independent normal random variables. Therefore, an immediate consequence of Theorem A is as follows.

Theorem 1.1. *Let $\{X_n, n \geqslant 1\}$ be independent normal random variables with mean zero. Assume (1.1) and*

$$t_n - t_{n-1} = \mathrm{o}(t_n / \log\log t_n) \quad \text{as} \quad n \to \infty \tag{1.5}$$

are satisfied. Then (1.2) holds.

Clearly, we can rewrite Theorem 1.1 in the following form.

Theorem 1.1*. *Let $\{W(t), t \geqslant 0\}$ be a standard Wiener process and $\{t_n, n \geqslant 1\}$ be an increasing sequence of positive numbers satisfying (1.1) and (1.5). Then we have*

$$\liminf_{n\to\infty} (t_n / \log\log t_n)^{-1/2} \max_{1 \leqslant i \leqslant n} |W(t_i)| = \pi/\sqrt{8} \quad \text{a.s.} \tag{1.6}$$

We remark that if

$$t_n = \mathrm{O}(t_{n-1}) \quad \text{as} \quad n \to \infty, \tag{1.7}$$

then we have

$$\limsup_{n\to\infty} (2t_n \log\log t_n)^{-1/2} \max_{1 \leqslant i \leqslant n} |W(t_i)| = 1 \quad \text{a.s.,} \tag{1.8}$$

$$\limsup_{n\to\infty} (2t_n \log\log t_n)^{-1/2} \sup_{0 \leqslant s \leqslant t_n} |W(s)| = 1 \quad \text{a.s.,} \tag{1.9}$$

and

$$\liminf_{n\to\infty} (t_n / \log\log t_n)^{-1/2} \sup_{0 \leqslant s \leqslant t_n} |W(s)| = \pi/\sqrt{8} \quad \text{a.s.} \tag{1.10}$$

(cf. Csörgő and Révész, 1981; Shao, 1989). Comparing (1.8) and (1.9), we see that, for the subsequence $\{t_n, n \geqslant 1\}$ satisfying $t_n = \mathrm{O}(t_{n-1})$, the almost sure limit superior for the maximum of that subsequence agrees with the almost sure limit superior for the maximum of the Wiener process, computed along the subsequence. It would be interesting to know if the same holds for limit inferior. In view of (1.10), this amounts to asking if (1.6) remains valid when (1.5) is replaced by (1.7).

The main aim of this paper is to show that (1.5) in Theorem 1.1* cannot be replaced by (1.7). We prove that o(·) in condition (1.5) of Theorem 1.1* cannot be weakened even to O(·).

In what follows, we always assume that $\{W(t), t \geqslant 0\}$ is a standard Wiener process.

Theorem 1.2. *Let* $c > 0$, $t_n = \exp(cn/\log n)$. *We have*

$$\liminf_{n\to\infty}(t_n/\log\log t_n)^{-1/2}\max_{1\leqslant i\leqslant n}|W(t_i)| \leqslant (6+(\pi c/2)^{1/2})\exp(-c^{1/2}) \quad \text{a.s.} \tag{1.11}$$

It is easy to check that

$$t_n - t_{n-1} \sim ct_n/\log\log t_n \quad \text{as } n\to\infty$$

under the hypothesis of Theorem 1.2, and that $(6+(\pi c/2)^{1/2})\exp(-c^{1/2}) < \pi/\sqrt{8}$ provided $c > 5$. Hence, (1.11) shows that the Chung LIL may fail if o(·) in (1.5) is replaced by O(·). Indeed, we obtain the following more general results.

Theorem 1.3. *Let* $\{c_n, n\geqslant 1\}$ *and* $\{(\log n)/c_n, n\geqslant 1\}$ *be non-decreasing sequences of positive numbers. Put*

$$t_n = \exp(n/c_n), \quad n = 1,2,\ldots, \tag{1.12}$$

$$d = \lim_{n\to\infty} c_n/\log n. \tag{1.13}$$

Assume

$$c_n \to \infty \quad as \quad n\to\infty. \tag{1.14}$$

Then we have

$$\liminf_{n\to\infty}(c_n/t_n)^{1/2}\exp((\log n/c_n)^{1/2})\max_{1\leqslant i\leqslant n}|W(t_i)| = \gamma \quad \text{a.s.,} \tag{1.15}$$

where $(\pi/2)^{1/2} \leqslant \gamma \leqslant 6d^{1/2} + (\pi/2)^{1/2}$.

Clearly, $\gamma = (\pi/2)^{1/2}$ if $d = 0$. That is, we have

Theorem 1.4. *Let* $\{c_n, n\geqslant 1\}$ *and* $\{(\log n)/c_n, n\geqslant 1\}$ *be non-decreasing sequences of positive numbers satisfying (1.14) and*

$$c_n/\log n \to 0 \quad as \quad n\to\infty. \tag{1.16}$$

Then

$$\liminf_{n\to\infty}(c_n/t_n)^{1/2}\exp((\log n/c_n)^{1/2})\max_{1\leqslant i\leqslant n}|W(t_i)| = (\pi/2)^{1/2} \quad \text{a.s.,} \tag{1.17}$$

where $t_n = \exp(n/c_n)$.

Let $0 < \theta < 1$, $c > 0$. Take $c_n = (\log n)^\theta / c$. It follows from Theorem 1.4 immediately that

Corollary 1.1. *We have*

$$\liminf_{n\to\infty} t_n^{-1/2} (\log n)^{\theta/2} \exp\left(c^{1/2} (\log n)^{(1-\theta)/2}\right) \max_{i\leqslant n} |W(t_i)| = (\pi c/2)^{1/2} \quad \text{a.s.,} \tag{1.18}$$

where $0 < \theta < 1$, $c > 0$, $t_n = \exp(cn/(\log n)^\theta)$.

Our next theorem concerns the case of $\theta = 0$.

Theorem 1.5. *We have, for every* $c > 0$,

$$\liminf_{n\to\infty} \exp\left(-c\,n/2 + (c\, \log n)^{1/2}\right) \max_{i\leqslant n} |W(\mathrm{e}^{c\,i})| = (\pi(1-\mathrm{e}^{-c})/2)^{1/2} \mathrm{e}^{c/4} \quad \text{a.s.} \tag{1.19}$$

In particular,

$$\liminf_{n\to\infty} 2^{-n/2} \cdot 2^{(\log_2 n)^{1/2}} \max_{i\leqslant n} |W(2^i)| = 2^{-3/4} \pi^{1/2} \quad \text{a.s.} \tag{1.20}$$

Remark 1.1. Another version of (1.19) is: For every $a > 1$, we have

$$\liminf_{n\to\infty} a^{-n/2} a^{(\log_a n)^{1/2}} \max_{i\leqslant n} |W(a^i)| = a^{1/4} (\pi(1-1/a)/2)^{1/2} \quad \text{a.s.} \tag{1.21}$$

Remark 1.2. It is easy to see that for independent normal random variables (1.3) is equivalent to (1.4). Theorem 1.2 in turn shows that (1.3) cannot be weakened for the Chung LIL in general.

Remark 1.3. Huggins (1990) claimed that (1.6) was true provided $t_n \to \infty$ and $t_n - t_{n-1} = o(t_n)$ as $n \to \infty$. Theorem 1.2 indicates that his conclusion is not correct.

We will give proofs of our theorems in Section 2. Section 3 is devoted to some further remarks and open problems. As a by-product of our Theorem 1.1*, we will establish

$$\lim_{n\to\infty} \frac{\sup_{0\leqslant s\leqslant t_n} |W(s)|}{\max_{i\leqslant n} |W(t_i)|} = 1 \quad \text{a.s.}$$

under the condition $t_n - t_{n-1} = o(t_n/(\log\log t_n)^2)$.

Throughout this paper we will use the following notations: $W(\cdot)$ denotes a standard Wiener process; $[x]$ denotes the integer part of x; $a \sim b$ means $\lim a/b = 1$.

2. Proofs

We start with some preliminary facts. Their proofs are elementary or well-known and so are omitted here (cf., for example, Csörgö and Révész, 1981)

(A) $1 - \mathrm{e}^{-x} \geqslant x/(1+x)$ for $x > -1$;

(B) $\sup_{-\infty<x<\infty} P(|W(b)+x| \leqslant a) = P(|W(b)| \leqslant a)$ for $a, b > 0$;

(C) $\inf_{|x|\leqslant a} P(|W(b)+x|\leqslant a) = P(|W(b)+a|\leqslant a)$ for $a, b > 0$;
(D) $P(|W(1)|\leqslant x)\leqslant 2x/\sqrt{2\pi}$ for $x > 0$;
(E) $P(\sup_{0\leqslant s\leqslant a}|W(s)|\leqslant x)\geqslant(1/2)\exp(-\pi^2 a/(8x^2))$ for $a, x > 0$;
(F) $\int_0^x e^{-t^2/2}\,dt\geqslant x\exp(-x^2/2)$ for $x > 0$.
To prove our theorems, we need the following lemma.

Lemma 2.1. *Let $\{s_n, n\geqslant 1\}$ be an increasing sequence of positive numbers. Then*

$$P\left(\max_{1\leqslant i\leqslant m}|W(s_i)|\leqslant x\right)\prod_{j=m+1}^{n}P\left(|W(s_j-s_{j-1})+x|\leqslant x\right)$$
$$\leqslant P\left(\max_{1\leqslant i\leqslant n}|W(s_i)|\leqslant x\right)$$
$$\leqslant P\left(\max_{1\leqslant i\leqslant m}|W(s_i)|\leqslant x\right)\prod_{j=m+1}^{n}P\left(|W(s_j-s_{j-1})|\leqslant x\right) \tag{2.1}$$

for every $1\leqslant m\leqslant n$ and $x > 0$.

Proof. Since $\{W(s), s\geqslant 0\}$ has independent increments, we obtain

$$P\left(\max_{1\leqslant i\leqslant n}|W(s_i)|\leqslant x\right)$$
$$= \int_{-x}^{x} P(|W(s_n)-W(s_{n-1})+t|\leqslant x)\,dP\left(\max_{1\leqslant i\leqslant n-1}|W(s_i)|\leqslant x, W(s_{n-1}) < t\right)$$
$$\leqslant \sup_{|t|\leqslant x} P(|W(s_n)-W(s_{n-1})+t|\leqslant x)P\left(\max_{1\leqslant i\leqslant n-1}|W(s_i)|\leqslant x\right)$$
$$= P(|W(s_n)-W(s_{n-1})|\leqslant x)P\left(\max_{1\leqslant i\leqslant n-1}|W(s_i)|\leqslant x\right)$$
$$= P(|W(s_n-s_{n-1})|\leqslant x)P\left(\max_{1\leqslant i\leqslant n-1}|W(s_i)|\leqslant x\right), \tag{2.2}$$

by (**B**). Similarly, we have

$$P\left(\max_{1\leqslant i\leqslant n}|W(s_i)|\leqslant x\right)$$
$$\geqslant \inf_{|t|\leqslant x} P(|W(s_n)-W(s_{n-1})+t|\leqslant x)P\left(\max_{1\leqslant i\leqslant n-1}|W(s_i)|\leqslant x\right)$$
$$= P(|W(s_n-s_{n-1})+x|\leqslant x)P\left(\max_{1\leqslant i\leqslant n-1}|W(s_i)|\leqslant x\right), \tag{2.3}$$

by (**C**). Now (2.1) follows from (2.2) and (2.3), by iteration. □

Proof of Theorem 1.3. According to the Kolmogorov Zero–One law, there must exist $0\leqslant\gamma\leqslant\infty$ such that (1.15) holds. So, we only need to show that for any $0 < \varepsilon < 0.01$

$$\liminf_{n\to\infty}(c_n/t_n)^{1/2}\exp((\log n/c_n)^{1/2})\max_{1\leqslant i\leqslant n}|W(t_i)|\geqslant(1-2\varepsilon)(\pi/2)^{1/2} \quad \text{a.s.} \tag{2.4}$$

and

$$\liminf_{n\to\infty}(c_n/t_n)^{1/2}\exp((\log n/c_n)^{1/2})\max_{1\leqslant i\leqslant n}|W(t_i)|\leqslant(1+2\varepsilon)(6d^{1/2}+(\pi/2)^{1/2})\text{ a.s.}\tag{2.5}$$

We prove (2.4) first. Put

$$y_n=c_n^{1/2}\exp((\log n/c_n)^{1/2}),\ \ m_n=n-[2(c_n\log n)^{1/2}],\ \ \gamma_1=(\pi/2)^{1/2}.\tag{2.6}$$

For $n>m\geqslant 1$, noting that

$$\begin{aligned}\frac{n}{c_n}-\frac{m}{c_m}&\geqslant\frac{n}{c_n}-\frac{m\log n}{c_n\log m}=\frac{n-m}{c_n}-\frac{m\log(n/m)}{c_n\log m}\\&\geqslant\left(1-\frac{1}{\log m}\right)\frac{n-m}{c_n},\end{aligned}\tag{2.7}$$

by the assumption that $c_n/\log n$ is non-increasing, we have

$$\begin{aligned}t_n-t_m&=t_n\left(1-\exp\left(\frac{m}{c_m}-\frac{n}{c_n}\right)\right)\\&\geqslant t_n\left(1-\exp\left(-\left(1-\frac{1}{\log m}\right)\frac{n-m}{c_n}\right)\right)\\&\geqslant t_n\left(1-\frac{1}{\log m}\right)\frac{n-m}{c_n+(1-\log^{-1}m)(n-m)}\\&\geqslant t_n\left(1-\frac{1}{\log m}\right)\frac{n-m}{n-m+c_n},\end{aligned}\tag{2.8}$$

by (**A**). To avoid cumbersome expressions, we always assume in what follows that n is sufficiently large. Using Lemma 2.1, we obtain

$$\begin{aligned}&P\left(\max_{1\leqslant i\leqslant n}|W(t_i)|\leqslant(1-2\varepsilon)\gamma_1\,(t_n/c_n)^{1/2}\exp(-(\log n/c_n)^{1/2})\right)\\&=P\left(\max_{1\leqslant i\leqslant n}|W(t_i)|\leqslant(1-2\varepsilon)\gamma_1\,t_n^{1/2}/y_n\right)\\&\leqslant\prod_{j=m_n+1}^{n}P\left(|W(t_j-t_{j-1})|\leqslant(1-2\varepsilon)\gamma_1\,t_n^{1/2}/y_n\right):=I_n.\end{aligned}\tag{2.9}$$

From (2.8), (1.14) and (**D**), we deduce that

$$\begin{aligned}I_n&=\prod_{j=m_n+1}^{n}P\left(|W(1)|\leqslant(1-2\varepsilon)\gamma_1 t_n^{1/2}/(y_n(t_j-t_{j-1})^{1/2})\right)\\&\leqslant\prod_{j=m_n+1}^{n}P\left(|W(1)|\leqslant(1-2\varepsilon)\gamma_1 t_n^{1/2}(1+c_j)^{1/2}/(y_n(t_j(1-\log^{-1}j))^{1/2})\right)\\&\leqslant\prod_{j=m_n+1}^{n}P\left(|W(1)|\leqslant(1-\varepsilon)\gamma_1\,t_n^{1/2}c_n^{1/2}/(y_n\,t_j^{1/2})\right)\\&\leqslant\prod_{j=m_n+1}^{n}P\left(|W(1)|\leqslant(1-\varepsilon)\gamma_1\exp\left(\frac{n-j}{2c_n}-\left(\frac{\log n}{c_n}\right)^{1/2}\right)\right)\end{aligned}$$

$$\leqslant \prod_{j=m_n+1}^{n} \frac{2(1-\varepsilon)\gamma_1}{\sqrt{2\pi}} \cdot \exp\left(\frac{n-j}{2c_n} - \left(\frac{\log n}{c_n}\right)^{1/2}\right)$$

$$= \exp\left((n-m_n)\left(\ln(1-\varepsilon) - \left(\frac{\log n}{c_n}\right)^{1/2}\right) + \sum_{j=m_n+1}^{n} \frac{n-j}{2c_n}\right). \tag{2.10}$$

An elementary argument shows that

$$(n-m_n)\left(\ln(1-\varepsilon) - \left(\frac{\log n}{c_n}\right)^{1/2}\right) + \sum_{j=m_n+1}^{n} \frac{n-j}{2c_n}$$

$$= (n-m_n)\left(\ln(1-\varepsilon) - \left(\frac{\log n}{c_n}\right)^{1/2}\right) + \frac{(n-m_n)(n-m_n-1)}{4c_n}$$

$$\leqslant (n-m_n)\left(\ln(1-\varepsilon) - \left(\frac{\log n}{c_n}\right)^{1/2}\right) + \frac{(n-m_n)^2}{4c_n}$$

$$\leqslant (2(c_n \log n)^{1/2} - 1)\left(\ln(1-\varepsilon) - \left(\frac{\log n}{c_n}\right)^{1/2}\right) + \frac{(2(c_n \log n)^{1/2} - 1)^2}{4c_n}$$

$$= -\log n + (2(c_n \log n)^{1/2} - 1)\ln(1-\varepsilon) + \frac{1}{4c_n}, \tag{2.11}$$

where the last inequality follows from the fact that $x(\ln(1-\varepsilon) - (\log n / c_n)^{1/2}) + x^2/(4c_n)$ is decreasing on $(-\infty, 2(c_n \log n)^{1/2}]$. Therefore,

$$I_n \leqslant \exp\left(-\log n + (c_1 \log n)^{1/2} \ln(1-\varepsilon)\right)$$

and

$$\sum_{n=1}^{\infty} I_n < \infty. \tag{2.12}$$

This proves (2.4), by (2.9), (2.12) and the Borel–Cantelli lemma.

In order to prove (2.5), we let

$$n_k = [4k \log k \cdot \log\log k \cdot \log\log\log k], \quad y_k = c_{n_k}^{1/2} \exp\left(\left(\frac{\log n_k}{c_{n_k}}\right)^{1/2}\right), \quad k = 1, 2, \ldots$$

We have

$$\liminf_{n\to\infty} (c_n/t_n)^{1/2} \exp((\log n / c_n)^{1/2}) \max_{1 \leqslant i \leqslant n} |W(t_i)|$$

$$\leqslant \liminf_{k\to\infty} (c_{n_k}/t_{n_k})^{1/2} \exp((\log n_k / c_{n_k})^{1/2}) \max_{1 \leqslant i \leqslant n_k} |W(t_i)|$$

$$\leqslant \liminf_{k\to\infty} (y_k / t_{n_k}^{1/2}) \max_{n_{k-1} < i \leqslant n_k} |W(t_i) - W(t_{n_{k-1}})|$$

$$+ \limsup_{k\to\infty} (c_{n_k}/t_{n_k})^{1/2} \exp\left((\log n_k / c_{n_k})^{1/2})\right) \max_{1 \leqslant i \leqslant n_{k-1}} |W(t_i)|$$

$$:= J_1 + J_2 \tag{2.13}$$

Noting that, by (2.7)

$$\begin{aligned}\frac{t_{n_{k-1}}}{t_{n_k}} &= \exp\left(\frac{n_{k-1}}{c_{n_{k-1}}} - \frac{n_k}{c_{n_k}}\right)\\ &\leqslant \exp\left(-\frac{n_k - n_{k-1}}{c_{n_k}}\left(1 - \frac{1}{\log n_{k-1}}\right)\right)\\ &\leqslant \exp\left(-\frac{n_k - n_{k-1}}{2c_{n_k}}\right)\\ &\leqslant \exp(-2\log k\cdot\log\log k\cdot\log\log\log k/c_{n_k})\end{aligned}$$

for every k sufficiently large, and applying the law of the iterated logarithm gives

$$\begin{aligned}J_2 &\leqslant 2\limsup_{k\to\infty}\left(\frac{c_{n_k}}{t_{n_k}}\right)^{1/2}\exp\left(\left(\frac{\log n_k}{c_{n_k}}\right)^{1/2}\right)(t_{n_{k-1}}\log\log t_{n_{k-1}})^{1/2}\\ &\leqslant 2\limsup_{k\to\infty} c_{n_k}^{1/2}\log n_k \exp\left(\left(\frac{\log n_k}{c_{n_k}}\right)^{1/2} - \frac{\log k\cdot\log\log k\cdot\log\log\log k}{c_{n_k}}\right)\\ &\leqslant 2\limsup_{k\to\infty}(\log k)^2\exp\left(\frac{\log n_k}{c_{n_k}} + 1 - \frac{\log k\cdot\log\log k\cdot\log\log\log k}{c_{n_k}}\right)\\ &\leqslant 6\limsup_{k\to\infty}(\log k)^2\exp\left(-\frac{\log k\cdot\log\log k\cdot\log\log\log k}{2c_{n_k}}\right)\\ &\leqslant 6\limsup_{k\to\infty}(\log k)^2\exp\left(-\frac{\log\log k\cdot\log\log\log k}{4c_1}\right)\\ &= 0,\end{aligned} \tag{2.14}$$

where, we have used the assumption that $c_n/\log n$ is non-increasing. Since $\{\max_{n_{k-1}<i\leqslant n_k}|W(t_i) - W(t_{n_{k-1}})|,\ k\geqslant 2\}$ are independent random variables, (2.5) follows from the Borel–Cantelli lemma if we can show

$$\sum_{k=2}^{\infty} P\left(\max_{n_{k-1}<i\leqslant n_k}|W(t_i) - W(t_{n_{k-1}})|\leqslant(1+2\varepsilon)\gamma_2 t_{n_k}^{1/2}/y_k\right) = \infty, \tag{2.15}$$

where $\gamma_2 := \gamma_2(d) = 6d^{1/2} + (\pi/2)^{1/2}$.

We divide the proof of (2.15) into two cases.

Case I. $d\geqslant 0.25$, where d is defined by (1.13). It is easy to see that

$$\begin{aligned}&P\left(\max_{n_{k-1}<i\leqslant n_k}|W(t_i) - W(t_{n_{k-1}})|\leqslant(1+2\varepsilon)\gamma_2 t_{n_k}^{1/2}/y_k\right)\\ &\geqslant P\left(\sup_{0\leqslant s\leqslant t_{n_k}}|W(s)|\leqslant(1+2\varepsilon)\gamma_2 t_{n_k}^{1/2}/y_k\right)\\ &\geqslant \frac{1}{2}\exp\left(-\frac{\pi^2 c_{n_k}}{8(1+2\varepsilon)^2\gamma_2^2}\exp\left(2\left(\frac{\log n_k}{c_{n_k}}\right)^{1/2}\right)\right)\\ &\geqslant \frac{1}{2}\exp\left(-\frac{\pi^2 d\exp(2/d^{1/2})\log n_k}{8((\pi/2)^{1/2}+6d^{1/2})^2}\right)\end{aligned} \tag{2.16}$$

for every k sufficiently large, by (**E**) and (1.13). Define

$$f(d)=\frac{\pi^2 d\exp(2/d^{1/2})}{8((\pi/2)^{1/2}+6d^{1/2})^2}=\frac{\pi^2\exp(-12)}{8}\left(\frac{\exp(6+(\pi/(2d))^{1/2})}{6+(\pi/(2d))^{1/2}}\right)^2.$$

Then, $f(d)$ is a decreasing function and hence

$$f(d)\leqslant f(0.25)\leqslant 0.94 \quad \text{for} \quad d\geqslant 0.25. \tag{2.17}$$

This proves (2.15), as desired.

Case II. $0\leqslant d\leqslant 0.25$. Set

$$m_k=n_k-[2(c_{n_k}\log n_k)^{1/2}-2c_{n_k}\ln\gamma_2]-2. \tag{2.18}$$

Clearly, we have

$$2(c_{n_k}\log n_k)^{1/2}-2c_{n_k}\ln\gamma_2\sim 2(1-d^{1/2}\ln\gamma_2)(c_{n_k}\log n_k)^{1/2}=\mathrm{o}(n_k-n_{k-1})$$

by (1.13) and the fact that

$$2(1-d^{1/2}\ln\gamma_2)>0 \quad \text{for each} \quad 0\leqslant d\leqslant 0.25.$$

Therefore, $n_k>m_k>n_{k-1}$ provided that k is sufficiently large. For the sake of convenience of expression, in what follows we will always let k be large enough. From Lemma 2.1 it follows that

$$\begin{aligned}
U_k &:= P\left(\max_{n_{k-1}<i\leqslant n_k}|W(t_i)-W(t_{n_{k-1}})|\leqslant(1+2\varepsilon)\gamma_2 t_{n_k}^{1/2}/y_k\right)\\
&\geqslant P\left(\max_{n_{k-1}<i\leqslant m_k}|W(t_i)-W(t_{n_{k-1}})|\leqslant(1+2\varepsilon)\gamma_2 t_{n_k}^{1/2}/y_k\right)\cdot\\
&\qquad \prod_{i=1+m_k}^{n_k} P\left(|W(t_i-t_{i-1})+(1+2\varepsilon)\gamma_2 t_{n_k}^{1/2}/y_k|\leqslant(1+2\varepsilon)\gamma_2 t_{n_k}^{1/2}/y_k\right)\\
&\geqslant P\left(\sup_{0\leqslant s\leqslant t_{m_k}}|W(s)|\leqslant(1+\varepsilon)\gamma_2 t_{n_k}^{1/2}/y_k\right)\cdot\\
&\qquad \prod_{i=1+m_k}^{n_k} P\left(|W(t_i-t_{i-1})+(1+2\varepsilon)\gamma_2 t_{n_k}^{1/2}/y_k|\leqslant(1+2\varepsilon)\gamma_2 t_{n_k}^{1/2}/y_k\right)\\
&\geqslant \frac{1}{2}\exp\left(-\frac{\pi^2 t_{m_k}y_k^2}{8(1+\varepsilon)\gamma_2^2 t_{n_k}}\right)\cdot\prod_{i=1+m_k}^{n_k}\frac{1}{\sqrt{2\pi}}\int_0^{2(1+2\varepsilon)\gamma_2 t_{n_k}^{1/2}/(y_k(t_i-t_{i-1})^{1/2})}\mathrm{e}^{-s^2/2}\,\mathrm{d}s,
\end{aligned} \tag{2.19}$$

using (**E**). Write

$$T_k=\prod_{i=1+m_k}^{n_k}\frac{1}{\sqrt{2\pi}}\int_0^{2(1+2\varepsilon)\gamma_2 t_{n_k}^{1/2}/(y_k(t_i-t_{i-1})^{1/2})}\mathrm{e}^{-s^2/2}\,\mathrm{d}s.$$

Since c_n and $\log n / c_n$ are non-increasing and $\lim_{k\to\infty} \log m_k / \log n_k = 1$, we have, for every $m_k < i \leqslant n_k$

$$t_i - t_{i-1} = t_i \left(1 - \exp\left(\frac{i-1}{c_{i-1}} - \frac{i}{c_i}\right)\right)$$
$$\leqslant \frac{t_i}{c_i} \leqslant \frac{t_i \log n_k}{c_{n_k} \log m_k} \sim \frac{t_i}{c_{n_k}}$$

and, by (2.7)

$$\frac{t_{n_k}^{1/2} c_{n_k}^{1/2}}{y_k t_i^{1/2}} = \exp\left(\frac{n_k}{2c_{n_k}} - \frac{i}{2c_i} - \left(\frac{\log n_k}{c_{n_k}}\right)^{1/2}\right)$$
$$\geqslant \exp\left(\frac{n_k - i}{2c_{n_k}}\left(1 - \frac{1}{\log m_k}\right) - \left(\frac{\log n_k}{c_{n_k}}\right)^{1/2}\right)$$
$$\geqslant \exp\left(\frac{n_k - i}{2c_{n_k}} - \left(\frac{\log n_k}{c_{n_k}}\right)^{1/2}\right) \cdot \exp\left(-\frac{n_k - m_k}{2\, c_{n_k} \log m_k}\right)$$
$$\geqslant \exp\left(\frac{n_k - i}{2c_{n_k}} - \left(\frac{\log n_k}{c_{n_k}}\right)^{1/2}\right) \cdot \exp\left(-\frac{(c_{n_k} \log n_k)^{1/2} + 2}{c_{n_k} \log m_k}\right)$$
$$\sim \exp\left(\frac{n_k - i}{2c_{n_k}} - \left(\frac{\log n_k}{c_{n_k}}\right)^{1/2}\right)$$

as $k \to \infty$ uniformly in $m_k < i \leqslant n_k$. Using the definitions of t_{n_k} and y_k and **(F)**, we obtain

$$T_k \geqslant \prod_{i=1+m_k}^{n_k} \frac{1}{\sqrt{2\pi}} \int_0^{2(1+\varepsilon)\gamma_2 \exp(\frac{n_k - i}{2c_{n_k}} - (\frac{\log n_k}{c_{n_k}})^{1/2})} \mathrm{e}^{-s^2/2}\, \mathrm{d}s$$
$$\geqslant \prod_{i=1+m_k}^{n_k} \left\{\frac{2(1+\varepsilon)\gamma_2}{\sqrt{2\pi}} \exp\left(\frac{n_k - i}{2\, c_{n_k}} - \left(\frac{\log n_k}{c_{n_k}}\right)^{1/2}\right) \cdot \right.$$
$$\left. \exp\left(-2(1+\varepsilon)^2 \gamma_2^2 \exp\left(\frac{n_k - i}{c_{n_k}} - 2\left(\frac{\log n_k}{c_{n_k}}\right)^{1/2}\right)\right)\right\}$$
$$\geqslant \exp\left(\sum_{i=1+m_k}^{n_k} \left\{\ln \frac{2(1+\varepsilon)\gamma_2}{\sqrt{2\pi}} + \frac{n_k - i}{2\, c_{n_k}} - \left(\frac{\log n_k}{c_{n_k}}\right)^{1/2}\right\}\right) \cdot$$
$$\exp\left(-2(1+\varepsilon)^2 \gamma_2^2 \sum_{i=1+m_k}^{n_k} \exp\left(\frac{n_k - i}{c_{n_k}} - 2\left(\frac{\log n_k}{c_{n_k}}\right)^{1/2}\right)\right)$$
$$:= \exp(A_k) \cdot \exp(B_k). \tag{2.20}$$

It is easy to see that

$$A_k = (n_k - m_k)\left(\ln \frac{2(1+\varepsilon)\gamma_2}{\sqrt{2\pi}} - \left(\frac{\log n_k}{c_{n_k}}\right)^{1/2}\right) + \sum_{i=1+m_k}^{n_k} \frac{n_k - i}{2\, c_{n_k}}$$
$$= (n_k - m_k)\left(\ln \frac{2(1+\varepsilon)\gamma_2}{\sqrt{2\pi}} - \left(\frac{\log n_k}{c_{n_k}}\right)^{1/2}\right) + \frac{(n_k - m_k)(n_k - m_k - 1)}{4\, c_{n_k}}$$

$$\geqslant (n_k - m_k)\left(\ln\frac{2(1+\varepsilon)\gamma_2}{\sqrt{2\pi}} - \left(\frac{\log n_k}{c_{n_k}}\right)^{1/2}\right) + \frac{(n_k - m_k - 1)^2}{4\,c_{n_k}}$$

$$\geqslant (2 + 2(c_{n_k}\log n_k)^{1/2} - 2c_{n_k}\ln\gamma_2)\left(\ln\frac{2(1+\varepsilon)\gamma_2}{\sqrt{2\pi}} - \left(\frac{\log n_k}{c_{n_k}}\right)^{1/2}\right)$$

$$+\frac{(2(c_{n_k}\log n_k)^{1/2} - 2c_{n_k}\ln\gamma_2)^2}{4\,c_{n_k}}$$

$$\geqslant -\log n_k + (2(c_{n_k}\log n_k)^{1/2} - 2c_{n_k}\ln\gamma_2)\ln\frac{2(1+\varepsilon)\gamma_2}{\sqrt{2\pi}}$$

$$+c_{n_k}(\ln\gamma_2)^2 - 2\left(\frac{\log n_k}{c_{n_k}}\right)^{1/2}$$

$$\geqslant -\log n_k + (2(c_{n_k}\log n_k)^{1/2} - 2c_{n_k}\ln\gamma_2)\ln\frac{2(1+\varepsilon/2)\gamma_2}{\sqrt{2\pi}} + c_{n_k}(\ln\gamma_2)^2$$

$$= -\log n_k + 2(c_{n_k}\log n_k)^{1/2}\ln\frac{2(1+\varepsilon/2)\gamma_2}{\sqrt{2\pi}} + c_{n_k}\ln\gamma_2\cdot\ln\frac{\pi}{2(1+\varepsilon/2)^2\gamma_2}. \quad (2.21)$$

by the fact that $\ln(2(1+\varepsilon)\gamma_2/\sqrt{2\pi}) \leqslant (\log n_k / c_{n_k})^{1/2}$ due to $d \leqslant 0.25$ and that $(\log n_k / c_{n_k})^{1/2} = o((c_{n_k}\log n_k)^{1/2})$.

We turn now to estimating B_k. Notice that

$$B_k \geqslant -2(1+\varepsilon)^2\gamma_2^2\int_{m_k}^{n_k}\exp\left(\frac{n_k - s}{c_{n_k}} - 2\left(\frac{\log n_k}{c_{n_k}}\right)^{1/2}\right)\mathrm{d}s$$

$$\geqslant -2(1+\varepsilon)^2\gamma_2^2\,c_{n_k}\,\exp\left(\frac{n_k - m_k}{c_{n_k}} - 2\left(\frac{\log n_k}{c_{n_k}}\right)^{1/2}\right)$$

$$\geqslant -2(1+\varepsilon)^2\gamma_2^2\,c_{n_k}\,\exp\left(\frac{2(c_{n_k}\log n_k)^{1/2} - 2c_{n_k}\ln\gamma_2 + 2}{c_{n_k}} - 2\left(\frac{\log n_k}{c_{n_k}}\right)^{1/2}\right)$$

$$\geqslant -2(1+\varepsilon)^2\gamma_2^2\,c_{n_k}\,\exp(-2\ln\gamma_2 + 2/c_{n_k})$$

$$\geqslant -2(1+\varepsilon)^3\,c_{n_k}, \quad (2.22)$$

where the last inequality is by (1.14). From (2.21) and (2.22) we conclude that

$$T_k \geqslant \exp\left(-\log n_k + 2(c_{n_k}\log n_k)^{1/2}\ln\frac{2(1+\varepsilon/2)\gamma_2}{\sqrt{2\pi}}\right.$$

$$\left.+c_{n_k}\ln\gamma_2\cdot\ln\frac{\pi}{2(1+\varepsilon/2)^2\gamma_2} - 2(1+\varepsilon)^3 c_{n_k}\right). \quad (2.23)$$

It follows from (2.7), (1.14) and (2.18) that

$$\frac{\pi^2 t_{m_k}\,y_k^2}{8\gamma_2^2 t_{n_k}} = \frac{\pi^2\,y_k^2}{8\gamma_2^2}\exp\left(\frac{m_k}{c_{m_k}} - \frac{n_k}{c_{n_k}}\right)$$

$$\leqslant \frac{\pi^2\,y_k^2}{8\gamma_2^2}\exp\left(\frac{m_k - n_k}{c_{n_k}}\left(1 - \frac{1}{\log m_k}\right)\right)$$

$$\sim \frac{\pi^2\,c_{n_k}}{8}.$$

Therefore, by (2.19) and (2.23)

$$U_k \geqslant \frac{1}{2}\exp\Bigg(-\log n_k + 2(c_{n_k}\log n_k)^{1/2}\ln\frac{2(1+\varepsilon/2)\gamma_2}{\sqrt{2\pi}} + c_{n_k}\ln\gamma_2\cdot\ln\frac{\pi}{2(1+\varepsilon/2)^2\gamma_2} - (1+\varepsilon)^3\left(2+\frac{\pi^2}{8}\right)c_{n_k}\Bigg) \tag{2.24}$$

Recall

$$\gamma_2 = \gamma_2(d) = 6\,d^{1/2} + (\pi/2)^{1/2} \leqslant \gamma_2(0.25) \quad \text{for} \quad 0 \leqslant d \leqslant 0.25 \quad \text{and} \quad 0 < \varepsilon \leqslant 0.01.$$

We have

$$\begin{aligned}
&2(c_{n_k}\log n_k)^{1/2}\ln\frac{2(1+\varepsilon/2)\gamma_2}{\sqrt{2\pi}} + c_{n_k}\ln\gamma_2\cdot\ln\frac{\pi}{2(1+\varepsilon/2)^2\gamma_2} - (1+\varepsilon)^3(2+\frac{\pi^2}{8})c_{n_k}\\
&\geqslant 2(c_{n_k}\log n_k)^{1/2}\ln\frac{2(1+\varepsilon/2)\gamma_2}{\sqrt{2\pi}}\\
&\quad + c_{n_k}\ln\gamma_2(0.25)\cdot\ln\frac{\pi}{2(1.01)^2\gamma_2(0.25)} - 1.01^3(2+\frac{\pi^2}{8})c_{n_k}\\
&\geqslant 2(c_{n_k}\log n_k)^{1/2}\ln\frac{2(1+\varepsilon/2)\gamma_2}{\sqrt{2\pi}} - 4.8\,c_{n_k}\\
&\sim (2\ln\frac{2(1+\varepsilon/2)\gamma_2}{\sqrt{2\pi}} - 4.8\,d^{1/2})(c_{n_k}\log n_k)^{1/2},\\
&= (2\ln((1+\varepsilon/2)(1+6(2/\pi)^{1/2}d^{1/2})) - 4.8\,d^{1/2})(c_{n_k}\log n_k)^{1/2},
\end{aligned} \tag{2.25}$$

by (1.13). Write

$$g(x) = \ln(1+6(2/\pi)^{1/2}x) - 4.8\,x, \quad x \geqslant 0.$$

Clearly, $g(x)$ is a concave function with $g(0)=0$ and $g(0.5) > 0.04$. Hence, $g(x) \geqslant 0$ for every $0 \leqslant x \leqslant 0.5$. Thus, we obtain

$$2\ln((1+\varepsilon/2)(1+6(2/\pi)^{1/2}d^{1/2})) - 4.8\,d^{1/2} > 0 \quad \text{for every } 0 \leqslant d \leqslant 0.25. \tag{2.26}$$

Therefore, by (2.24), (2.20), and (1.4),

$$U_k \geqslant \frac{1}{2}\exp(-\log n_k),$$

which yields (2.15) immediately, as desired.

The proof of Theorem 1.3 is now complete. □

Proof of Theorem 1.2. Take $c_n = (1/c)\log n$, $n = 1,2,\ldots$ Now (1.11) is a direct consequence of (1.15). □

Proof of Theorem 1.5. The proof is along the same lines of that of Theorem 1.3. Put $\gamma_3 = (\pi(1-\mathrm{e}^{-c})/2)^{1/2}$. It suffices to show that for any $0 < \varepsilon \leqslant 0.01$

$$\liminf_{n\to\infty}\exp\left(-\frac{cn}{2} + (c\,\log n)^{1/2}\right)\max_{i\leqslant n}|W(\mathrm{e}^{c\,i})| \geqslant (1-\varepsilon)\mathrm{e}^{c/4}\gamma_3 \quad \text{a.s.} \tag{2.27}$$

and

$$\liminf_{n\to\infty} \exp\left(-\frac{cn}{2}+(c\log n)^{1/2}\right)\max_{i\leqslant n}|W(e^{ci})|\leqslant(1+\varepsilon)e^{c/4}\gamma_3 \quad \text{a.s.} \tag{2.28}$$

Let $m_n = n-[2(\log n/c)^{1/2}]$. Applying Lemma 2.1 again, we have

$$\begin{aligned} &P\left(\max_{i\leqslant n}|W(e^{ci})|\leqslant(1-\varepsilon)e^{c/4}\gamma_3\exp\left(\frac{cn}{2}-(c\log n)^{1/2}\right)\right) \qquad (2.29)\\ &\leqslant \prod_{i=m_n+1}^{n} P\left(|W(e^{ci}-e^{c(i-1)})|\leqslant(1-\varepsilon)e^{c/4}\gamma_3\exp\left(\frac{cn}{2}-(c\log n)^{1/2}\right)\right)\\ &= \prod_{i=m_n+1}^{n} P\left(|W(1)|\leqslant(1-\varepsilon)(1-e^{-c})^{-1/2}\gamma_3 e^{c/4}\exp\left(\frac{c(n-i)}{2}-(c\log n)^{1/2}\right)\right)\\ &\leqslant \prod_{i=m_n+1}^{n}\left\{\frac{2(1-\varepsilon)(1-e^{-c})^{-1/2}\gamma_3 e^{c/4}}{\sqrt{2\pi}}\exp\left(\frac{c(n-i)}{2}-(c\log n)^{1/2}\right)\right\}\\ &= \exp\left((n-m_n)\left(\ln(1-\varepsilon)-(c\log n)^{1/2}+\frac{c}{4}\right)+\sum_{i=m_n+1}^{n}\frac{c(n-i)}{2}\right). \end{aligned}$$

Similarly to (2.11), we have

$$\begin{aligned} &\exp\left((n-m_n)\left(\ln(1-\varepsilon)-(c\log n)^{1/2}+\frac{c}{4}\right)+\sum_{i=m_n+1}^{n}\frac{c(n-i)}{2}\right)\\ &=\exp\left((n-m_n)\left(\ln(1-\varepsilon)-(c\log n)^{1/2}+\frac{c}{4}\right)+\frac{c}{4}(n-m_n)(n-m_n-1)\right)\\ &=\exp\left((n-m_n)(\ln(1-\varepsilon)-(c\log n)^{1/2})+\frac{c}{4}(n-m_n)^2\right)\\ &\leqslant\exp\left(\left(2\left(\frac{\log n}{c}\right)^{1/2}-1\right)(\ln(1-\varepsilon)-(c\log n)^{1/2})+\frac{c}{4}\left(2\left(\frac{\log n}{c}\right)^{1/2}-1\right)^2\right)\\ &=\exp\left(-\log n+\left(2\left(\frac{\log n}{c}\right)^{1/2}-1\right)\ln(1-\varepsilon)+\frac{c}{4}\right), \end{aligned}$$

where the last inequality is from the fact that $x(\ln(1-\varepsilon)-(c\log n)^{1/2})+cx^2/4$ is decreasing on $(-\infty, 2(\log n/c)^{1/2}]$. Therefore

$$\sum_{n=1}^{\infty} P\left(\max_{i\leqslant n}|W(e^{ci})|\leqslant(1-\varepsilon)e^{c/4}\gamma_3\exp\left(\frac{cn}{2}-(c\log n)^{1/2}\right)\right)<\infty,$$

which yields (2.27) immediately, by the Borel–Cantelli lemma.

We prove below (2.28). Set

$$n_k=[k\log k\cdot\log\log k],\quad m_k=n_k-\left[2\left(\frac{\log n_k}{c}\right)^{1/2}\right]-1,\quad k=1,2,\ldots$$

Using the law of the iterated logarithm, we obtain

$$\limsup_{k\to\infty}\exp\left(-\frac{cn_k}{2}+(c\log n_k)^{1/2}\right)\max_{i\leqslant n_{k-1}}|W(e^{ci})|=0 \quad \text{a.s.}$$

and hence

$$\liminf_{n\to\infty} \exp\left(-\frac{cn}{2}+(c\log n)^{1/2}\right)\max_{i\leqslant n}|W(e^{ci})|$$
$$\leqslant \liminf_{k\to\infty} \exp\left(-\frac{cn_k}{2}+(c\log n_k)^{1/2}\right)\max_{i\leqslant n_k}|W(e^{ci})|$$
$$\leqslant \liminf_{k\to\infty} \exp\left(-\frac{cn_k}{2}+(c\log n_k)^{1/2}\right)\max_{n_{k-1}<i\leqslant n_k}|W(e^{ci})-W(e^{cn_{k-1}})|. \quad (2.30)$$

Thus, to finish the proof of (2.28), by the Borel–Cantelli lemma, it is enough to show that

$$\sum_{k=2}^{\infty} P\left(\max_{n_{k-1}<i\leqslant n_k}|W(e^{ci})-W(e^{cn_{k-1}})|\leqslant(1+\varepsilon)e^{c/4}\gamma_3 z_k\right)=\infty, \quad (2.31)$$

where $z_k=\exp(cn_k/2-(c\log n_k)^{1/2})$.

Similarly to (2.19), we have

$$U_k := P\left(\max_{n_{k-1}<i\leqslant n_k}|W(e^{ci})-W(e^{cn_{k-1}})|\leqslant(1+\varepsilon)e^{c/4}\gamma_3 z_k\right)$$
$$\geqslant P\left(\sup_{0\leqslant s\leqslant e^{cm_k}}|W(s)|\leqslant\gamma_3 z_k\right)$$
$$\prod_{i=1+m_k}^{n_k} P\left(|W(e^{ci}-e^{c(i-1)})+(1+\varepsilon)e^{c/4}\gamma_3 z_k|\leqslant(1+\varepsilon)e^{c/4}\gamma_3 z_k\right)$$
$$\geqslant\frac{1}{2}\exp\left(-\frac{\pi^2\exp(2(c\log n_k)^{1/2})}{8\gamma_3^2\exp(c(n_k-m_k))}\right)$$
$$\times\prod_{i=1+m_k}^{n_k}\left\{\frac{2(1+\varepsilon)e^{c/4}\gamma_3}{(2\pi(1-e^{-c}))^{1/2}}\exp\left(\frac{c(n_k-i)}{2}-(c\log n_k)^{1/2}\right)\right.$$
$$\left.\times\exp\left(-\frac{2(1+\varepsilon)^2e^{c/2}\gamma_3^2}{1-e^{-c}}\exp(c(n_k-i)-2(c\log n_k)^{1/2})\right)\right\}$$
$$\geqslant\frac{1}{2}\exp\left(-\frac{\pi^2}{8\gamma_3^2}\right)$$
$$\times\exp\left((n_k-m_k)\left(\ln(1+\varepsilon)-(c\log n_k)^{1/2}+\frac{c}{4}\right)+\sum_{i=1+m_k}^{n_k}\frac{c(n_k-i)}{2}\right)\cdot$$
$$\times\exp\left(-\frac{2(1+\varepsilon)^2e^{c/2}\gamma_3^2}{1-e^{-c}}\sum_{i=1+m_k}^{n_k}\exp(c(n_k-i)-2(c\log n_k)^{1/2})\right). \quad (2.32)$$

Similarly to (2.21), we have

$$\exp\left((n_k-m_k)\left(\ln(1+\varepsilon)-(c\log n_k)^{1/2}+\frac{c}{4}\right)+\sum_{i=1+m_k}^{n_k}\frac{c(n_k-i)}{2}\right)$$
$$=\exp\left((n_k-m_k)\left(\ln(1+\varepsilon)-s(c\log n_k)^{1/2}+\frac{c}{4}\right)+\frac{c(n_k-m_k)(n_k-m_k-1)}{4}\right)$$
$$=\exp\left((n_k-m_k)(\ln(1+\varepsilon)-(c\log n_k)^{1/2})+\frac{c(n_k-m_k)^2}{4}\right)$$

$$\geqslant \exp\left(2\left(\frac{\log n_k}{c}\right)^{1/2}(\ln(1+\varepsilon)-(c\log n_k)^{1/2})+\log n_k\right)$$
$$=\exp\left(-\log n_k+2\left(\frac{\log n_k}{c}\right)^{1/2}\ln(1+\varepsilon)\right), \tag{2.33}$$

where the last inequality uses the fact that $x(\ln(1+\varepsilon)-(c\log n_k)^{1/2})+cx^2/4$ is increasing on $[2(\log n_k/c)^{1/2}, \infty)$.

Along the same lines of the estimation of B_k in (2.22) one can easily obtain

$$\exp\left(-\frac{2(1+\varepsilon)^2 e^{c/2}\gamma_3^2}{1-e^{-c}}\sum_{i=1+m_k}^{n_k}\exp\left(c(n_k-i)-2(c\log n_k)^{1/2}\right)\right)$$
$$\geqslant \exp\left(-\frac{2(1+\varepsilon)^2 e^{c/2}\gamma_3^2}{c(1-e^{-c})}\exp(c(n_k-m_k)-2(c\log n_k)^{1/2})\right)$$
$$\geqslant \exp\left(-\frac{2(1+\varepsilon)^2 e^{2c}\gamma_3^2}{c(1-e^{-c})}\right). \tag{2.34}$$

Now (2.31) follows from (2.32), (2.33) and (2.34), as desired.

This completes the proof of Theorem 1.5. □

3. Further remarks and open questions

As mentioned in the introduction (see (1.6), (1.8), (1.9) and (1.10)), $\sup_{0\leqslant s\leqslant t_n}|W(s)|$ and $\max_{i\leqslant n}|W(t_i)|$ have the same upper and lower bound if the gap in the sequence $\{t_n, n\geqslant 1\}$ does not grow too fast, i.e., $t_n - t_{n-1} = o(t_n/\log\log t_n)$. This leads to the idea that $\max_{i\leqslant n}|W(t_i)|$ and $\sup_{0\leqslant s\leqslant t_n}|W(s)|$ may share same sample behaviour as $n\to\infty$. The following theorem makes this notion precise.

Theorem 3.1. *Let $\{t_n,\ n\geqslant 1\}$ be an increasing sequence of positive numbers with $t_n\to\infty$. Assume*

$$t_n - t_{n-1} = o(t_n/(\log\log t_n)^2) \quad \textit{as} \quad n\to\infty. \tag{3.1}$$

Then, we have

$$\lim_{n\to\infty}\sup_{0\leqslant s\leqslant t_n}|W(s)|/\max_{1\leqslant i\leqslant n}|W(t_i)| = 1 \quad \text{a.s.} \tag{3.2}$$

Corollary 3.1. *We have*

$$\lim_{n\to\infty}\sup_{0\leqslant s\leqslant n^p}|W(s)|/\max_{1\leqslant i\leqslant n}|W(i^p)| = 1 \quad \text{a.s. } \textit{for every} \quad p>0. \tag{3.3}$$

The significance of Theorem 3.1 as well as Corollary 3.1 is that in order to estimate $\sup_{0\leqslant s\leqslant t_n}|W(s)|$, one only need to calculate $|W(\cdot)|$ at points $t_i, i = 1,2,\ldots,n$. For example, to estimate $\max_{i\leqslant 24025}|W(i)|$, it is enough to calculate 155 values of $|W(\cdot)|$

at $t_i = i^2, i = 1,2,\ldots,155$. This phenomenon exists in many other situations, as we state in the next theorem.

Theorem 3.2. *Let $\{X_n, n\geqslant 1\}$ be i.i.d. random variables with $EX_1 = 0$, $EX_1^2 = 1$ and $E|X_1|^{2+\delta} < \infty$ for some $\delta > 0$. Let $\{t_n,\ n\geqslant 1\}$ be an increasing sequence of positive numbers satisfying (3.1) and $t_n \to \infty$. Then, we have*

$$\lim_{n\to\infty} \sup_{0\leqslant s\leqslant t_n} |S(s)|/\max_{1\leqslant i\leqslant n} |S(t_i)| = 1 \quad \text{a.s.,} \tag{3.4}$$

Here, and in the sequel $S(s) = \sum_{1\leqslant i\leqslant s} X_i$.

Corresponding to Theorems 1.4 and 1.5 (cf. Remark 1.1), we have

Theorem 3.3. *Let $\{X_n, n\geqslant 1\}$ be i.i.d. random variables with $EX_1 = 0$, $EX_1^2 = 1$ and $E|X_1|^{2+\delta} < \infty$ for some $\delta > 0$. Let $\{c_n, n\geqslant 1\}$ and $\{(\log n)/c_n, n\geqslant 1\}$ be non-decreasing sequences of positive numbers with $c_n \to \infty$ and $c_n/\log n \to 0$ as $n\to\infty$. Then*

$$\liminf_{n\to\infty} (c_n/t_n)^{1/2} \exp((\log n/c_n)^{1/2}) \max_{i\leqslant n} |S(t_i)| = (\pi/2)^{1/2} \quad \text{a.s.,} \tag{3.5}$$

where $t_n = \exp(n/c_n)$.

Theorem 3.4. *Let $\{X_n, n\geqslant 1\}$ be i.i.d. random variables with $EX_1 = 0$, $EX_1^2 = 1$ and $E|X_1|^{2+\delta} < \infty$ for some $\delta > 0$. We have*

$$\liminf_{n\to\infty} a^{-n/2} \cdot a^{(\log_a n)^{1/2}} \max_{i\leqslant n} |S(a^i)| = a^{1/4} \left(\frac{\pi}{2}\left(1-\frac{1}{a}\right)\right)^{1/2} \quad \text{a.s.} \tag{3.6}$$

for every $a > 1$ and, in particular

$$\liminf_{n\to\infty} 2^{-n/2} \cdot 2^{(\log_2 n)^{1/2}} \max_{i\leqslant n} |S(2^i)| = 2^{-3/4}\pi^{1/2} \quad \text{a.s.} \tag{3.7}$$

Proof of Theorem 3.1. Put $t_0 = 0$, $d_n = \max_{1\leqslant n}(t_i - t_{i-1})$. Clearly, by (3.1), we have

$$d_n = o(t_n/(\log\log t_n)^2) \quad \text{as} \quad n\to\infty. \tag{3.8}$$

Noting that

$$\begin{aligned}\sup_{0\leqslant s\leqslant t_n} |W(s)| &\leqslant \max_{1\leqslant i\leqslant n} |W(t_i)| + \max_{1\leqslant i\leqslant n} \sup_{t_{i-1}<s\leqslant t_i} |W(t_i) - W(s)| \\ &\leqslant \max_{1\leqslant i\leqslant n} |W(t_i)| + \sup_{0\leqslant u\leqslant t_n} \sup_{0<v\leqslant d_n} |W(u+v) - W(u)|,\end{aligned}$$

we obtain

$$\begin{aligned}&\limsup_{n\to\infty} \sup_{0\leqslant s\leqslant t_n} |W(s)|/\max_{i\leqslant n} |W(t_i)| \\ &\quad\leqslant 1 + \limsup_{n\to\infty} \sup_{0\leqslant u\leqslant t_n} \sup_{0<v\leqslant d_n} |W(u+v) - W(u)|/\max_{i\leqslant n} |W(t_i)|\end{aligned}$$

$$\leqslant 1 + \limsup_{n\to\infty} \frac{t_n^{1/2}}{(\log\log t_n)^{1/2} \max_{i\leqslant n} |W(t_i)|} \cdot$$

$$\limsup_{n\to\infty} (\frac{\log\log t_n}{t_n})^{1/2} \sup_{0\leqslant u\leqslant t_n} \sup_{0\leqslant v\leqslant d_n} |W(u+v) - W(u)|$$

$$\leqslant 1 + \frac{\sqrt{8}}{\pi} \limsup_{n\to\infty} \left(\frac{(\log\log t_n) d_n (\log(t_n/d_n) + \log\log t_n)}{t_n} \right)^{1/2} \cdot$$

$$\limsup_{n\to\infty} \frac{\sup_{0\leqslant u\leqslant t_n} \sup_{0\leqslant v\leqslant d_n} |W(u+v) - W(u)|}{(d_n(\log(t_n/d_n) + \log\log t_n))^{1/2}}$$

$$\leqslant 1 + 2 \limsup_{n\to\infty} \left(\frac{(\log\log t_n) d_n (\log(t_n/d_n) + \log\log t_n)}{t_n} \right)^{1/2}$$

$$= 1, \tag{3.9}$$

where the third inequality is from Theorem 1.1*, the fourth is by Theorem 3.2 A of Hanson and Russo (1983), and the last equality follows from (3.8). On the other hand, it is trivial that $\sup_{0\leqslant s\leqslant t_n} |W(s)| \geqslant \max_{i\leqslant n} |W(t_i)|$. This proves (3.2). □

Proof of Theorem 3.2. By the well-known strong approximation theorem of Komlós et al. (1976), without changing the distribution of $\{S(t), t\geqslant 0\}$ we can redefine $\{S(t), t\geqslant 0\}$ on a richer probability space together with a standard Wiener process $\{W(t), t\geqslant 0\}$ such that

$$S(t) - W(t) = \mathrm{o}(t^{1/(2+\delta)}) \quad \text{a.s. as} \quad t\to\infty. \tag{3.10}$$

From (3.10) and (1.6), we derive that

$$\begin{aligned} \frac{\max_{1\leqslant i\leqslant t_n} |S(i)|}{\max_{1\leqslant i\leqslant n} |S(t_i)|} &= \frac{\max_{1\leqslant i\leqslant t_n} |W(i)| + \mathrm{o}(t_n^{1/(2+\delta)})}{\max_{1\leqslant i\leqslant n} |W(t_i)| + \mathrm{o}(t_n^{1/(2+\delta)})} \\ &= (1+\mathrm{o}(1)) \frac{\max_{1\leqslant i\leqslant t_n} |W(i)|}{\max_{1\leqslant i\leqslant n} |W(t_i)|}. \end{aligned} \tag{3.11}$$

Now (3.4) follows from (3.11) and (3.2) immediately. □

Proof of Theorems 3.3 and 3.4. The conclusion is a direct consequence of (3.10), (1.17) and (1.21). □

We conclude with the following remarks and open questions.

Remark 3.1. Let $\{S(t), t\geqslant 0\}$ be a stochastic process. If (3.10) holds for some $\delta > 0$ in the Strassen sense, then (3.4)–(3.7) remain true.

Question 1. We are not sure if the condition (3.1) is the best one for (3.2). Can (3.1) in Theorem 3.1 be replaced by (1.5)?

Question 2. What is the exact convergence rate to 1 in (3.3)?

Question 3. If we only assume that the second moment of X_1 is finite, do Theorems 3.2–3.4 remain true? If the answer is no, what are the normalizing constants in these theorems?

Question 4. Let $c > 0$, $t_n = \exp(c\,n/\log n)$. By Theorem 1.3, we have

$$\left(\frac{\pi c}{2}\right)^{1/2} \mathrm{e}^{-c^{1/2}} \leqslant \liminf_{n\to\infty} \left(\frac{\log\log t_n}{t_n}\right)^{1/2} \max_{i\leqslant n} |W(t_i)| \leqslant \left(6 + \left(\frac{\pi c}{2}\right)^{1/2}\right) \mathrm{e}^{-c^{1/2}}.$$

What is the precise constant for the above lim inf?

References

K.L. Chung, On the maximum partial sums of sequences of independent random variables, Trans. Amer. Math. Soc. 64 (1948) 205–233.

E. Csáki, On the lower limits of maximuma and minima of Wiener process and partial sums, Z. Wahr. verw. Gebiete 43 (1978) 205–221.

M. Csörgő and P. Révész, Strong Approximations in Probability and Statistics (Academic Press, New York, 1981).

D.L. Hanson and P. Russo, Some results on increments of the Wiener process with applications to log sums of iid r.v., Ann. Probab. 11 (1983) 609–623.

R.M. Huggins, The other law of the iterated logarithm for martingales, Bull. Austral. Math. Soc. 41 (1990) 307–311.

N.C. Jain and W.E. Pruitt, The other law of the iterated logarithm, Ann. Probab. 3 (1975) 1046–1049.

J. Koml'os, P. Major and G. Tusnády, An approximation of partial sums of independent R. V.'s and the sample D.F. II., Z. Wahr. verw. Gebiete 34 (1976) 33–58.

P. Major, The approximation of partial sums of independent rv's, Z. Wahr. verw. Gebiete 35 (1976) 213–220.

A.I. Martikainen, On Chung's law for nonidentically distributed independent summands, Theory Probab. Appl. 31 (1986) 498–504.

Q.M. Shao, Limit theorems for sums of dependent and independent random variables, Ph.D. Thesis, Uni. of Sci. and Tech. of China, P.R. China, 1989.

Q.M.Shao, How small are the increments of partial sums of non-iid random variables? Scientia Sinica (Series A) 35 (1992) 675–689.

V.A. Strassen, Almost sure behaviour of sums of independent random variables and martingales, Proc. Fifth Berkley Symp. Stat. Prob. 2 (1965) 315–343.