Inverse Problems for Hankel and Toeplitz Matrices

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ABSTRACT

The problem is considered how to construct a Toeplitz or Hankel matrix $A$ from one or two equations of the form $Au = g$. The general solution is described explicitly. Special cases are inverse spectral problems for Hankel and Toeplitz matrices and problems from the theory of orthogonal polynomials.

INTRODUCTION

When we speak of inverse problems we have in mind the following type of problems: Given vectors $u_j \in \mathbb{C}^n$, $g_j \in \mathbb{C}^m$ ($j = 1, \ldots, r$), we ask for an $m \times n$ matrix $A$ of a certain matrix class such that

$$Au_j = g_j \quad (j = 1, \ldots, r). \quad (0.1)$$

In the present paper we deal with inverse problems with the additional condition that $A$ is a Hankel matrix $[s_{1+j}]$ or a Toeplitz matrix $[c_{i-j}]$.

Inverse problems for Hankel and Toeplitz matrices occur, for example, in the theory of orthogonal polynomials when a measure $\mu$ on the real line or the unit circle is wanted such that given polynomials are orthogonal with respect to this measure. The moment matrix of $\mu$ is just the solution of a certain inverse problem and is Hankel (in the real line case) or Toeplitz (in the unit circle case); here the $g_j$ are unit vectors.
Inverse problems for Toeplitz matrices were considered for the first time in the paper [10] of M. G. Krein. Before formulating the corresponding result let us adopt a convenient notation. If \( x = (x_j)^n \in \mathbb{C}^{n+1} \), then \( x(\lambda) \) will denote the polynomial

\[
x(\lambda) = x_0 + x_1\lambda + \cdots + x_n\lambda^n,
\]

where \( \lambda \) is regarded as a complex variable.

**Theorem 0.1** [10]. Suppose that a vector \( u = (u_j)^n \in \mathbb{C}^{n+1} \) is given and its first coordinate satisfies \( u_0 = u_1 \neq 0 \). Then there exists a regular hermitian Toeplitz matrix \( T \) such that \( Tu \) equals the first unit vector if and only if the polynomial \( u(\lambda) \) has \( n_c \) roots on the unit circle and \( n_r \) roots symmetric with respect to the unit circle.

A generalization of this result to the nonhermitian case is presented in the paper [7] of I. Gohberg and A. Semencul.

**Theorem 0.2** [7]. Suppose that two vectors \( u, v \in \mathbb{C}^{n+1} \) are given and \( u_0 \neq 0 \). Then there exists a regular Toeplitz matrix \( T \) such that \( Tu \) equals the first and \( Tv \) equals the last unit vector if and only if

\[
u_0 = v_n
\]

and the polynomials \( u(\lambda) \) and \( v(\lambda) \) are coprime. If the latter conditions are fulfilled, then \( T \) is uniquely determined.

Another inverse problem is considered in a paper of I. Gohberg and N. Krupnik.

**Theorem 0.3** [6]. Suppose that two vectors \( u, v \in \mathbb{C}^{n+1} \) are given and \( u_n \neq 0 \). Then there exists a regular Toeplitz matrix \( T \) such that \( Tu \) equals the first and \(Tv \) equals the second unit vector if and only if

\[
v_n = u_{n-1}
\]

and the polynomials \( u(\lambda) \) and \( v(\lambda) \) are coprime. If the latter conditions are fulfilled, then \( T \) is uniquely determined.

Generalizations to the block case and other special inverse problems are considered in [2], [3], [4], [5], and [9]. In [9] it is shown (Section 1.5.3) that
Theorems 0.2 and 0.3 are consequences of the following theorem concerning an inverse problem for homogeneous equations.

**Theorem 0.4** [9]. Let \( u \in \mathbb{C}^{n+1}, \ v \in \mathbb{C}^{m+1} \) be given vectors such that the polynomials \( u(\lambda) \) and \( v(\lambda) \) are coprime and the last component of one of the vectors \( u \) or \( v \) is nonzero. Then there exist numbers \( s_j, j = 0, \ldots, n + m - 2 \), such that the matrices

\[
H_{m-2,n} = \begin{bmatrix} s_{i+j} \end{bmatrix}_{0}^{m-2n}, \quad H_{n-2,m} = \begin{bmatrix} s_{i+j} \end{bmatrix}_{0}^{n-2m}
\]

satisfy

\[
H_{m-2,n}u = 0 \quad \text{and} \quad H_{n-2,m}v = 0.
\]

The matrices are uniquely determined up to a multiplicative constant.

In the present paper we consider the inverse problem for the Toeplitz and Hankel matrices in its very general setting for \( r = 1 \) and \( r = 2 \) in (0.1). Since an \( n \times n \) Toeplitz or Hankel matrix has \( 2n - 1 \) degrees of freedom, the restriction to the case \( r < 2 \) is reasonable.

Section 1 is dedicated to the Hankel problem for \( r = 1 \). In order to represent the general solution we consider the corresponding Hankel matrices as generated by rational functions. That means we look for rational functions \( f(\lambda) \) such that \( f(\lambda) = s_0 \lambda^{-1} + s_1 \lambda^{-2} + \cdots \) in a neighborhood of infinity and \([s_{i+j}]\) is a solution of the inverse problem. In Section 2 a more general version of the inverse problem for \( r = 2 \) is considered. The solvability conditions are formulated in terms of divisability of certain polynomials, and the general solution is described explicitly using the solution of a polynomial congruence. Special cases of our main theorem are Theorems 0.1–0.4 and results concerning inverse spectral problems for Hankel and Toeplitz matrices.

All results on Hankel matrices can be immediately transferred to Toeplitz matrices. However, proceeding in such a way, special features of the matrix like hermitian symmetry cannot be taken into account. For this reason we also present a typical Toeplitz approach for the inverse problem in Sections 3–5; in Section 4 special attention is given to the hermitian Toeplitz case.

Let us remark that our approaches based on polynomial congruences allow us to construct fast algorithms for solving the inverse problems. For \( n = m \) the complexity will be \( O(n^2) \) flops if the euclidean algorithm is applied in its usual recursive form. Utilizing divide and conquer strategies, the complexity can be reduced to \( O(n \log^2 n) \) (see [1]).
1. INVERSE PROBLEMS FOR ONE HANKEL EQUATION

Our first aim is to give a parametrization of the general solution of the following inverse problem.

**Problem H1.** Given vectors \( u \in \mathbb{C}^{n+1}, \ g \in \mathbb{C}^{m+1} \), find all \((m+1) \times (n+1)\) Hankel matrices \( H \) satisfying

\[
Hu = g. \tag{1.1}
\]

Without loss of generality we may assume that the last component of \( u \) is different from zero.

The solutions of Problem H1 will be represented with the help of rational functions (in system theory language, that means in the frequency domain). For this let us introduce some notation.

Suppose that \( f(\lambda) \) is a rational function and

\[
f(\lambda) = \sum_k s_k \lambda^{-k-1} \tag{1.2}
\]

is its Laurent expansion at infinity. The numbers \( s_k \) are called Markov parameters of \( f(\lambda) \). We define operators \( Q_n \) by

\[
Q_n f(\lambda) := \begin{bmatrix} s_0 & \cdots & s_n \end{bmatrix}^T \in \mathbb{C}^{n+1}
\]

and introduce the Hankel matrices

\[
H_{mn}(f) := [s_{i+j}]_{0,0}^{m,n}, \quad H_n(f) := H_{nn}(f). \tag{1.3}
\]

The matrices \( H_{mn}(f) \) will be referred to as Hankel matrices generated by the function \( f(\lambda) \). Obviously, any Hankel matrix can be regarded as generated by certain rational function. Therefore, we shall look for solutions to Problem H1 of the form \( H = H_{mn}(f) \).

For fixed \( u = (u_i)_{0}^{n} \in \mathbb{C}^{n+1} \) we introduce the difference operator matrices

\[
D_m(u) = \begin{bmatrix} u_0 & u_1 & \cdots & u_n & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & u_n \\
\end{bmatrix}_{m+1}.
\]
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The equation (1.1) with $H = [s_{i+j}]$ can be written in the form

$$D_m(u) s = g, \quad s = (s_i)_0^{m+n}.$$  \hspace{1cm} (1.4)

Representing the matrix $H$ as $H = H_m(f)$, (1.1) is equivalent to

$$D_m(u) Q_{m+n} f(\lambda) = g.$$  \hspace{1cm} (1.5)

Next we need the following relation.

**Lemma 1.1.** For rational $f(\lambda)$ and $u \in \mathbb{C}^{n+1}$,

$$D_m(u) Q_{m+n} f(\lambda) = Q_m u(\lambda) f(\lambda).$$  \hspace{1cm} (1.6)

The proof is a straightforward computation.

Now it remains to solve the equation

$$Q_m u(\lambda) f(\lambda) = g.$$  \hspace{1cm} (1.7)

The general solution of this equation is given by

$$f(\lambda) = \frac{g(\lambda^{-1}) \lambda^{-1} + p(\lambda) + \lambda^{-m-2} r(\lambda)}{u(\lambda)},$$  \hspace{1cm} (1.8)

where $p(\lambda)$ is a polynomial and $r(\lambda)$ is a function rational and analytic at infinity.

Now we are able to formulate and prove our first theorem.

**Theorem 1.1.** The general solution of Problem $H_1$ is given by

$$H = H_m \left( \frac{g(\lambda^{-1}) \lambda^{-1} + p(\lambda)}{u(\lambda)} \right),$$  \hspace{1cm} (1.9)

where $p(\lambda)$ is an arbitrary polynomial with degree less than $n$.

**Proof.** As shown above, (1.1) with $H = H_m(f)$ is equivalent to (1.7).
Hence any function (1.8) provides a solution \( H_{mn}(f) \) of Problem H1. Since 
\[ Q_{m+n} \lambda^{-m-2} r(\lambda) / u(\lambda) = 0, \]
we may assume that \( r(\lambda) = 0 \). For similar reasons \( p(\lambda) \) can be assumed to be of degree \( n - 1 \) at most.

We consider now some special cases.

(1) Case \( g = 0 \)

**Corollary 1.1.** The general form of an \((m + 1) \times (n + 1)\) Hankel matrix
for which \( Hu = 0 \) is given by

\[
H = H_{mn} \left( \frac{p(\lambda)}{u(\lambda)} \right),
\]

where \( \deg p(\lambda) < n \).

(2) Inverse Spectral Problem

**Corollary 1.2.** Suppose that a vector \( u \in \mathbb{C}^{n+1} \) and a number \( \alpha \in \mathbb{C} \)
are given. Then the general form of a Hankel matrix \( H \) for which \( u \) is an
eigenvector corresponding to the eigenvalue \( \alpha \) is given by

\[
H = H_n \left( \frac{\alpha u(\lambda^{-1}) \lambda^{-1} + p(\lambda)}{u(\lambda)} \right),
\]

where \( \deg p(\lambda) < n \).

(3) Classical Inverse Problem

**Corollary 1.3.** Suppose that a vector \( u \in \mathbb{C}^{n+1} \) is given. The general
form of a \((n + 1) \times (n + 1)\) Hankel matrix \( H \) for which \( Hu \) equals the first
[last] unit vector is given by

\[
H = H_n \left( \frac{1 + \lambda p(\lambda)}{\lambda u(\lambda)} \right)
\]

\[
H = H_n \left( \frac{\lambda^{-n-1} + p(\lambda)}{u(\lambda)} \right),
\]

where \( \deg p(\lambda) < n \).
In the first case of Corollary 1.3 one can easily describe all regular solutions of the inverse problem. This follows from the following fact.

**Lemma 1.2** [8]. Suppose that \( H = H_n(b(\lambda)/a(\lambda)) \) and \( \deg a(\lambda) = n \). Then \( H_n \) is regular if and only if \( b(\lambda) \) and \( a(\lambda) \) are coprime.

Hence the following is true.

**Corollary 1.4.** The general form of a regular Hankel matrix the first column of whose inverse equals \( u \) is given by (1.12), where \( 1 + \lambda p(\lambda) \) and \( u(\lambda) \) are coprime.

In the general case of Theorem 1.1 the description of all regular solutions is not so easy. But one can decide whether a given solution is regular or not after applying some steps of the euclidean algorithm to the pair \( \lambda^{n+1}u(\lambda), g(\lambda^{-1})\lambda^n + \lambda^{n+1}p(\lambda) \). In that way a proper rational function \( b(\lambda)/a(\lambda) \) with \( \deg a(\lambda) < n + 1 \) is obtained, generating also the Hankel matrix \( H \). Now \( H \) is regular if and only if \( \deg a(\lambda) = n + 1 \) and \( a(\lambda) \) and \( b(\lambda) \) are coprime (cf. [8]).

In many applications one is interested only in hermitian solutions of the inverse problem. A Hankel matrix is hermitian if it is square and real. If \( u \) and \( g \) in Problem H1 are real vectors, then the real solutions of H1 are also given by the formula (1.9), where \( p(\lambda) \) is assumed to be real. Let now \( u \) and \( g \) be complex vectors with real and imaginary parts \( u_{re}, g_{re}, u_{im}, g_{im} \), respectively. Then \( H \) is an hermitian solution of Problem H1 iff

\[
Hu_{re} = g_{re}, \quad Hu_{im} = g_{im}.
\]

That means we have an inverse problem with \( r = 2 \). This problem will be considered in the next section.

Sometimes one is interested only in positive definite or semidefinite solutions of the inverse problem. In the first case of Corollary 1.3 the signature of the solutions \( H \) equals the Cauchy index of the function \( [1 + \lambda p(\lambda)]/\lambda u(\lambda) \) with respect to the real line (see [9]). Therefore, a positive definite solution exists iff \( \lambda u(\lambda) \) has only real simple roots. In the general situation the analogous condition is only sufficient. An explicit description of all positive definite solutions is not so easy. For a fixed solution one can decide whether it is positive definite or not after applying Euclid’s algorithm as described above.
2. INVERSE PROBLEMS FOR TWO HANKEL EQUATIONS

In this section we solve the following inverse problem.

**Problem H2.** Given vectors \( u \in \mathbb{C}^{n+1}, v \in \mathbb{C}^{m+1}, g \in \mathbb{C}^{l+1}, h \in \mathbb{C}^{k+1}, \) find \( s = (s_j)_0^N \in \mathbb{C}^{N+1}, \) where \( N = \max\{k + m, l + n\}, \) such that the Hankel matrices

\[
H_{ln} := [s_{i+j}]_{00}^{l} \quad \text{and} \quad H_{km} := [s_{i+j}]_{00}^{m}
\]

satisfy

\[
H_{ln} u = g \quad \text{and} \quad H_{km} v = h. \tag{2.1}
\]

Without loss of generality we may assume that the last components of the vectors \( u \) and \( v \) are different from zero. For definiteness we assume that \( N = n + 1 \geq k + m. \)

We introduce the projection \( P \) defined for rational functions \( f(\lambda) \) with Laurent expansion \( f(\lambda) = \sum_{i = -\infty}^{\infty} f_i \lambda^i \) at infinity by

\[
Pf(\lambda) = \sum_{i = 0}^{\infty} f_i \lambda^i.
\]

Furthermore, we define

\[
\omega(\lambda) := [h(\lambda^{-1}) u(\lambda) - g(\lambda^{-1}) v(\lambda)] \lambda^{-1}.
\]

We distinguish the cases \( n < k + 1 \) and \( n > k + 1. \)

**Theorem 2.1.** Suppose that \( n < k + 1. \) Then Problem H2 is solvable if and only if:

(i) \( Q_{k-n} \omega(\lambda) = 0, \) where \( Q_j \) is defined in Section 1 for \( j > 0 \) and \( Q_{-1} := 0. \)

(ii) The g.c.d. of \( u(\lambda) \) and \( v(\lambda) \) is a divisor of \( P\omega(\lambda). \)

The solution is unique iff \( u(\lambda) \) and \( v(\lambda) \) are coprime. If (i) and (ii) are
fulfilled, then the general solution is given by

$$s = Q_N \frac{g(\lambda^{-1})\lambda^{-1} + p(\lambda)}{u(\lambda)},$$

(2.2)

where $p(\lambda)$ runs over all solutions (deg $p < n$) of the congruence

$$v(\lambda) p(\lambda) \equiv Pw(\lambda) \mod u(\lambda).$$

(2.3)

Remark 2.1. If $n = k$ and $m = 1$, then (i) goes over into the relation

$$h^T u = g^T v.$$

(2.4)

Proof of Theorem 2.1. Let $s = (s_j)_{j=0}^N$ be a solution of Problem H2. Then, by Theorem 1.1, $s$ is of the form (2.2) for some polynomial $p(\lambda)$ with degree less than $n$. We introduce the subsequence $s' := (s_0, \ldots, s_{k+n})$ of $s$. Then, again by Theorem 1.1,

$$s' = Q_{k+m} \frac{h(\lambda^{-1})\lambda^{-1} + g(\lambda)}{v(\lambda)},$$

for some polynomial $q(\lambda)$ with deg $q(\lambda) < m$. Hence

$$Q_{k+m} \left( \frac{g(\lambda^{-1})\lambda^{-1} + p(\lambda)}{u(\lambda)} - \frac{h(\lambda^{-1})\lambda^{-1} + g(\lambda)}{v(\lambda)} \right) = 0.$$  

(2.5)

This implies

$$\frac{g(\lambda^{-1})\lambda^{-1} + p(\lambda)}{u(\lambda)} + \frac{h(\lambda^{-1})\lambda^{-1} + g(\lambda)}{v(\lambda)} = r(\lambda)\lambda^{-k-m-2}$$

for some rational and analytic at infinity function $r(\lambda)$. Thus

$$p(\lambda)v(\lambda) - q(\lambda)u(\lambda) = \omega(\lambda) + \lambda^{n-k-2}t(\lambda)$$

(2.6)

for a function $t(\lambda)$ that is rational and analytic at infinity. Applying the
operators $Q_{k-m}$ and $P$ we get (i) and (2.3). The last includes the condition (ii).

Let now (i) and (ii) be fulfilled. Then (2.3) is solvable. If $p(\lambda)$ is a solution of (2.3), then (2.6) holds for some $t(\lambda)$, which implies (2.5). Hence (2.2) is a solution of Problem H2.

**Theorem 2.2.** Suppose that $n > k + 1$ and let $d(\lambda)$ denote the g.c.d. of $u(\lambda)$ and $v(\lambda)$. Then Problem H2 is solvable iff the remainder of $Pw(\lambda)$ modulo $d(\lambda)$ has degree $n - k - 2$ at most. If the latter condition is fulfilled, the general solution of Problem H2 is given by (2.2), where $p(\lambda)$ runs over all polynomials with degree less than $n$ for which the remainder of $p(\lambda)v(\lambda) - Pw(\lambda)$ modulo $u(\lambda)$ has degree $n - k - 2$ at most.

**Proof.** Let $s$ be a solution of Problem H2. Then, by the arguments of the proof of Theorem 2.1, (2.5) holds. Hence

$$p(\lambda)v(\lambda) - q(\lambda)u(\lambda) = Pw(\lambda) + c(\lambda),$$

where $c(\lambda)$ is a polynomial with degree less than or equal to $n - k - 2$. Dividing by $d(\lambda)$, we obtain

$$Pw(\lambda) \equiv -c(\lambda) \mod d(\lambda). \quad (2.7)$$

Furthermore,

$$p(\lambda)v(\lambda) - Pw(\lambda) \equiv c(\lambda) \mod u(\lambda). \quad (2.8)$$

Conversely, let (2.7) be fulfilled, and let $p(\lambda)$ be a solution of (2.8). Then (2.6) holds. Hence (2.5) is true. This implies that $s$ is a solution of Problem H2.

**Special cases:**

(1) **Inverse Problem for Schmidt Pairs**

Let $\sigma > 0$ be a singular value of a square matrix $A$. A pair of vectors $(x, y)$ is said to be a Schmidt pair of $A$ corresponding to $\sigma$ if and only if

$$Ax = \sigma y \quad \text{and} \quad A^*y = \sigma x. \quad (2.9)$$
Our aim is to construct Hankel matrices $H$ with given Schmidt pair and singular value. Due to the symmetry of Hankel matrices, the second equation of (2.9) is equivalent to $H\bar{y} = \sigma \bar{x}$. Hence the inverse problem for Schmidt pairs is a special case of Problem H2, where $n = l = m = k$, $v = \bar{y}$, $u = x$, $g = \sigma y$, and $h = \sigma \bar{x}$. The condition (2.4) goes over into

$$
\|y\| = \|x\|,
$$

where $\|\cdot\|$ denotes the euclidean norm. Furthermore, we have

$$
Pw(\lambda) = P\left[\bar{x}(\lambda^{-1})x(\lambda) - \sigma(\lambda^{-1})\bar{y}(\lambda)\right]\lambda^{-1}.
$$

\textbf{Corollary 2.1.} Suppose that vectors $x, y \in \mathbb{C}^{n+1}$ and a positive number $\sigma$ are given, and let $d(\lambda)$ denote the g.c.d. of $x(\lambda)$ and $\bar{y}(\lambda)$. Assume that the last coefficient of $x$ is nonzero. Then there exists a Hankel matrix $H$ such that $\{x, y\}$ is a Schmidt pair of $H$ corresponding to $\sigma$ if and only if the condition (2.10) is fulfilled and $d(\lambda)$ is a divisor of $Pw(\lambda)$ defined by (2.11). The solution is unique iff $x(\lambda)$ and $\bar{y}(\lambda)$ are coprime. The general solution is given by

$$
H = H_n \left( \frac{p(\lambda) + \sigma y(\lambda^{-1})\lambda^{-1}}{x(\lambda)} \right),
$$

where $p(\lambda)$ runs over all solutions of

$$
p(\lambda)\bar{y}(\lambda) \equiv Pw(\lambda) \mod x(\lambda).
$$

\textbf{(2) Inverse spectral problems}

We ask now for Hankel matrices $H$ such that two given vectors $u, v$ are eigenvectors corresponding to given eigenvalues $\alpha$ and $\beta$, $\alpha \neq \beta$. This is a special case of Problem H2, where $g = \alpha u$ and $h = \beta v$, and

$$
Pw(\lambda) = P\left[\beta v(\lambda^{-1})u(\lambda) - \alpha u(\lambda^{-1})v(\lambda)\right]\lambda^{-1}.
$$

The condition (2.4) is equivalent to $u^Tv = 0$. Hence from Theorem 2.1 we get the following.

\textbf{Corollary 2.2.} Suppose that vectors $u, v \in \mathbb{C}^{n+1}$ and numbers $\alpha, \beta \in \mathbb{C}$, $\alpha \neq \beta$, are given. Then there exists a Hankel matrix with the eigenvectors $u$
and \( v \) corresponding to eigenvalues \( \alpha \) and \( \beta \), respectively, iff \( u^T v = 0 \) and the g.c.d. of \( u(\lambda) \) and \( v(\lambda) \) is a divisor of \( Pw(\lambda) \) defined by (2.12). The general solution is given by

\[
H = H_n \left( \frac{p(\lambda) + u(\lambda^{-1})\lambda^{-1}}{u(\lambda)} \right),
\]

where \( p(\lambda) \) runs over all solutions of

\[
p(\lambda)v(\lambda) = Pw(\lambda) \mod u(\lambda).
\]

(3) **Inverse Spectral Problem for Toeplitz Matrices**

The Toeplitz analogue of the problem considered in the preceding subsection is obviously equivalent to Problem H2 for \( g = \alpha J_n u \) and \( h = \beta J_n v \), where \( J_n \) denotes the flip operator in \( \mathbb{C}^{n+1} \).

\[
J_n(x_0,\ldots,x_n) := (x_n,\ldots,x_0)
\]

In this case

\[
Pw(\lambda) = (\beta - \alpha) Pu(\lambda)v(\lambda)\lambda^{-n-1}.
\]

We consider the inverse Toeplitz spectral problem once more in Section 5.

(4) **Classical Inverse Problems**

We show now that Theorem 0.2 and 0.3 are consequences of Theorem 2.1. Moreover, Theorem 1.2 includes a representation of the general solution.

In the case of Theorem 0.2 we have \( g = e_n, \ h = e_0 \). Furthermore,

\[
Pw(\lambda) = P[\lambda^{-1}u(\lambda) - \lambda^{-n-1}v(\lambda)] = [u(\lambda) - u_0]\lambda^{-1},
\]

and the condition (2.4) means \( v_n = u_0 \). If now \( u(0) = u_0 \neq 0 \), then the congruence (2.3) is equivalent to

\[
\lambda p(\lambda)v(\lambda) = -u_0 \mod u(\lambda).
\]

This congruence is solvable iff \( v(\lambda) \) and \( u(\lambda) \) are coprime. In particular, the
necessity part of Theorem 0.2 and the representation for the general solution

\[ H = H_n \left( \frac{\lambda^{n-1} + p(\lambda)}{u(\lambda)} \right) \]

follow. Here \( p(\lambda) \) runs over all solutions of (2.14). An alternative representation is

\[ H = H_n \left( \frac{1 + q(\lambda)}{\lambda v(\lambda)} \right), \]

where \( q(\lambda) \) runs over all solutions of the congruence

\[(\lambda q(\lambda) + 1) u(\lambda) \equiv u_0 \mod v(\lambda).\]

From the latter representation the regularity of \( H \) follows by Lemma 2.2.

Next we consider the case of Theorem 0.3: \( g = e_n \), \( h = e_n \). In this case we have

\[ Pw(\lambda) = P(\lambda^{-n}u - \lambda^{-n-1}v) = u_n, \]

and (2.4) goes over into \( u_n = v_{n-1} \). Now Theorem 0.3 can immediately be deduced from Theorem 2.1. Moreover we have representations for the general solution.

(5) Inverse Problem for Homogeneous Equations

In case \( g = 0 \) and \( h = 0 \), Problem H2 has always the trivial solution \( H = 0 \). It follows from Theorems 2.1 and 2.2 that the problem has a nontrivial solution if and only if \( n > k + 1 \) or \( u(\lambda) \) and \( v(\lambda) \) have a nontrivial common divisor. We show now that Theorem 0.4 follows from Theorem 2.2. In the situation of Theorem 0.4 we have \( l = m - 2 \) and \( k = n - 2 \).

According to Theorem 2.2 the general form of a vector \( s = (s_j)_{0}^{m+n} \) satisfying (0.1) is given by \( s = Q_{m+n} p(\lambda)/u(\lambda) \), where \( p(\lambda) \) runs over all solutions of the congruence \( p(\lambda)v(\lambda) \equiv \text{const} \mod u(\lambda) \). The claim of Theorem 0.4 is now an immediate consequence.
3. INVERSE PROBLEM FOR ONE TOEPLITZ EQUATION

We discuss now inverse problems for Toeplitz matrices. At first glance one observes that the Toeplitz problems can readily transformed into Hankel problems by changing the order of the columns or rows. However, proceeding in such a way, special properties of the Toeplitz matrix like hermitian symmetry will not be reflected in the solution. We present now another approach better fitted to the Toeplitz structure. The problem under consideration in this section is the following one.

**Problem T1.** Given vectors \( u, g \in \mathbb{C}^{n+1} \), find Toeplitz matrices \( T \) satisfying

\[
Tu = g. \tag{3.1}
\]

For simplicity we shall assume always in the sequel that the first and the last components of \( u \) are nonzero.

We denote \( \hat{x} := J_n x \), where \( J_n \) is defined by (2.13). Now (3.1) is equivalent to

\[
D_n(\hat{u}) c = g \quad \text{and} \quad D_n(u) \hat{c} = \hat{g}, \tag{3.2}
\]

where \( T = [c_{i-j}] \), \( c = (c_{-n}, \ldots, c_n) \).

We introduce some operators acting on the space of rational functions. Let \( f(\lambda) \) be rational, and

\[
f(\lambda) = \sum_{j = -\infty}^{\infty} f_j^+ \lambda^j, \tag{3.3}
\]

\[
f(\lambda) = \sum_{j = -\infty}^{\infty} f_j^- \lambda^j \tag{3.4}
\]

its Laurent series expansions at 0 and at infinity, respectively. We define

\[
P_n^+ f := (f_0^+, \ldots, f_n^+) \quad \text{and} \quad P_n^- f := (f_0^-, \ldots, f_n^-)
\]

and

\[
R_n f = (f_{-n}^+, \ldots, f_0^+, \ldots, f_n^+, f_{-n}^- \ldots, f_0^- \ldots, f_n^-).
\]
REMARK 3.1. Obviously, any vector \( c \in \mathbb{C}^{2n+1} \) can be represented in the form \( c = R_n f \) for a function \( f(\lambda) \) that is rational and analytic at 0 and infinity. This follows, for example, from the fact that the system \( R_n (\lambda - \alpha)^{-1} \) \( (\alpha \in \mathbb{C}, \alpha \neq 0) \) is complete.

We shall say that the Toeplitz matrix \( T = [c_{i-j}] \) is generated by \( f(\lambda) \) if \( c = (c_{-n}, \ldots, c_n) = R_n f \). We denote

\[
T := T_n(f).
\]

Let us point out that this notation, though convenient for us, is different from the common notation in the theory of Toeplitz operators, where \( T(f) \) denotes the Toeplitz operator with symbol \( f \).

Now we look for solutions of Problem T1 of the form \( T = T_n(f) \). We shall need the following relation.

**Lemma 3.1.** For a rational function \( f(\lambda) \),

\[
D_n(\hat{u}) R_n f(\lambda) = (P_n^+ - P_n^-) u(\lambda) f(\lambda).
\]  

(3.5)

The proof is an elementary computation.

**Theorem 3.1.** The general solution of Problem T1 is given by \( T = T_n(f) \), where

\[
f(\lambda) = \frac{p(\lambda)}{u(\lambda)} + \frac{g(\lambda)}{2u(\lambda)} \frac{1 - \lambda^{n+1}}{1 + \lambda^{n+1}}
\]  

(3.6)

and \( p(\lambda) \) is an arbitrary polynomial with degree less than \( n \).

**Proof.** As remarked above, any solution of Problem T1 is of the form \( T = T_n(f) \) for a function \( f(\lambda) \) that is rational and analytic at 0 and infinity. Furthermore, \( T_n(f) \) is a solution of this problem if and only if

\[
D_n(\hat{u}) R_n f = g,
\]  

(3.7)

which is, in view of Lemma 3.1, equivalent to

\[
(P_n^+ - P_n^-) u(\lambda) f(\lambda) = g.
\]  

(3.8)
Put

\[ f_0(\lambda) = \frac{g(\lambda)}{2u(\lambda)} \frac{1 - \lambda^{n+1}}{1 + \lambda^{n+1}}. \]

Then, for \(|\lambda| < 1\),

\[ u(\lambda) f_0(\lambda) = g(\lambda) \left( \frac{1}{2} - \lambda^{n+1} - \lambda^{2(n+1)} - \cdots \right). \]

Hence

\[ P_n^+ u(\lambda) f_0(\lambda) = \frac{1}{2} g(\lambda). \]

Analogously, for \(|\lambda| > 1\),

\[ u(\lambda) f_0(\lambda) = g(\lambda) \left( -\frac{1}{2} + \lambda^{-n-1} + \lambda^{-2(n+1)} + \cdots \right); \]

thus

\[ P_n^- u(\lambda) f_0(\lambda) = -\frac{1}{2} g(\lambda). \]

Consequently, \( f_0(\lambda) \) is a solution of (3.8), which means that \( R_n f_0 \) is a special solution of (3.7).

Furthermore, for any polynomial \( p(\lambda) \) with \( \deg p(\lambda) < n \), \( f(\lambda) := p(\lambda) / u(\lambda) \) is a solution of the homogeneous equation

\[ (P_n^+ - P_n^-) u(\lambda) f(\lambda) = 0. \]

That means \( c = R_n p(\lambda) / u(\lambda) \) is a solution of

\[ D_n (\hat{u}) c = 0. \]

In order to see that any solution is of this form we have to show, in view of \( \dim \ker D_n(\hat{u}) = n \), that the vectors \( R_n \lambda^i / u(\lambda) \) \((i = 0, \ldots, n - 1)\) are linearly independent. But this follows from the fact that \( R_n p(\lambda) / u(\lambda) = 0 \) together with \( \deg p(\lambda) < n \) implies \( p(\lambda) = 0 \).

Thus we have proved that the general solution of Problem T1 is given by (3.6).
4. INVERSE PROBLEMS FOR ONE HERMITIAN TOEPLITZ EQUATION

We consider now Problem T1 with the additional condition that the matrix \( T \) is hermitian. We introduce the involution \( \# \) in \( \mathbb{C}^{n+1} \) by \( u^\# := \overline{J_n u} \), where the bar denotes complex conjugate. A vector \( u \) is said to be symmetric if \( u^\# = u \), and antisymmetric if \( u^\# = -u \). Obviously, \( u \) is antisymmetric if \( iu \) is symmetric. Any vector \( u \) can be represented in the form \( u = u_+ + iu_- \), where \( u_+ = (u + u^\#)/2 \), \( u_- = (u - u^\#)/2i \), and \( u_+ \) and \( u_- \) are symmetric. This representation is unique.

Since for an \((n+1) \times (n+1)\) Toeplitz matrix \( T \), \( J_n T J_n \) equals the transpose of \( T \), the equation

\[
T u = g
\]

is equivalent to \( T^* u^\# = g^\# \)—that is, for hermitian \( T \), equivalent to \( T u^\# = g^\# \). Therefore, in the case of Hermitian \( T \) (4.1) can be replaced by the two equations

\[
T u_+ = g_+, \quad T u_- = g_-,
\]

where \( u_+, u_- \) are defined above and \( g_+ = (g + g^\#)/2 \), \( g_- = (g - g^\#)/2i \). Hence it is reasonable to consider the following problem first.

**Problem T1\(^*\)**. Given two symmetric vectors \( u \) and \( g \in \mathbb{C}^{n+1} \), find hermitian Toeplitz matrices \( T \) satisfying \( T u = g \).

In order to solve Problem T1\(^*\) we have to look for all symmetric solutions \( c \) of the equation

\[
D_n(\tilde{u}) c = g.
\]

We observe that \( c = R_n f \) is symmetric if \( f(\lambda) \) fulfills

\[
f(\lambda^{-1}) = -\overline{f(\lambda)}.
\]

The function

\[
f_0(\lambda) = \frac{g(\lambda)}{u(\lambda)} \frac{1 - \lambda^{n+1}}{1 + \lambda^{n+1}}
\]
satisfies this condition (provided that \( u \) and \( g \) are symmetric). That means \( T_n(f_0) \) is a solution of Problem \( T_1^* \).

It remains to find all symmetric solutions of the homogeneous equation

\[
D_n(\bar{u})c = 0. \quad (4.1)
\]

**Lemma 4.1.** The general symmetric solution of (4.4) is given by

\[
c = R_n \frac{p(\lambda)}{u(\lambda)}, \quad (4.5)
\]

where \( p(\lambda) \) is antisymmetric and \( \deg p(\lambda) \leq n \).

**Proof.** Since \( p(\lambda)/u(\lambda) \) fulfills (4.3), the vector \( c \) given by (4.5) is symmetric. Furthermore, in view of (3.5) \( c \) is a solution of (4.4). On the other hand, let \( c \) be a solution of (4.4). Then \( c \) is of the form \( c = R_n[q(\lambda)/u(\lambda)] \) for some polynomial \( q(\lambda) \) with degree less than \( n \). This follows from the proof of Theorem 3.1. We have

\[
c^\# = -R_n \frac{q^\#(\lambda)}{u(\lambda)}.
\]

If now \( c = c^\# \), then \( c = R_n p(\lambda)/u(\lambda) \), where \( p = (q - q^\#)/2 \). But \( p \) is antisymmetric.

Thus we have proved the following theorem.

**Theorem 4.1.** The general solution of Problem \( T_1^* \) is given by \( T = T_n(f) \), where

\[
f(\lambda) = \frac{p(\lambda)}{u(\lambda)} + \frac{g(\lambda)}{2u(\lambda)} \frac{1 - \lambda^{n+1}}{1 + \lambda^{n+1}},
\]

and \( p(\lambda) \) runs over all antisymmetric polynomials with degree less than or equal to \( n \).
5. INVERSE PROBLEM FOR TWO TOEPLITZ EQUATIONS

We consider now the following problem.

**Problem T2.** Given vectors \( u, v, g, h \in \mathbb{C}^{n+1} \) find Toeplitz matrices \( T \) satisfying \( Tu = g \) and \( Tv = h \).

For simplicity we assume that \( u \) and \( v \) have nonvanishing first and last components.

Introduce the function

\[
Z(\lambda) = \frac{1}{2} \left( 1 - \lambda^{n+1} \right).
\]

Then we have at 0

\[
Z(\lambda) = \frac{1}{2} + O(\lambda^{n+1}) \quad (5.1a)
\]

and at infinity

\[
Z(\lambda) = -\frac{1}{2} + O(\lambda^{-n-1}). \quad (5.1b)
\]

According to Theorem 3.1 every solution \( T \) of Problem T2 admits representations \( T = T_n(f_1) \) and \( T = T_n(f_2) \), where

\[
f_1(\lambda) = \frac{p(\lambda) + g(\lambda)Z(\lambda)}{u(\lambda)} \quad \text{and} \quad f_2(\lambda) = \frac{g(\lambda) + h(\lambda)Z(\lambda)}{v(\lambda)} \quad (5.2)
\]

and \( p(\lambda) \) and \( q(\lambda) \) are polynomials with degree less than \( n \). In order to solve Problem T2 we have to answer the question under which conditions on \( p(\lambda) \) and \( q(\lambda) \) the two functions in (5.2) generate the same Toeplitz matrix \( T_n \).

**Lemma 5.1.** Let \( f_1(\lambda) \) and \( f_2(\lambda) \) be defined by (5.2), \( \deg p < n \), \( \deg q < n \). Then \( T_n(f_1) = T_n(f_2) \) if and only if

\[
\hat{T}u = \hat{T}v, \quad (5.3)
\]

\[
p(\lambda)v(\lambda) - q(\lambda)u(\lambda) - \frac{1}{2} (U_n - V_n) w(\lambda), \quad (5.4)
\]

where

\[
w(\lambda) := h(\lambda)u(\lambda) - g(\lambda)v(\lambda)
\]
and $U_n, V_n$ are operators defined for polynomials $x(\lambda) = x_0 + x_1\lambda + \cdots + x_{2n}\lambda^{2n}$ by

$$U_n x(\lambda) := \sum_{j=0}^{n} x_j \lambda^j, \quad V_n x(\lambda) := \sum_{j=-n}^{2n} x_j \lambda^j. \quad (5.5)$$

Proof. By definition, $T_n(f_1) = T_n(f_2)$ if and only if

$$R_n(f_1) = R_n(f_2). \quad (5.6)$$

Suppose that (5.6) holds. Then there are two representations

$$\frac{p}{u} - \frac{q}{v} = \alpha + \left(\frac{h}{v} - \frac{g}{u}\right)Z + \lambda^{n+1}r_+(\lambda) \quad (5.7a)$$

and

$$\frac{p}{u} - \frac{q}{v} = \alpha + \left(\frac{h}{v} - \frac{g}{u}\right)Z + \lambda^{-n-1}r_-(\lambda) \quad (5.7b)$$

where $\alpha \in \mathbb{C}$ and $r_+(\lambda), r_-(\lambda)$ are rational functions analytic at 0 and infinity, respectively. Since the degrees of $p(\lambda)$ and $q(\lambda)$ are assumed to be less than $n$, we obtain $\alpha = 0$ from the second equality. In view of (5.1) we have now representations

$$pv - qu = \frac{1}{2}w + \lambda^{n+1}t_+(\lambda)uv \quad (5.8a)$$

and

$$pv - qu = -\frac{1}{2}w + \lambda^{-n-1}t_- (\lambda)uv \quad (5.8b)$$

for certain rational functions $t_+(\lambda)$ and $t_- (\lambda)$, where $t_+(\lambda)$ is analytic at 0 and $t_-(\lambda)$ at infinity. Applying the operator $X_n x(\lambda) := x_n$ defined for polynomials $x(\lambda) = \sum x_j\lambda^j$ to (5.8a) and (5.8b), we obtain $X_n w(\lambda) = 0$, which is (5.3). Applying now $U_n$ to (5.8a) and $V_n$ to (5.8b) and adding the results, we obtain, in view of $U_n + V_n - X_n = I_n$, the equality (5.4).
Vice versa, let (5.3) and (5.4) be fulfilled. We apply the operators $X_n, U_n, V_n$ to $y := pv - qu$ and get $X_n y = 0$ and

$$U_n y = \frac{1}{2} U_n w, \quad V_n y = -\frac{1}{2} V_n w.$$ 

Hence there are representations

$$y = \frac{1}{2} w + \lambda^{n+1} p_1 \quad \text{and} \quad y = -\frac{1}{2} + p_2,$$

where $p_1$ and $p_2$ are polynomials with degree less than $n$. This implies (5.8), from which (5.7) and (5.6) follow.

**Remark 5.1.** If $p(\lambda)$ and $q(\lambda)$ are assumed to be of degree less than or equal to $n$ (this will be done for the hermitian Toeplitz problem), and condition (5.4) has to be replaced by

$$p(A)v(A) - q(A)u(A) = -(U_n - V_n)w(A) + w(A)v(A),$$

where $\alpha$ is a constant.

From Lemma 5.1 we obtain the following.

**Theorem 5.1.** Problem T2 is solvable iff the following conditions are satisfied:

(a) $\hat{h}^T u = \hat{g}^T v$;

(b) the g.c.d. of $u(\lambda)$ and $v(\lambda)$ is a divisor of $w(\lambda) := h(\lambda)u(\lambda) - g(\lambda)v(\lambda)$.

The solution is unique if and only if $u(\lambda)$ and $v(\lambda)$ are coprime. The general solution is given by

$$T = T_n \left( \frac{p(\lambda) + g(\lambda) Z(\lambda)}{u(\lambda)} \right), \quad (5.9)$$

where $p(\lambda)$ runs over all solutions of

$$p(\lambda)v(\lambda) \equiv \frac{1}{2}(U_n - V_n)w(\lambda) \mod u(\lambda). \quad (5.10)$$

Comparing Theorems 5.1 and 4.1, we get the solution of the following problem.
Problem T2*. Given symmetric vectors \( u, v, g, h \in \mathbb{C}^{n+1} \), find hermitian Toeplitz matrices \( T \) such that \( Tu = g \) and \( Tb = h \).

Theorem 5.2. Problem T2* is solvable iff conditions (a) and (b) of Theorem 5.1 are satisfied. The solution is unique iff \( u(\lambda) \) and \( v(\lambda) \) are coprime. The solution is given by (5.9), where \( p(\lambda) \) runs over all antisymmetric solutions of (5.10), \( \deg p \leq n \).

Proof. It remains to show that Problem T2 admits a hermitian solution under the conditions of the theorem. This follows from the facts that any solution \( p(\lambda) \) of (5.10) provides the antisymmetric solution \( \frac{p(\lambda) - p^*(\lambda)}{2} \) of this congruence and any antisymmetric solution of (5.10) provides a solution of Problem T2*.

Let us consider two special cases. First we assume that vectors \( u, v \in \mathbb{C}^{n+1} \) and numbers \( \alpha, \beta, \alpha \neq \beta \), are given. We look for a Toeplitz matrix such that \( u \) and \( v \) are eigenvectors corresponding to eigenvalues \( \alpha \) and \( \beta \), respectively. In this case we have \( h = \beta v, g = \alpha u \), and \( \omega(\lambda) = (\beta - \alpha)u(\lambda)v(\lambda) \). Furthermore, condition (a) in Theorem 5.1 holds if \( \beta^TU = 0 \). If this condition is satisfied, then, according to Theorem 5.1, the problem admits a solution iff the g.c.d. of \( u(\lambda) \) and \( v(\lambda) \) is a divisor of \( (U_n - V_n)u(\lambda)v(\lambda) \) or, equivalently, of \( V_nu(\lambda)v(\lambda) \). Since \( V_nu(\lambda)v(\lambda) = \lambda^{n+1}Pu(\lambda)v(\lambda)\lambda^{-n-1} \), this result coincides with that of Section 2(3). If \( u \) and \( v \) are symmetric, \( \alpha \) and \( \beta \) real, then the general form of a hermitian Toeplitz matrix solving the inverse spectra problem is given by

\[
T = \alpha I_n + T_n\left( \frac{p(\lambda)}{u(\lambda)} \right),
\]

where \( p \) runs over all antisymmetric solutions of the congruence

\[
p(\lambda)v(\lambda) \equiv \frac{\beta - \alpha}{2} (U_n - V_n)u(\lambda)v(\lambda) \mod u(\lambda).
\]

Let us note that for real or purely imaginary \( u \) and \( v \) the general real symmetric solution of the problem can be described.

Finally we consider the classical problem \( g = e_n, h = e_0 \). In this case we have \( \omega(\lambda) = v(\lambda) - \lambda^n u(\lambda) \). Hence

\[
(U_n - V_n)\omega(\lambda) = v(\lambda) + \lambda^n u(\lambda) + (u_0 + v_n)\lambda^n.
\]
The congruence (5.10) goes over into

\[ [p(\lambda) - 1] v(\lambda) \equiv \frac{1}{2} (u_0 + v_n) \lambda^n \mod u(\lambda). \]

Again we obtain Theorem 0.2 and, moreover, a Toeplitz representation of the solution.

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