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Reflection groups and polytopes over finite fields, III

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With best wishes for our friend and colleague, Jörg Wills

Abstract

When the standard representation of a crystallographic Coxeter group Γ is reduced modulo an odd prime p, one obtains a finite group G^p acting on some orthogonal space over $\mathbb{Z}_p$. If Γ has a string diagram, then G^p will often be the automorphism group of a finite abstract regular polytope. In parts I and II we established the basics of this construction and enumerated the polytopes associated to groups of rank at most 4, as well as all groups of spherical or Euclidean type. Here we extend the range of our earlier criteria for the polytopality of G^p. Building on this we investigate the class of '3-infinity' groups of general rank, and then complete a survey of those locally toroidal polytopes which can be described by our construction.

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1. Introduction and notation

The regular polytopes continue to be a rich source of beautiful mathematical ideas. Their combinatorial features, for instance, have been generalized in the theory of *abstract regular polytopes*. Here we conclude a series of three papers concerning the properties of (abstract) regular polytopes, as constructed from orthogonal groups over finite fields. Our main goal is to complete

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a description of the *locally toroidal polytopes* provided by our construction (see Section 4). To that end, in Section 2 we establish some new structural theorems concerning the 'polytopality' of orthogonal groups. As a test case, we also apply our methods in Section 3 to an interesting family of polytopes of general rank n.

Let us begin with a review of the basic set up and key results from parts I and II ([14] and [15], respectively). In [14], we first surveyed some of the essential properties of an abstract regular polytope $\mathcal{P}$, referring to [12] for details. Crucially, for each such $\mathcal{P}$ the automorphism group $\Gamma(\mathcal{P})$ is equipped with a natural list of involutory generators and is further a very special quotient of a certain Coxeter group G. (We say that $\Gamma(\mathcal{P})$ is a *string C-group*.) Since $\mathcal{P}$ can be uniquely reconstructed from $\Gamma(\mathcal{P})$, we may therefore shift our focus.

Throughout, then, $G = \langle r_0, \dots, r_{n-1}\rangle$ will be a possibly infinite, crystallographic Coxeter group $[p_1, p_2, \dots, p_{n-1}]$ with a string Coxeter diagram $\Delta_c(G)$ (with branches labeled $p_1, p_2, \dots, p_{n-1}$, respectively), obtained from the corresponding abstract Coxeter group $\Gamma = \langle \rho_0, \dots, \rho_{n-1}\rangle$ via the standard representation on real n-space V. (Very often G will be infinite.) For any odd prime p, we may reduce G modulo p to obtain a subgroup G^p of $GL_n(\mathbb{Z}_p)$ generated by the modular images of the r_i's. We shall abuse notation by referring to the modular images of objects by the same name (such as r_i, b_i, $B = [b_i \cdot b_j]$, V, etc.). In particular, $\{b_i\}$ will denote the standard basis for $V = \mathbb{Z}_p^n$. In any event, G^p is a subgroup of the orthogonal group $O(\mathbb{Z}_p^n)$ of isometries for the (possibly singular) symmetric bilinear form $x \cdot y$, the latter being defined on $\mathbb{Z}_p^n$ by means of the Gram matrix B. Likewise, each r_i remains a reflection, although we may write

$$r_i(x) = x - 2\frac{x \cdot b_i}{b_i \cdot b_i} b_i$$

only if $b_i^2 := b_i \cdot b_i \not\equiv 0 \bmod p$. Concerning this situation, we now make a convenient definition: if $p \geqslant 5$, or $p = 3$ but no branch of $\Delta_c(G)$ is marked 6, then we say that p is *generic* for G. Indeed, in such cases, no node label b_i^2 of the diagram $\Delta(G)$ (for a basic system) is zero $\bmod p$, and the corresponding *root* b_i is anisotropic. Also, a change in the underlying basic system for G has the effect of merely conjugating G^p in $GL_n(\mathbb{Z}_p)$. On the other hand, in the non-generic case, in which $p = 3$ and $\Delta_c(G)$ has some branch marked 6, the group G^p may depend essentially on the actual diagram $\Delta(G)$ taken for the reduction mod p. (Note that p generic does not necessarily mean that $p \nmid |G|$, or that certain subspaces of V are non-singular, etc.)

Now we confront two questions: what exactly is the finite reflection group G^p and when is it a string C-group (i.e. the automorphism group of a finite, abstract regular n-polytope $\mathcal{P} = \mathcal{P}(G^p)$)? To help answer the first question, we recall from [14, Thm. 3.1] that an irreducible group G^p of the above sort, generated by $n \geqslant 3$ reflections, must necessarily be one of the following:

- an orthogonal group $O(n, p, \epsilon) = O(V)$ or $O_j(n, p, \epsilon) = O_j(V)$, excluding the cases $O_1(3, 3, 0)$, $O_2(3, 5, 0)$, $O_2(5, 3, 0)$ (supposing for these three that $\mathrm{disc}(V) \sim 1$), and also excluding the case $O_j(4, 3, -1)$; or
- the reduction mod p of one of the finite linear Coxeter groups of type A_n ($p \nmid n + 1$), B_n, D_n, E_6 ($p \neq 3$), E_7, E_8, F_4, H_3 or H_4.

We shall say in these two cases that G^p is of *orthogonal* or *spherical type*, respectively, although there is some overlap for small primes. Concerning our groups G^p, it is only a slight abuse of notation to let $[p_1, \ldots, p_{n-1}]^p$ denote the modular representation of a group $[p_1, \ldots, p_{n-1}]$, so long as p is generic for the group.

Let us turn to our second question. The generators r_i of G^p certainly satisfy the Coxeter-type relations inherited from G. Thus G^p is a *string C-group* if and only if it satisfies the following intersection property on standard subgroups:

$$\langle r_i \mid i \in I\rangle \cap \langle r_i \mid i \in J\rangle = \langle r_i \mid i \in I \cap J\rangle, \tag{1}$$

for all $I, J \subseteq \{0, \ldots, n-1\}$ (see [12, §2E]). Our main problem is therefore to determine when G^p satisfies (1). Before reviewing a few preliminary results in this direction, we establish some notation.

For any $J \subseteq \{0, \ldots, n-1\}$, we let $G_J^p := \langle r_j \mid j \notin J\rangle$; in particular, for $k, l \in \{0, \ldots, n-1\}$ we let $G_k^p := \langle r_j \mid j \neq k\rangle$ and $G_{k,l}^p := \langle r_j \mid j \neq k, l\rangle$. We also let V_J be the subspace of $V = \mathbb{Z}_p^n$ spanned by $\{b_j \mid j \notin J\}$, and similarly for $V_k, V_{k,l}$. Note that V_J is G_J^p-invariant. In particular, G_j^p acts on V_j, for $j = 0$ or $n-1$. The upshot of Lemma 3.1 in [15] is that this action is faithful when p is generic for G. Referring to [14, Eq. 12], we record here a useful rule for inductively computing the determinant of the Gram matrix $B = [b_i \cdot b_j]$. Letting B_J be the submatrix obtained by deleting all rows and columns indexed by J, we have, for example,

$$\det(B) = b_0^2 \det(B_0) - (b_0 \cdot b_1)^2 \det(B_{0,1}). \tag{2}$$

We will frequently refer to the following general properties of string C-groups, here as they apply to the groups G^p:

Proposition 1.1.

(a) *G^p is a string C-group if and only if G_0^p, G_{n-1}^p are string C-groups and $G_0^p \cap G_{n-1}^p = G_{0,n-1}^p$.*
(b) *If G^p is a string C-group, then so too is any subgroup G_J^p, for $J \subseteq \{0, \ldots, n-1\}$.*

Proof. See [12, 2E16 and 2E12]. □

In the next section we extract from [14,15] various more specialized criteria for G^p to be a string C-group. These concern the features of V as an orthogonal space, as well as the action of standard subgroups of G^p on V. Using them, we were able in [14] to classify all groups G^p, and their polytopes, whenever $n \leqslant 3$, as well as when G is of spherical or Euclidean type, for all ranks n. Then in [15] we extended the classification to all cases in rank 4. After generalizing these criteria, it will be clear that we have enough machinery to systematically extend our efforts to polytopes of still higher rank. However, already in rank 4 there is a bewildering variety of possibilities, so that below we shall investigate only a few families of special interest.

2. More on the intersection property

Let us review various situations in which G^p is guaranteed to be a string C-group. First of all, this will be the case if one of the subgroups G_0^p or G_{n-1}^p is spherical and the other is a string C-group:

Theorem 2.1. *(See [14, Th. 4.2].) Let* $G = \langle r_0, \ldots, r_{n-1}\rangle$ *be a crystallographic linear Coxeter group with string diagram, and suppose the prime* $p \geqslant 3$. *If* G_{n-1} *is of spherical type and* G_0^p *is a string* C*-group, or* (*dually*) *if* G_0 *is of spherical type and* G_{n-1}^p *is a string* C*-group, then* G^p *is a string* C*-group.*

We note that the proof supplied in [14] is inadequate for the groups $G = [6, k]$ with $p = 3$, though only for some of the possible basic systems (which need not be equivalent in these non-generic cases). A familiar example, taking $k = 3$, is the Euclidean group with diagram

$$\overset{1}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{3}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{3}{\bullet}.$$

Nevertheless, the intersection condition can be verified for all these groups, using GAP [4] or by hand. In fact, such peculiar exceptions appear only peripherally in this paper.

The next two theorems utilize the occurrence of groups of orthogonal type.

Theorem 2.2. *(See [14, Th. 4.1].) Let* $G = \langle r_0, \ldots, r_{n-1}\rangle$ *be a crystallographic linear Coxeter group with string diagram, and suppose the prime* $p \geqslant 3$. *Suppose that* G_0^p *and* G_{n-1}^p *are string* C*-groups, and that the subspace* $V_{0,n-1}$ *is non-singular. Then if* $G_{0,n-1}^p$ *is the full orthogonal group* $O(n-2, p, \epsilon)$ *on* $V_{0,n-1}$, G^p *must be a string* C*-group.*

We now take a closer look at ways in which the geometry of the various subspaces V, V_0, V_{n-1} or $V_{0,n-1}$ affects the interaction of the corresponding subgroups of G^p. The *fully non-singular* case, proved in [15, Th. 3.2] and generalized in part (a) below, sometimes allows us to *reject* large classes of groups G^p as C-groups because of the size of their subgroups $G_0^p \cap G_{n-1}^p$. When we leave the fully non-singular case, we must adjust our approach in various ways, depending on which of the various subspaces is singular. The case in which just the *middle section* is singular, proved in [15, Th. 3.3] and repeated in (b) below, can sometimes be used to affirm the polytopality of G^p (see [15, Cor. 3.2]).

In any ambient space V, each non-singular subspace W induces an orthogonal direct sum $V = W \perp W^{\perp}$ [2, Ch. 6, Lemma 2.1]. Now consider $O(W)$, the orthogonal group for W (equipped with the bilinear form inherited from V). It is easy to check that the mapping

$$\begin{aligned} \lambda : O(W) &\to \mathrm{Stab}_{O(V)}\, W^{\perp}, \\ g &\to g \perp 1_{W^{\perp}} \end{aligned} \tag{3}$$

establishes an isomorphism between $O(W)$ and a subgroup of the *pointwise* stabilizer of $W^{\perp}$ in $O(V)$. We may therefore identify $O(W)$ with this subgroup; this is done without much comment for several subspaces W in Theorem 2.3 below. If V happens to be non-singular, then the spinor norm on $O(W)$ is also invariant under this identification [1, Th. 5.13], and we clearly have $O(W) \simeq \mathrm{Stab}_{O(V)} W^{\perp}$.

Let us now turn to the subgroup $O_1(W) := \langle r_a \mid a \in W,\ a^2 = 1\rangle$. By [14, Prop. 3.1], $O_1(W)$ almost always coincides with the kernel of the spinor norm on $O(W)$ and so then has index 2 in $O(W)$. However, for $\dim(W) \geqslant 2$ there are two exceptions to this: if $O(W)$ is isomorphic to either

$$[B_3]^3 \simeq O(3, 3, 0) \quad (\text{with disc } \sim 1) \quad \text{or} \quad [F_4]^3 \simeq O(4, 3, +1), \tag{4}$$

then $O_1(W)$ has index 3 in the spinor kernel [14, p. 301].

In similar fashion we can work with a *singular* subspace W of a non-singular space V. Here we let $\widehat{O}(W)$ denote the subgroup of $O(W)$ consisting of those isometries which act trivially on $\operatorname{rad} W$ (see [15, Section 3]). It is not hard to show that $\widehat{O}(W)$ contains and is generated by all reflections with non-isotropic roots in W. Furthermore, we may define a *spinor norm* θ on $O(W)$; and $\widehat{O}_1(W)$ will denote the subgroup of $\widehat{O}(W)$ generated by reflections in $O(W)$ with square spinor norm. In the proof of [15, Th. 3.3], we employed a variant of the mapping in (3) to show that $\widehat{O}(W)$ can also be identified with a suitable subgroup of the pointwise stabilizer of $W^{\perp}$ in $O(V)$, so long as W is a subspace $V_{0,n-1}$ (of codimension 2 in V); this is the only case that we require. Again we find that $\widehat{O}_1(W)$ usually has index 2 in $\widehat{O}(W)$; for $\dim(W) \geqslant 2$, exceptions occur when $O(W/\operatorname{rad}(W))$ is either $O(2,3,+1)$ or one of the groups in (4).

Let us assemble our old results, along with some new criteria, into one package:

Theorem 2.3. *Let $G = \langle r_0, \ldots, r_{n-1}\rangle$ be a crystallographic linear Coxeter group with string diagram. Suppose that $n \geqslant 3$, that the prime p is generic for G and that there is a square among the labels of the nodes $1, \ldots, n-2$ of the diagram $\Delta(G)$ (this can be achieved by readjusting the node labels). For various subspaces W of V we identify $O(W)$, $\widehat{O}(W)$, etc., with suitable subgroups of the pointwise stabilizer of $W^{\perp}$ in $O(V)$.*

(a) *Let the subspaces V_0, V_{n-1} and $V_{0,n-1}$ be non-singular, and let G_0^p, G_{n-1}^p be of orthogonal type.*
  (i) *Then $G_0^p \cap G_{n-1}^p$ acts trivially on $V_{0,n-1}^{\perp}$ and $O_1(V_{0,n-1}) \leqslant G_0^p \cap G_{n-1}^p \leqslant O(V_{0,n-1})$.*
  (ii) *If $G_0^p = O(V_0)$ and $G_{n-1}^p = O(V_{n-1})$, then $G_0^p \cap G_{n-1}^p = O(V_{0,n-1})$.*
  (iii) *If either $G_0^p = O_1(V_0)$ or $G_{n-1}^p = O_1(V_{n-1})$, then $G_0^p \cap G_{n-1}^p = O_1(V_{0,n-1})$.*

(b) *Let V, V_0, V_{n-1} be non-singular, let $V_{0,n-1}$ be singular (so that $n \geqslant 4$), and let G_0^p, G_{n-1}^p be of orthogonal type.*
  (i) *Then $G_0^p \cap G_{n-1}^p$ acts trivially on $V_{0,n-1}^{\perp}$, and $\widehat{O}_1(V_{0,n-1}) \leqslant G_0^p \cap G_{n-1}^p \leqslant \widehat{O}(V_{0,n-1})$.*
  (ii) *If $G_0^p = O(V_0)$ and $G_{n-1}^p = O(V_{n-1})$, then $\widehat{O}(V_{0,n-1}) = G_0^p \cap G_{n-1}^p$.*
  (iii) *If either $G_0^p = O_1(V_0)$ or $G_{n-1}^p = O_1(V_{n-1})$, then $\widehat{O}_1(V_{0,n-1}) = G_0^p \cap G_{n-1}^p$.*

(c) *Suppose V, $V_{0,n-1}$ are non-singular while at least one of V_0, V_{n-1} is singular. Also suppose that $G_{0,n-1}^p$ is of orthogonal type, with $G^p = O_1(V)$ when $G_{0,n-1}^p = O_1(V_{0,n-1})$. Then $G_0^p \cap G_{n-1}^p = G_{0,n-1}^p$.*

Proof. When V is non-singular, parts (a)(i), (ii) appear as Theorem 3.2 in [15]. For V singular, $\operatorname{rad}(V) = \langle c\rangle$ is 1-dimensional, and we may choose a basis w, w' for $V_{0,n-1}^{\perp}$ so that $c = w + w'$, $V_{n-1} = V_{0,n-1} \perp \langle w\rangle$, $V_0 = V_{0,n-1} \perp \langle w'\rangle$ and $V = V_{n-1} \perp \langle v\rangle = V_0 \perp \langle v'\rangle$, with $v = v' = c$. Then $g \in G_0^p \cap G_{n-1}^p$ implies that $g(w) = \alpha w$, $g(w') = \alpha' w'$, where $\alpha, \alpha' \in \{\pm 1\}$. Since $g(c) = c$, we have $\alpha = \alpha' = 1$, so that $g \in O(V_{0,n-1})$. The rest of the proof of (i) and (ii) proceeds as in [15, Th. 3.2].

For (a)(iii) we may suppose $G_0^p = O_1(V_0)$. When $n = 3$ there is nothing to prove, since node 1 has a square label and so $O(V_{0,2}) = O_1(V_{0,2})$. Now suppose $n - 2 \geqslant 2$, so that there exists a reflection $r \in O(V_{0,n-1})$ with *non-square* spinor norm. Since $r \notin O_1(V_0) = G_0^p$, we must by (i) have $G_0^p \cap G_{n-1}^p = O_1(V_{0,n-1})$, so long as $O_1(V_{0,n-1})$ has index 2 in $O(V_{0,n-1})$. As we observed earlier, this almost always holds. In fact, neither of the groups indicated in (4) can occur in our setup (as $O(V_{0,4})$, $O(V_{0,5})$, respectively). Indeed, since $p = 3$ in either case and since $G_0^p = G_0^3 = O_1(V_0)$, nodes $1, 2, \ldots, n-1$ must all be labeled by squares mod 3. Also, p is generic for G. These two restrictions imply that each of the standard rotations $r_{j-1}r_j$

in G, except possibly for $j = 1$, must have period 3 or ∞ (or 2, if $\Delta(G)$ is disconnected). In any case, $G_0^3 \simeq S_{a_1} \times \cdots \times S_{a_k}$ is a direct product of $k \geqslant 1$ symmetric groups, where $(a_1 - 1) + \cdots + (a_k - 1) = n - 1$ ($= 4$ or 5 in the two cases). A direct check of the possible orders shows that G_0^3 could not then be of orthogonal type in dimension 4 or 5 respectively.

In (b) we have $n \geqslant 4$; indeed, for $n = 3$ we note that $V_{0,2}$ must be non-singular, since p is generic for G. Parts (i), (ii) appear as Theorem 3.3 in [15]. We settle part (iii) in much the same way as for (a)(iii) above. Suppose that $G_0^p = O_1(V_0)$ and let $X := V_{0,n-1}/\mathrm{rad}(V_{0,n-1})$, a non-singular space of dimension $n - 3$. (Note that $X \simeq V_{0,1,n-1} \simeq V_{0,n-2,n-1}$.) If $n = 4$, the invariant quadratic form induced on X must be equivalent to x_1^2; then $\widehat{O}(V_{0,3}) = \widehat{O}_1(V_{0,3})$ and (b)(iii) follows trivially. Now suppose that $n \geqslant 5$. By our earlier remarks, $\widehat{O}_1(V_{0,n-1})$ usually has index 2 in $\widehat{O}(V_{0,n-1})$, in which case (b)(iii) follows easily. The three exceptional cases have $p = 3$ with $n = 5, 6, 7$. But as in part (a)(iii) above, these groups cannot occur as $O(X)$ when $G_0^p = O_1(V_0)$ or $G_{n-1}^p = O_1(V_{n-1})$.

In part (c) there are two very similar cases, depending on whether one or both of V_0, V_{n-1} are singular. To begin with, each $g \in G_0^p \cap G_{n-1}^p$ certainly fixes $\mathrm{rad}(V_0)$ and $\mathrm{rad}(V_{n-1})$ pointwise. It is then easy to show in the two cases that g fixes $V_{0,n-1}^{\perp}$ pointwise, so that $g \in O(V_{0,n-1})$. Next one shows that $O_1(V) \cap O(V_{0,n-1}) = O_1(V_{0,n-1})$, using the assumption on square labels and [1, Th. 5.13]. (Here, too, we must consider, and again exclude, the possibility that $O(V_{0,n-1})$ is one of the groups in (4).) Since either $G^p = O_1(V)$ or $G_{0,n-1}^p = O(V_{0,n-1})$, we now have $g \in G_{0,n-1}^p$. $\square$

Theorem 2.3 has several immediate and useful consequences. For example, in [15, Cor. 3.2], we used a preliminary version of part (b)(iii) to prove that $[k, \infty, m]^p$ is a C-group for any odd prime p and integers $k, m \geqslant 2$. On the other hand, part (a)(iii) led just as easily to a proof that $[\infty, 3, \infty]^p$ is a C-group only when $p = 3, 5, 7$.

Next we generalize [15, Th. 3.4], which concerns 4-polytopes for which the facet group G_3 (say) is Euclidean and so situated that the 'point group' acts on the middle section of the polytope. Our first step is a closer look at the geometric action of groups of affine Euclidean isometries. In the background we typically have an abstract Coxeter group of Euclidean (or 'affine') type, faithfully represented in the standard way as a linear reflection group $E = \langle r_0, \ldots, r_m\rangle$ on real $(m+1)$-space W. Recall that E preserves a positive semidefinite form $x \cdot y$, so that $\mathrm{rad}(W) = \langle c\rangle$ is 1-dimensional. Since $r_j(c) = c$, for $0 \leqslant j \leqslant m$, E is in fact a subgroup of $\widehat{O}(W)$.

To actually exploit the structure of E as a group of isometries on Euclidean m-space, we pass to the contragradient representation of E in the dual space $\check{W}$ (as in [7, 5.13]). Since c is fixed by E, we see that E leaves invariant any translate of the m-space

$$U = \bigl\{\mu \in \check{W}\colon\ \mu(c) = 0\bigr\}.$$

Next, for each $w \in W$ define $\mu_w \in \check{W}$ by $\mu_w(x) := w \cdot x$. The mapping $w \mapsto \mu_w$ factors to a linear isomorphism between $W/\mathrm{rad}(W)$ and U, and so we transfer to U the *positive definite* form induced by W on $W/\mathrm{rad}(W)$. Now choose any $\alpha \in \check{W}$ such that $\alpha(c) = 1$, and let $\mathbb{A}^m := U + \alpha$. Putting all this together we may now think of $\mathbb{A}^m$ as *Euclidean m-space*, with U as its *space of translations*. Indeed, each fixed $\tau \in U$ defines an isometric translation on $\mathbb{A}^m$:

$$\mu \mapsto \mu + \tau, \quad \forall \mu \in \mathbb{A}^m.$$

It is easy to check that this mapping on $\mathbb{A}^m$ is induced by a unique isometry $t \in \widehat{O}(W)$, namely the *transvection*

$$\begin{aligned} t(x) &= x - \tau(x)c, \\ &= x - (x \cdot a)c, \end{aligned}$$

where $\tau = \mu_a$ for suitable $a \in W$. (Remember here that we employ the contragradient representation of $\widehat{O}(W)$ on $\check{W}$, not just that of E.) In summary, we can therefore safely think of translations as transvections.

In the following table we list those Euclidean Coxeter groups which are relevant to our analysis (see [14, §6B]). Concerning the group $E = [4, 3^{m-2}, 4]$ (for the familiar cubical tessellation of $\mathbb{A}^m$), we recall our convention that 3^{m-2} indicates a string of $m - 2 \geqslant 0$ consecutive 3's.

An investigation of the action of these discrete reflection groups on the Euclidean m-space $\mathbb{A}^m$ shows, in each case, that E splits as the semidirect product of the (normal) subgroup T of translations with a certain (finite) *point group* group H:

$$E \simeq T \rtimes H. \tag{5}$$

(See [7, Prop. 4.2].) We can and do display each group in the table so that $H = E_0 = \langle r_1, \ldots, r_m \rangle$.

Returning now to our generalization of [15, Th. 3.4], we suppose that G_{n-1} is of Euclidean type. Of course, a dual result holds when G_0 is Euclidean.

Theorem 2.4. *Let $G = \langle r_0, \ldots, r_{n-1} \rangle$ be a crystallographic linear Coxeter group with string diagram. Suppose that G_{n-1} is Euclidean, with $G_{n-1} = T \rtimes G_{0,n-1}$, where T is the translation subgroup of G_{n-1}. Suppose also that the prime p is generic for G, and that G_0^p is a C-group. Then G^p is a C-group.*

Proof. The subgroup G_{n-1}^p of G^p leaves invariant the subspace V_{n-1} of V. Since p is generic for G, we may conclude from [15, Lemma 3.1] that this action is faithful. Thus G_{n-1}^p is a string C-group of Euclidean type, as described in [14, §6B]. By Proposition 1.1(a), our task is therefore to show that $G_0^p \cap G_{n-1}^p = G_{0,n-1}^p$; so consider any $g \in G_0^p \cap G_{n-1}^p$. Now since $G_{n-1} = T \rtimes G_{0,n-1}$ projects onto G_{n-1}^p, we can multiply g by a suitable element of $G_{0,n-1}^p$, and thereby assume that $g \in T^p$. We want to show that $g = e$.

We observed earlier that g acts as a transvection on V_{n-1}, with $g(x) - x \in \langle c \rangle = \mathrm{rad}(V_{n-1})$ for all $x \in V_{n-1}$. On the other hand, since $g \in G_0^p \cap G_{n-1}^p$ we have $g(x) - x \in V_{0,n-1}$. Finally, we observe that $\langle c \rangle \cap V_{0,n-1} = \{0\}$ by direct inspection of the various cases exhibited in Table 1,

Table 1
Euclidean Coxeter groups

The group E	$m = \dim(\mathbb{A}^m)$	One possible diagram $\Delta(E)$	The corresponding vector $c \in \mathrm{rad}(W)$
$[4, 3^{m-2}, 4]$	$m \geqslant 2$	$\overset{2}{\bullet}\!-\!\!-\!\overset{1}{\bullet}\!-\!\!-\!\overset{1}{\bullet}\ \cdots\ \overset{1}{\bullet}\!-\!\!-\!\overset{1}{\bullet}\!-\!\!-\!\overset{2}{\bullet}$	$c = b_0 + 2(b_1 + \cdots + b_{m-1}) + b_m$
$[3, 3, 4, 3]$	4	$\overset{1}{\bullet}\!-\!\!-\!\overset{1}{\bullet}\!-\!\!-\!\overset{1}{\bullet}\!-\!\!-\!\overset{2}{\bullet}\!-\!\!-\!\overset{2}{\bullet}$	$c = b_0 + 2b_1 + 3b_2 + 2b_3 + b_4$
$[3, 6]$	2	$\overset{1}{\bullet}\!-\!\!-\!\overset{1}{\bullet}\!-\!\!-\!\overset{3}{\bullet}$	$c = b_0 + 2b_1 + b_2$
$[\infty]$	1	$\overset{1}{\bullet}\!=\!\!=\!\overset{1}{\bullet}$	$c = b_0 + b_1$

taking $m = n-1$. (In most cases this trivial intersection is implied directly by the fact that $G_{0,n-1}$ is of spherical type.)

Thus $g(x) = x$ for all $x \in V_{n-1}$. Invariably for us the final node $n-1$ in the diagram $\Delta(G)$ will be connected to node $n-2$, so that $\operatorname{disc}(V) \sim -\operatorname{disc}(V_{n-2,n-1})$ by a dual version of (2). The latter discriminant is non-zero for all groups G_{n-1} encountered here, again because p is generic for G. Since V_{n-1} is therefore a singular subspace of the non-singular space V, we conclude from [1, Th. 3.17] that $g = e$. This completes the proof in all important cases. (It is possible that nodes $n-1$, $n-2$ be non-adjacent; but then it is easy to check directly that $G^p \simeq G^p_{n-1} \times C_2$ is a C-group.) $\square$

3. The 3-infinity groups

The large number of crystallographic Coxeter groups $G = [p_1, \ldots, p_{n-1}]$ of higher ranks makes it difficult to fully enumerate the regular polytopes obtained by our method. However, it is clear that many interesting examples occur. As a test of our methods, we survey in this section groups

$$G = \left[\ldots, 3^k, \infty^l, 3^m, \ldots\right],$$

of general rank n and having all periods $p_j \in \{3, \infty\}$.

When $p_j = \infty$, it is convenient to employ the basic system defined by the subdiagram $\cdots - \overset{1}{\bullet} = \overset{1}{\bullet} - \cdots$ on nodes $j-1, j$. (Thus, $1 = b_{j-1}^2 = b_j^2 = -b_{j-1} \cdot b_j$.) Typically then, $\Delta(G)$ consists of alternating strings of single and doubled branches, as in

$$\cdots - \overset{1}{\bullet} = \overset{1}{\bullet} - \overset{1}{\bullet} - \overset{1}{\bullet} = \overset{1}{\bullet} = \overset{1}{\bullet} = \overset{1}{\bullet} - \cdots.$$

For the prime $p = 3$, each rotation $r_{j-1} r_j$ in G^3 has period 3, and we clearly obtain

$$G^3 \simeq A_n \simeq S_{n+1},$$

regardless of the allocation of branches [14, 6.1]. Likewise, if no $p_j = \infty$, then $G^p \simeq A_n$ for any prime $p \geqslant 3$. Thus, we may henceforth assume when it suits us that $p \geqslant 5$ and that $\Delta(G)$ has at least one doubled branch. If in this case V is non-singular, then $G^p \simeq O_1(V)$ is of orthogonal type (see [14, Th. 3.1]).

Our approach now must be inductive on the size of certain classes of subdiagrams in $\Delta(G)$; but first we must determine the orthogonal structures on V, V_0, V_{n-1} and $V_{0,n-1}$.

For $n \geqslant 1$ we let $d_n := \operatorname{disc}(V)$ be the discriminant of the underlying basic system for $G = [\infty^{n-1}]$, as encoded in the diagram

$$\Delta(G) = \overset{1}{\bullet} = \overset{1}{\bullet} = \overset{1}{\bullet} = \cdots = \overset{1}{\bullet} = \overset{1}{\bullet} = \overset{1}{\bullet}$$

(on n nodes). From (2) we have $d_n = 1 d_{n-1} - 1^2 d_{n-2} = d_{n-1} - d_{n-2}$, for $n \geqslant 2$ and taking $d_0 := 1$. Thus

$$d_n = \begin{cases} 1 & \text{if } n \equiv 0, 1 \bmod 6, \\ 0 & \text{if } n \equiv 2, 5 \bmod 6, \\ -1 & \text{if } n \equiv 3, 4 \bmod 6. \end{cases} \tag{6}$$

Again using (2), we find that the basic system underpinning $[3^{n-1}]$ has $\mathrm{disc}(V) = (n+1)/2^n$. A routine induction then gives the discriminant $e_{k,l,m}$ corresponding to the basic system for the group $G = [3^k, \infty^l, 3^m]$, with $k+l+m = n-1$ and $k, l, m \geqslant 0$. Thus

$$e_{k,l,m} = \frac{1}{2^{k+m+2}}\big[d_{l+1}(4+2k+2m) - d_{l-1}(2k+2m+3km)\big]. \tag{7}$$

In certain singular cases we have this

Lemma 3.1. *Let $G = [3^k, \infty^l]$, with $k+l = n-1$ and $l \geqslant 1$. Suppose that the corresponding space V is singular for the prime p. Then $G^p = \widehat{O}_1(V)$.*

Proof. Clearly $G^p \leqslant \widehat{O}_1(V)$. Now suppose that $c = \sum_{j=0}^{n-1} x_j b_j \in \mathrm{rad}(V)$. Note that each scalar $b_{j-1} \cdot b_j \in \{-1/2, -1\}$ and is therefore invertible in $\mathbb{Z}_p$. Thus, from

$$0 = b_0 \cdot c = x_0 + (b_0 \cdot b_1)x_1$$

we obtain $x_1 = \alpha_1 x_0$, where $\alpha_1 \in \{1, 2\}$. Since $\Delta(G)$ is a tree, we can continue to solve for $x_2, \dots, x_{n-1}$ as multiples of x_0 to obtain $x_j = \alpha_j x_0$ for various α_j (with $\alpha_0 := 1$), where $\alpha_0, \dots, \alpha_{n-1}$ are determined only by the basic system for G. In the end, as V is singular, the equation $0 = b_{n-1} \cdot c$ must be redundant, so we have $\mathrm{rad}(V) = \langle c \rangle$ (with $x_0 \neq 0$). Anyway, we may now take $c = 1b_0 + \alpha_1 b_1 + \cdots + \alpha_{n-1} b_{n-1}$. Then $V = \langle c \rangle \perp V_0$, where V_0 is non-singular and $G_0^p \simeq O_1(V_0)$ (because $l \geqslant 1$). We thus have

$$\widehat{O}_1(V) = T \rtimes G_0^p, \tag{8}$$

where $T \simeq \mathbb{Z}_p^{n-1}$ is the abelian group generated by transvections $t_1, \dots, t_{n-1}$ satisfying $t_j(b_i) = b_i + \delta_{i,j} c$ and $t_j(c) = c$, for $1 \leqslant i, j \leqslant n-1$. Now $r_0 \in \widehat{O}_1(V)$ induces an isometry on $V/\mathrm{rad}(V) \simeq V_0$. Let $h \in G_0^p$ be the isometry corresponding to r_0 under the natural isomorphism between $O_1(V/\mathrm{rad}(V))$ and $O_1(V_0) \simeq G_0^p$. A short calculation shows that $t_1 = (h^{-1} r_0)^q \in G^p$, where $q = 1$ or $(p+1)/2$, according as $r_0 r_1$ has period 3 or ∞ (in characteristic 0). Finally we show inductively that $t_j \in G^p$ for $1 \leqslant j \leqslant n-1$; this implies that $G^p = \widehat{O}_1(V)$. Fixing $j < n-1$ we may suppose $t_i \in G^p$ for all $1 \leqslant i \leqslant j$. From [14, Eq. (9)] we note that $r_j(b_{j-1}) = b_{j-1} + \alpha b_j$, $r_j(b_j) = -b_j$ and $r_j(b_{j+1}) = b_{j+1} + \beta b_j$, where in our case the *Cartan integers* $\alpha, \beta \in \{1, 2\}$; otherwise, for $|k-j| > 1$, $r_j(b_k) = b_k$. It is then a routine matter to check that

$$t_{j-1}^{-\alpha} t_j r_j t_j r_j = t_{j+1}^{\beta}.$$

It follows by induction that $t_{j+1}^{\beta} \in G^p$ and hence $t_{j+1} \in G^p$. □

From [14, §5] and [15, §5], we already know that $[3^k, \infty^l]^p$ is a string C-group for all $p \geqslant 3$ and ranks $n \leqslant 4$ (so that $k+l \leqslant 3$). For example, $[\infty, \infty, \infty]^p$ is the automorphism group of a self-dual regular 4-polytope $\mathcal{P}$ of type $\{p, p, p\}$; when $p = 5$ we find that $\mathcal{P}$ is isomorphic to the classical star-polytope $\{5, \frac{5}{2}, 5\}$ (see [3, Ch. XIV] and [8]). Similarly, $[3, \infty, \infty]^5$ gives back the regular star-polytope $\{3, 5, \frac{5}{2}\}$.

Let us take stock of our progress so far. Keep in mind that whenever G^p is a string C-group, so too is the dually generated group, with k and m interchanged.

Theorem 3.1. *Let $G = [3^k, \infty^l]$, with $k + l = n - 1$. Then for all primes $p \geqslant 3$, the group $G^p = [3^k, \infty^l]^p$ is a string C-group.*

Proof. For $l = 0$ we have already observed that $G^p = [3^{n-1}]^p \simeq S_{n+1}$, the group of the n-simplex. Let us now dispose of the case $l = 1$. Since the facet group $[3^k]$ is spherical, an induction on k, together with Theorem 2.1, shows that $G^p = [3^k, \infty]^p$ is a string C-group.

Now we may suppose $l \geqslant 2$. From (7), with $m = 0$, we have

$$e_{k,l,0} = \frac{1}{2^{k+1}}\left[(k+2)d_{l+1} - kd_{l-1}\right]. \tag{9}$$

We may also suppose that $p \geqslant 5$. Then it is easy to check that if any one of the spaces $V, V_0, V_{n-1}, V_{0,n-1}$ is singular for a given prime p, all the others must be non-singular. We also know, as a basis for induction, that $[3^k, \infty^l]^p$ is a string C-group whenever $k + l \leqslant 3$. Thus we may suppose that G_0^p and G_{n-1}^p are string C-groups. If $V_0, V_{n-1}, V_{0,n-1}$ are non-singular, then the corresponding subgroups of G^p are all of type O_1, since $l \geqslant 2$. By Theorem 2.3(a)(iii) we have $G_0^p \cap G_{n-1}^p = O_1(V_{0,n-1}) = G_{0,n-1}^p$. Thus G^p is a string C-group by Proposition 1.1(a). If V_0 or V_{n-1} is singular, we similarly apply Theorem 2.3(c). Finally, if just $V_{0,n-1}$ is singular, we employ Theorem 2.3(b)(iii) and apply Lemma 3.1 to the subspace $V_{0,n-1}$. □

As an example, consider the group $G = [3, 3, 3, \infty]$ of rank 5. Here we obtain regular 5-polytopes of type $\{3, 3, 3, p\}$ with group $O_1(5, p, 0)$. A particularly interesting case occurs when $p = 5$. Then the polytope can be viewed as a regular tessellation of type $\{3, 3, 3, 5\}$ on a hyperbolic 4-manifold whose 78000 tiles (facets) are 4-simplices and whose 650 vertex-figures are 600-cells (see [12, 6J]). The dual tessellation has 120-cells as tiles and 4-simplices as vertex-figures. (As an aside, when $p = 5$, the group $G = [4, 3, 3, \infty]$ similarly gives a regular tessellation of type $\{4, 3, 3, 5\}$ with group $O(5, p, 0)$ on a hyperbolic 4-manifold, whose facets are 4-cubes and whose vertex-figures are 600-cells. The dual tessellation again has 120-cells as tiles and 4-crosspolytopes as vertex-figures.)

Now let us generalize a little and consider $G = [3^k, \infty^l, 3^m]$, with $k + l + m = n - 1$. All cases with $klm = 0$ are covered by Theorem 3.1, so we assume $k, l, m \geqslant 1$. Then the only new string C-group of small rank is $[3, \infty, 3]$. Again, we tackle G^p inductively; but since the details are more complicated, we shall settle for slightly less comprehensive results.

Theorem 3.2. *Let $G = [3^k, \infty^l, 3^m]$, with $k + l + m = n - 1$ and $k, l, m \geqslant 1$; and suppose p is an odd prime. Then*

(a) *$G^p = [3, \infty^l, 3]^p$, with $n = l + 3$, $l \geqslant 1$ is a string C-group, except possibly when $p = 7$ and $l \geqslant 4$, with $l \equiv 1 \bmod 3$.*
(b) *$G^p = [3^k, \infty^l, 3^m]^p$, with $k > 1$ or $m > 1$, and $l \geqslant 1$, is a string C-group for all but finitely many primes p.*

Proof. The details in part (a) are very similar to those for Theorem 3.1, which also serves as a basis for our induction. The analysis there fails only when the subspaces $V, V_0, V_{n-1}, V_{0,n-1}$ are singular in more than one of dimensions $n, n-1$ and $n-2$; this can only occur when $l \equiv 1 \bmod 3$ and $p = 7$ (in this case exactly V and $V_{0,n-1}$ are singular). In fact, when $p = 7$, $l = 1$, we can use GAP to verify that G^7 is a string C-group anyway [14, p. 347].

For part (b), we notice in (7) that $e_{k,l,m} = 0$ (in characteristic 0) only when $k = m = 0$ and $l \equiv 1 \bmod 3$. Typically then, G^p falls under the fully non-singular case described in Theorem 2.3(a)(iii). □

Remarks. It may well be in the previous theorem that G^p is a C-group for all primes. Certainly in specific cases, one can explicitly list the 'doubtful' primes; but there seems to be little served by trying to do more here.

Now we hunt for contrary cases in which G^p is *not* a string C-group. Once again we may assume $p \geqslant 5$. Evidently we should examine the groups $G = [\infty, 3^k, \infty]$, with $k \geqslant 1$, for which we have $\operatorname{disc}(V) = (k-2)/2^{k+1}$. By Theorem 3.1, both G_0^p and G_{n-1}^p are string C-groups; and $\operatorname{disc}(V_0) = \operatorname{disc}(V_{n-1}) = -k/2^{k+1}$. Finally, $G_{0,n-1}^p = [3^k]^p$ is the symmetric group of order $(k+2)!$, and $\operatorname{disc}(V_{0,n-1}) = (k+2)/2^{k+1}$.

Lemma 3.2. *Suppose $G = [3^k]$, acting as usual on the $(k+1)$-dimensional space V; and let $p \geqslant 5$.*

(a) *If $p \nmid (k+2)$, then V is non-singular and $G^p \leqslant O_1(V)$, with equality only when $O(V) = O(2,5,-1)$ or $O(2,7,+1)$.*
(b) *If $p \mid (k+2)$, then V is singular and G^p is always a proper subgroup of $\widehat{O}_1(V)$.*

Proof. Recall that $G^p \leqslant O_1(V), \widehat{O}_1(V)$ in (a), (b) respectively. We have equality in (a) only when $|O_1(V)| = (k+2)!$. But the highest power of p dividing $(k+2)!$ is p^ν, where

$$\nu = \left\lfloor \frac{k+2}{p} \right\rfloor + \left\lfloor \frac{k+2}{p^2} \right\rfloor + \cdots < \frac{k+2}{p}\left(1 + \frac{1}{p} + \cdots\right) = \frac{k+2}{p-1},$$

(see [5, Prop. 2.3.2]). Taking $n = k+1$ in [14, §3.1], we find that the highest power p^μ dividing $|O_1(V)|$ for a non-singular space V has $\mu = \lfloor \frac{k^2}{4} \rfloor$. (Since $p > 3$ we again ignore the groups in (4).) Clearly, we usually have $\nu < \mu$, and it is easy by inspection to determine the two cases with $G^p = O_1(V)$. The situation for (b) is similar. The splitting in (8) continues to hold in the present context (in which $l = 0$); from this we get $\mu = k + \lfloor \frac{(k-1)^2}{4} \rfloor$ and ultimately no cases of equality at all. □

Now we can show that G^p is very often *not* a string C-group:

Theorem 3.3. *Suppose that G has a string subgroup of the form $[\ldots, \infty, 3^k, \infty, \ldots]$, with $k \geqslant 1$. Let $p \geqslant 5$. Then G^p is not a string C-group, except possibly when $p = 5$ or 7 and $k \leqslant 1$ for all such string subgroups $[\ldots, \infty, 3^k, \infty, \ldots]$.*

Proof. Since our intention is to show that G^p fails to be a string C-group, we may assume by Proposition 1.1(b) that $G = [\infty, 3^k, \infty]$.

If $V_{0,n-1}$ is singular, then $p \mid (k+2)$, so that $p \nmid k(k-2)$, implying that V, V_0, V_{n-1} are non-singular. By Theorem 2.3(b)(iii) and Lemma 3.2(b) (applied to $V_{0,n-1}$), we have $G_0^p \cap G_{n-1}^p = \widehat{O}_1(V_{0,n-1}) \neq G_{0,n-1}^p$. Thus G^p is not a string C-group.

Suppose that $V_{0,n-1}$ is non-singular, with $p > 7$ when $k = 1$. Thus, $p \nmid (k+2)$, and $G^p_{0,n-1} \neq O_1(V_{0,n-1})$ by Lemma 3.2(a). If also $p \nmid k$, then we have $G^p_0 \cap G^p_{n-1} = O_1(V_{0,n-1})$ by Theorem 2.3(a)(iii), so that G^p is not a string C-group.

Finally, suppose $p \mid k$. Thus, $k \geqslant 5$, and this gives enough wiggle room to destroy polytopality in another way. Let $I = \{0, 1\}$, $J = \{5, \dots, n-1\}$. Recall, for example, that G^p_J denotes the subgroup generated by the complementary set of reflections $r_0, \dots, r_4$; these reflections leave invariant the subspace V_J spanned by $\{b_j \mid j \notin J\} = \{b_0, \dots, b_4\}$. But $\mathrm{disc}(V_I) \sim (1-k)/2^k$, $\mathrm{disc}(V_J) \sim -3$ and $\mathrm{disc}(V_{I\cup J}) \sim 1/2$, so that these subspaces are all non-singular subspaces of the non-singular space V. Hence, $G^p_I = O_1(V_I) \geqslant O_1(V_{I\cup J})$ and $G^p_J = O_1(V_J) \geqslant O_1(V_{I\cup J})$. Thus, if G^p is a string C-group, we must have

$$O_1(V_{I\cup J}) \leqslant G^p_I \cap G^p_J = G^p_{I\cup J} = \langle r_2, r_3, r_4\rangle^p \simeq S_4.$$

But $O_1(3, p, 0)$ has order $p(p^2-1) > 24$, for $p \geqslant 5$. $\square$

We can summarize the results of this section as follows: if $G = [p_1, \dots, p_{n-1}]$ has each period $p_j \in \{3, \infty\}$, then except for a few small primes, we cannot expect G^p to be a string C-group if two of the p_j's are ∞'s separated by a string of 3's. That is, only the groups $[3^k, \infty^l, 3^m]$ can give a string C-group, and they do for most primes p.

4. Locally toroidal polytopes of ranks 5 or 6

The crystallographic string Coxeter groups of spherical or Euclidean type, along with the associated modular polytopes, were described in [14, §§5–6]. When the group is spherical with connected diagram on $m+1$ nodes, we obtain (up to isomorphism) familiar convex regular $(m+1)$-polytopes. After central projection, such polytopes can usefully be viewed as regular spherical tessellations of the circumsphere $\mathbb{S}^m$.

Likewise, each Euclidean group E acts as the full symmetry group of a certain regular tessellation of Euclidean space $\mathbb{A}^m$. Indeed, E must be one of the Coxeter groups displayed in Table 1, though perhaps with generators specified in dual order. A *regular* $(m+1)$*-toroid* $\mathcal{P}$ is the quotient of such a tessellation by a non-trivial normal subgroup L of translations in E. Thus every toroid can be viewed as a finite, regular tessellation of the m-torus. We refer to [12, 1D and 6D-E] for a complete classification; briefly, for each group E the distinct toroids are indexed by a *type vector* $\mathbf{q} := (q^k, 0^{m-k}) = (q, \dots, q, 0, \dots, 0)$, where $q \geqslant 2$ and $k = 1, 2$ or m. (For $G = [3, 3, 4, 3]$, the case $k = 4$ is subsumed by the case $k = 1$.) Anyway, L is generated (as a normal subgroup of G) by the translation

$$\bar{t} := t_1^q \cdots t_k^q,$$

where $\{t_1, \dots, t_m\}$ is a standard set of generators for the full group T of translations in E. The modular toroids $\mathcal{P}(E^p)$ described in [14, §6B] are special instances; with one exception, we had there $\mathbf{q} = (p, 0, \dots, 0)$.

In this section, we consider *locally toroidal* regular polytopes, that is, polytopes in which the facets and vertex-figures are globally spherical or toroidal, as described above (with at least one kind toroidal). The n-polytopes of this kind have not yet been fully classified, although quite a lot has been discovered since Grünbaum first proposed the problem in the 1970s (see [6]). What is known rests on a broad range of ideas, including frequent use of unitary reflection groups, and 'twisting' and 'mixing' operations on presentations for string C-groups. We refer to [9–11] for some of the original investigations, or to [12, Chs. 7–12] for a detailed survey of the project.

As usual, we begin our own investigation with a crystallographic linear Coxeter group $G = \langle r_0, \ldots, r_m \rangle$, but immediately discard degenerate cases in which the underlying diagram $\Delta(G)$ is disconnected. (In such cases G^p is reducible; and $\mathcal{P}(G^p)$ has the sort of 'flatness' described in [12, 4E].)

In [15] we discussed all locally toroidal 4-polytopes $\mathcal{P}(G^p)$ which arise from our construction. Turning to higher rank $n > 4$, we observe that any spherical facet, or vertex-figure, must be of type $\{3^{n-2}\}$, $\{4, 3^{n-3}\}$, $\{3^{n-3}, 4\}$ or $\{3, 4, 3\}$ ($n = 5$ only). Likewise, the required Euclidean section must have type $\{4, 3^{n-4}, 4\}$ or when $n = 6$, $\{3, 3, 4, 3\}$ or $\{3, 4, 3, 3\}$. As described in [12, Lemma 10A1], these constraints severely limit the possibilities: in rank 5, we have just $G = [4, 3, 4, 3]$ acting on hyperbolic space $\mathbb{H}^4$; and in rank 6 we have $G = [4, 3, 3, 4, 3]$, $[3, 4, 3, 3, 3]$ or $[3, 3, 4, 3, 3]$, all acting on $\mathbb{H}^5$. Thus we may complete our discussion by examining the modular polytopes which result from these groups in ranks 5 or 6.

4.1. Rank 5: the group $G = [4, 3, 4, 3]$

We may suppose that G has diagram

$$\overset{2}{\bullet}\!\!-\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!-\!\!\overset{2}{\bullet}\!\!-\!\!-\!\!\overset{2}{\bullet}.$$

Note that $G_0^p \simeq F_4$ is a C-group by [14, 6.3]. It follows at once from Theorem 2.4, and a look at Table 1, that G^p is a C-group for any prime $p \geqslant 3$. (Alternatively, since the vertex-figures are spherical, we can appeal to Theorem 2.1.) Thus $\mathcal{P}(G^p)$ is a locally toroidal regular polytope of type $\{4, 3, 4, 3\}$. Its vertex-figures are copies of the 24-cell $\{3, 4, 3\}$. The facets are toroids $\{4, 3, 4\}_{(p,0,0)}$, which one could construct by identifying opposite square faces of a $p \times p \times p$ cube. (See [14, 6.4]; we note that the facet and vertex number mentioned there should be p^{n-1} rather than p^n.) Of course, by flipping the diagram end-for-end, we just as easily obtain the dual polytope of type $\{3, 4, 3, 4\}$.

In order to identify G^p we consult the list of irreducible reflection groups in [14, Table 1]. Since G^p has an abelian subgroup of order p^3 (generated by translations in the facet), we immediately rule out the long shots A_5^p, B_5^p and D_5^p of orders $6!$, $2^5 5!$ and $2^4 5!$, respectively. Note also that the node label 2 is a square mod p if and only if $p \equiv \pm 1 \pmod 8$. Our hand is now forced: $\mathcal{P} = \mathcal{P}(G^p)$ has automorphism group

$$\Gamma(\mathcal{P}) = \begin{cases} O_1(5, p, 0), & \text{if } p \equiv \pm 1 \pmod 8, \\ O(5, p, 0), & \text{if } p \equiv \pm 3 \pmod 8. \end{cases} \tag{10}$$

For any prime $p \geqslant 3$, $O_1(5, p, 0)$ has order $p^4(p^4 - 1)(p^2 - 1)$ and index two in $O(5, p, 0)$ (see [14, pp. 300–301]).

The universal locally toroidal polytopes of rank 5 are completely described in [12, 12B]. All but three are infinite. (The three finite instances have facets with type vector $(2, 0, 0)$, $(2, 2, 0)$ or $(2, 2, 2)$; but clearly these do not occur when the modulus p is an odd prime.) In short, $\mathcal{P}(G^p)$, being finite, is never universal for its type.

Using the fact that $\operatorname{disc}(V) \sim -2$, we compute that the central isometry $-e \in G^p$, except when $p \equiv -1 \pmod 8$. When $-e \in G^p$, the polytope $\mathcal{P}(G^p)$ doubly covers the quotient polytope

$\mathcal{P}(G^p/\{\pm e\})$. The latter polytope still has the same facets and vertex-figures as $\mathcal{P}(G^p)$. To verify these claims, we apply [12, 2E19], so we must show that

$$\{\pm e\} \cap G^p_{n-1} G^p_0 = \{e\}.$$

Suppose, on the contrary, that $-e \in G^p_{n-1} G^p_0$. Since $G^p_{n-1} = G^p_4 = T \rtimes G^p_{0,4}$, we can assume $-e = th$, where $h \in G^p_0 \simeq F_4$ and t is some transvection. Thus $h = -t^{-1}$, which has period $2p$. This is already impossible if $p > 3$, since $|F_4| = 3 \cdot 2^4 \cdot 4!$. Even when $p = 3$ it is easy to verify the contradiction directly.

4.2. Rank 6*: the groups* [3, 4, 3, 3, 3]*,* [3, 3, 4, 3, 3] *and* [4, 3, 3, 4, 3]

In rank 6 we must consider three closely related groups, beginning with

$$G = \langle r_0, r_1, r_2, r_3, r_4, r_5 \rangle \simeq [3, 4, 3, 3, 3].$$

We may describe a basic system (of roots) for G by the diagram

$$\overset{1}{\bullet}\text{——}\overset{1}{\bullet}\text{——}\overset{2}{\bullet}\text{——}\overset{2}{\bullet}\text{——}\overset{2}{\bullet}\text{——}\overset{2}{\bullet}. \tag{11}$$

Now the subgroup $H = \langle s_0, \ldots, s_5 \rangle$ generated by the reflections

$$(s_0, s_1, s_2, s_3, s_4, s_5) := (r_1, r_0, r_2 r_1 r_2, r_3, r_4, r_5) \tag{12}$$

has index 5 in G and is isomorphic to [3, 3, 4, 3, 3]. The basic system of roots attached to the s_j's provides the diagram

$$\overset{1}{\bullet}\text{——}\overset{1}{\bullet}\text{——}\overset{1}{\bullet}\text{——}\overset{2}{\bullet}\text{——}\overset{2}{\bullet}\text{——}\overset{2}{\bullet} \tag{13}$$

for H. Another subgroup $K = \langle t_0, \ldots, t_5 \rangle$ generated by

$$(t_0, t_1, t_2, t_3, t_4, t_5) := (r_2, r_1, r_0, r_3 r_2 r_1 r_2 r_3, r_4, r_5) \tag{14}$$

has index 10 in G, is isomorphic to [4, 3, 3, 4, 3] and has diagram

$$\overset{2}{\bullet}\text{——}\overset{1}{\bullet}\text{——}\overset{1}{\bullet}\text{——}\overset{1}{\bullet}\text{——}\overset{2}{\bullet}\text{——}\overset{2}{\bullet}. \tag{15}$$

(See [12, 12A2]. Each group acts on $\mathbb{H}^5$ with a simplicial fundamental domain of finite volume. In [13], these indices were computed by dissecting a simplex for H (or K) into copies of the simplex for G.)

Let us now survey the three families of locally toroidal polytopes arising from reducing these groups mod p. Since H^p_0 and K^p_0 are C-groups by [14, 6B], we conclude from Theorem 2.4 that H^p and K^p are string C-groups. Similarly, G^p is a string C-group because of Theorem 2.1.

In each case the underlying space V is non-singular for any prime $p \geqslant 3$ and has

$$\operatorname{disc}(V) = 2 \cdot 0 - (-1)^2 (1/4) \sim -1.$$

Thus $\epsilon = +1$ (indicating that the Witt index is 3). As in rank 5, we conclude, for *each* of the three types, that the polytope $\mathcal{P}$ has automorphism group

$$\Gamma(\mathcal{P}) = \begin{cases} O_1(6, p, +1), & \text{if } p \equiv \pm 1 \pmod 8, \\ O(6, p, +1), & \text{if } p \equiv \pm 3 \pmod 8. \end{cases} \tag{16}$$

Of course, this group is differently generated in the three cases, as indicated above. We observe that the indices 5 and 10 in characteristic 0 must collapse to 1 under reduction mod p.

For any prime $p \geqslant 3$, $O_1(6, p, +1)$ has order $p^6(p^4-1)(p^3-1)(p^2-1)$ and index two in $O(6, p, +1)$ (see [14, pp. 300–301]). We have $-e \in \Gamma(\mathcal{P})$ *except when* $p \equiv -1 \pmod 8$.

Remark. Some of the results established below were already announced in [12, 12C,D,E].

The Polytopes $\mathcal{P} = \mathcal{P}(G^p)$.

By [14, 6.2] the vertex-figures of $\mathcal{P}(G^p)$ are 5-cubes $\{4,3,3,3\}$. The facets of $\mathcal{P}(G^p)$ are toroids $\{3,4,3,3\}_{(p,0,0,0)}$, each with $3p^4$ vertices (which corrects the $3p^n$ mentioned in [14, 6.5]).

The universal polytope covering $\mathcal{P}(G^p)$ is

$$\mathcal{U}_{G^p} := \big\{\{3,4,3,3\}_{(p,0,0,0)}, \{4,3,3,3\}\big\}.$$

Recall that $\mathcal{U}_{G^p}$ is conjectured to be finite only when $p = 3$ (see the discussion in [12, 12C1], restricted to the class of polytopes under consideration here). As for $p = 3$, it is known that $\mathcal{U}_{G^3}$ has automorphism group $\Gamma(\mathcal{U}_{G^3})$ of order $3|G^3| = 3|O(6,3,+1)| = 72783360$. Using GAP we find that $\Gamma(\mathcal{U}_{G^3})$ is a split extension of the additive cyclic group $(\mathbb{Z}_3, +)$ by G^3, that is,

$$\Gamma(\mathcal{U}_{G^3}) = \mathbb{Z}_3 \rtimes O(6,3,+1). \tag{17}$$

To check this directly we exploit the spinor norm, which for $p = 3$ we may view as a homomorphism $\theta : G^3 \to \{\pm 1\} = \mathbb{Z}_3^*$. Using this we define an action of G^3 on $\mathbb{Z}_3$ by $gz := \theta(g)\det(g)z$, for $g \in G^3$, $z \in \mathbb{Z}_3$. We obtain the semidirect product $\Lambda := \mathbb{Z}_3 \rtimes G^3$, with identity $(0, e)$ and

$$(y, g)(z, h) = (y + gz, gh),$$

for all $y, z \in \mathbb{Z}_3$, $g, h \in G^3$. Note that $r_i z = \eta_i z$, where

$$\eta_i = \begin{cases} -1, & i \leqslant 1, \\ +1, & i > 1. \end{cases}$$

Now let $\rho_0 := (1, r_0)$ and $\rho_i := (0, r_i)$, for $i \geqslant 1$; in brief, $\rho_i = (\delta_{i0}, r_i)$. It is then a routine matter to check that the ρ_i's satisfy the standard relations for the Coxeter group $[3,4,3,3,3]$. Indeed,

$$\rho_i\rho_j = (\delta_{i0}, r_i)(\delta_{j0}, r_j) = (\delta_{i0} + \eta_i\delta_{j0}, r_ir_j),$$

so that $\rho_i^2 = (0, e)$, for all i. Next we get

$$(\rho_i\rho_j)^2 = \big(\delta_{i0}(1 + \eta_i\eta_j) + \delta_{j0}(\eta_i + \eta_j), (r_ir_j)^2\big),$$

so that $(\rho_i\rho_j)^2 = (0, e)$, whenever $i < j - 1$. Similarly, $\rho_{i-1}\rho_i$ has the same period as $r_{i-1}r_i$ for each i. In particular, we note that $(\rho_0\rho_1)^3 = (0, e)$, since $3 = 0$. Last of all we must verify the required extra relation for the facet, namely

$$(\rho_4\sigma\tau\sigma)^3 = (0, e),$$

where $\sigma := \rho_3\rho_2\rho_1\rho_2\rho_3$, $\tau := \rho_0\rho_1\rho_2\rho_1\rho_0$; see [14, 6.5]. Here check first that $\sigma = (0, s)$ and $\tau = (0, t)$, where $s := r_3r_2r_1r_2r_3$, $t := r_0r_1r_2r_1r_0$, then observe that r_4sts has period 3 in G^3.

Finally, we observe that the *Petrie element* $h := r_0r_1r_2r_3r_4r_5$ of G has characteristic polynomial $x^6 - x^4 - x^3 - x^2 + 1$, so that $h^{13} = 6h^5 + 9h^4 + 6h^3 + 3h^2 - 3h - 5e$ and hence $h^{13} = e$ in G^3. The corresponding element of Λ is $\pi := \rho_0\rho_1\rho_2\rho_3\rho_4\rho_5 = (1, h)$. Since $\pi^j = (j, h^j)$, we have $\pi^{13} = (1, e)$; thus the ρ_i's generate Λ, and π has period 39. Since $\Gamma(\mathcal{U}_{G^3})$ and Λ have equal orders, we conclude that the two groups are isomorphic (and that Λ is a string C-group with respect to the ρ_i's). Observe that we obtain the modular polytope $\mathcal{P}(G^3)$ from $\mathcal{U}_{G^3}$ by identifying vertices separated by 13 steps along *Petrie polygons* of $\mathcal{U}_{G^3}$.

The Polytopes $\mathcal{P} = \mathcal{P}(K^p)$.

The facets of $\mathcal{P}(K^p)$ are toroids $\{4, 3, 3, 4\}_{(p,0,0,0)}$, while the vertex-figures are toroids $\{3, 3, 4, 3\}_{(p,0,0,0)}$ of another type. Consulting [12, pp. 466–467], we note that the universal polytope

$$\mathcal{U}_{K^p} := \bigl\{\{4, 3, 3, 4\}_{(p,0,0,0)}, \{3, 3, 4, 3\}_{(p,0,0,0)}\bigr\}$$

is conjectured to exist for all primes $p \geqslant 3$ and to be infinite for $p > 3$; it is known to be finite for $p = 3$ (and $p = 2$, which again is outside our discussion). Our construction of $\mathcal{P}(K^p)$ establishes the existence part of this conjecture for all primes $p \geqslant 3$. In fact, restricting ourselves to $p = 3$ for the moment, $\Gamma(\mathcal{U}_{K^3})$ also has the same order as $\Lambda = \langle\rho_0, \ldots, \rho_5\rangle$ [12, Table 12E1]. Taking a cue from (14), we let

$$(\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5) := (\rho_2, \rho_1, \rho_0, \rho_3\rho_2\rho_1\rho_2\rho_3, \rho_4, \rho_5).$$

It is routine to check that the τ_i's satisfy the defining relations for $\mathcal{U}_{K^3}$. Next consider the new Petrie element

$$\pi_1 := \tau_0\tau_1\tau_2\tau_3\tau_4\tau_5 = (-1, h_1),$$

where $h_1 := t_0t_1t_2t_3t_4t_5$. Here h_1 has characteristic polynomial $x^6 - x^5 - x^4 - x^2 - x + 1$, so that

$$h_1^{13} = 60h_1^5 + 48h_1^4 + 24h_1^3 + 42h_1^2 + 15h_1 - 34e,$$

and hence $h_1^{13} \equiv -e \bmod 3$. Thus $\pi_1^{13} = (-1, -e)$, and both π_1 and h_1 have period 26. Moreover, we find that the subgroup generated by the τ_i's contains the crucial element

$$\bigl(\tau_1\pi_1^{13}\bigr)^2 = (1, -t_1)(1, -t_1) = (-1, e).$$

Thus, $\Gamma(\mathcal{U}_{K^3}) \simeq \Lambda = \langle\tau_0, \ldots, \tau_5\rangle$.

The Polytopes $\mathcal{P} = \mathcal{P}(H^p)$.

For each prime $p \geqslant 3$, the polytope $\mathcal{P}(H^p)$ inherits self-duality from the hyperbolic tessellation $\{3,3,4,3,3\}$. To verify this claim, we first use (12) to establish a basic system of roots c_i for the s_i, namely

$$c_0 := b_1, \qquad c_1 := b_0, \qquad c_2 := r_2(b_1) = b_1 + b_2, \qquad c_3 := b_3, \qquad c_4 := b_4, \qquad c_5 := b_5.$$

(The Gram matrix $[c_i \cdot c_j]$ for these is encoded in diagram (13).) Over a suitable extension of the field $\mathbb{Z}_p$, we may now define an isometry w on V by mapping $c_i \mapsto \alpha_i c_{5-i}$, where $\alpha_i = 1/\sqrt{2}$ for $i \leqslant 2$, $\alpha_i = \sqrt{2}$ for $i > 2$. Thus, $w^2 = e$, $ws_jw = s_{5-j}$, and w induces a *polarity* in the polytope $\mathcal{P}(H^p)$. The facets of $\mathcal{P}(H^p)$ are toroids $\{3,3,4,3\}_{(p,0,0,0)}$; its vertex-figures are the dual toroids $\{3,4,3,3\}_{(p,0,0,0)}$.

Consulting [12, 12D3], we note that the self-dual universal polytope

$$\mathcal{U}_{H^p} := \bigl\{\{3,3,4,3\}_{(p,0,0,0)}, \{3,4,3,3\}_{(p,0,0,0)}\bigr\}$$

is conjectured to exist for all primes $p \geqslant 3$ and to be infinite for $p > 3$, but is actually known to be finite only for $p = 3$ (and $p = 2$). Our construction of $\mathcal{P}(H^p)$ again establishes the existence part of the conjecture for all primes $p \geqslant 3$. Unexpectedly, considering our previous look at $\mathcal{U}_{G^3}$ and $\mathcal{U}_{K^3}$, we have that $\mathcal{U}_{H^3}$ is a 9-fold cover of $\mathcal{P}(H^3)$ (see [12, Table 12D1]). After constructing $\mathcal{U}_{H^3}$, we will see that the group Λ from above reappears here as the automorphism group for a *non-self-dual* 3-fold cover of $\mathcal{P}(H^3)$; see (21) below.

To start the construction we use the automorphism induced on H^3 by w to extend the earlier action of H^3 $(= G^3)$ on $(\mathbb{Z}_3, +)$ to an action on $\mathbb{Z}_3 \oplus \mathbb{Z}_3$:

$$g(y_1, y_2) := \bigl(\theta(wgw)\det(g)y_1, \theta(g)\det(g)y_2\bigr),$$

for all $y_1, y_2 \in \mathbb{Z}_3$, $g \in H^3$. In the semidirect product

$$\Sigma := (\mathbb{Z}_3 \oplus \mathbb{Z}_3) \rtimes H^3,$$

with multiplication given by

$$(y_1, y_2, g) \cdot (z_1, z_2, h) = \bigl(y_1 + \theta(wgw)\det(g)z_1, y_2 + \theta(g)\det(g)z_2, gh\bigr),$$

we define

$$\sigma_i := (\delta_{i4}, \delta_{i1}, s_i), \quad 0 \leqslant i \leqslant 5.$$

It is a straightforward calculation to check that these σ_i satisfy the defining relations for the automorphism group $\Gamma(\mathcal{U}_{H^3})$. The work is halved by first noting that the map

$$\begin{aligned} \delta : \Sigma &\to \Sigma, \\ (y_1, y_2, g) &\mapsto (y_2, y_1, wgw) \end{aligned} \tag{18}$$

defines an involutory automorphism which transposes each pair σ_i, σ_{5-i}. (Thus δ must induce the standard polarity on the self-dual universal polytope $\mathcal{U}_{H^3}$.) It remains only to check that the σ_i's generate Σ, since then Σ and $\Gamma(\mathcal{U}_{H^3})$, having equal orders, must be isomorphic.

So consider

$$\pi_2 := \sigma_0\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5 = (-1, -1, h_2),$$

where $h_2 := s_0s_1s_2s_3s_4s_5$ has period 26 and satisfies $h_2^{13} = -e$ in H^3. Then from $\gamma_0 := \sigma_0\pi_2^{13} = (-1, 1, -s_0)$ and dually $\gamma_5 := \sigma_5\pi_2^{-13} = (1, -1, -s_5)$ we obtain

$$\gamma_0^2 = (0, -1, e), \qquad \gamma_5^2 = (-1, 0, e), \tag{19}$$

so that $\Sigma = \langle\sigma_0, \ldots, \sigma_5\rangle \simeq \Gamma(\mathcal{U}_{H^3})$. Furthermore, it is clear that the projection

$$\begin{aligned} \varphi : \Sigma &\to \Lambda, \\ (y_1, y_2, g) &\mapsto (y_2, g) \end{aligned} \tag{20}$$

yields yet another set of generators $\bar{\sigma}_i = \varphi(\sigma_i)$ for the group Λ. Thus $\Lambda = \langle\bar{\sigma}_0, \ldots, \bar{\sigma}_5\rangle$ is the automorphism group for an intermediate polytope $\mathcal{P}(\Lambda)$, still of type

$$\bigl\{\{3, 3, 4, 3\}_{(3,0,0,0)}, \{3, 4, 3, 3\}_{(3,0,0,0)}\bigr\},$$

but now a 3-fold cover of $\mathcal{P}(H^3)$. From (19) we get that $\varphi(\gamma_0^2)$ and $\varphi(\gamma_5^2)$ have different periods in Λ, so that $\mathcal{P}(\Lambda)$ is not self-dual. Evidently the other projection $\varphi^* : (y_1, y_2, g) \mapsto (y_1, g)$ yields the automorphism group of the dual polytope $\mathcal{P}(\Lambda)^*$. The situation is summarized here:

$$\begin{tikzcd} & \mathcal{P}(\Lambda) \arrow[dr, "3{:}1"] & \\ \mathcal{U}_{H^3} \arrow[ur, "\varphi", "3{:}1"'] \arrow[dr, "\varphi^*", "3{:}1"'] & & \mathcal{P}(H^3) \\ & \mathcal{P}(\Lambda)^*. \arrow[ur, "3{:}1"'] & \end{tikzcd} \tag{21}$$

As interesting as the results in this section are, it seems that in order to make further progress with the conjectures in [12, §12 C,D,E] concerning locally toroidal polytopes, we must relax our restriction to a prime modulus p in favor of a more general (composite) modulus s. This necessitates a somewhat different plan of attack, which we shall pursue in [16].

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