Cycle-finite algebras

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Abstract

Let $A$ be a finite-dimensional $K$-algebra over an algebraically closed field $K$ and $\text{mod} A$ be the category of finitely generated right $A$-modules. Following [1], $A$ is said to be cycle-finite if, for every cycle $M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_n = M_0$ of non-zero non-isomorphisms between indecomposable modules in $\text{mod} A$, the morphisms on this cycle do not belong to the infinite power of the Jacobson radical of $\text{mod} A$. In this article we describe the supports of stable tubes of the Auslander–Reiten quivers of cycle-finite algebras. As a consequence we get that every cycle-finite algebra is of polynomial growth. Moreover, we prove some characterizations of domestic cycle-finite algebras.

0. Introduction

Let $K$ be an algebraically closed field, and $A$ be a finite-dimensional $K$-algebra. Denote by $\text{mod} A$ the category of finite generated right $A$-modules. By a cycle in $\text{mod} A$ is meant a sequence $M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_n = M_0$ of non-zero non-isomorphisms between indecomposable modules. Recent investigations in the representation theory of algebras showed that study of cycles in $\text{mod} A$ leads to important information on indecomposable $A$-modules, the Auslander–Reiten quiver of $A$, and the ring structure of $A$ (see the author's survey article [19]). In this work we are concerned with the representation type of cycle-finite algebras. Recall that following [1] an algebra $A$ is called cycle-finite if, for every cycle in $\text{mod} A$, all morphisms on this cycle do not belong to the infinite power $\text{rad}^\infty(\text{mod} A)$ of the Jacobson radical $\text{rad}(\text{mod} A)$ of $\text{mod} A$. Examples of cycle-finite algebras are all representation-finite algebras, tame tilted algebras [7, 14], tubular algebras [14], iterated tubular algebras [12], and multicoil algebras [2, 3]. It is known (see [1]) that every cycle-finite algebra is of tame representation type. Then, by [4], for any dimension $d > 0$, all but a finite number of isomorphism classes of indecomposable $A$-modules of dimension $d$ lie in stable tubes of rank 1.
The main result of this paper describes the supports of stable tubes in the Auslander–Reiten quivers of cycle-finite algebras. Namely, we prove that the Auslander–Reiten quiver $\Gamma_A$ of a cycle-finite algebra $A$ admits a sincere stable tube if and only if $A$ is either tame concealed or tubular. As a consequence we get that every cycle-finite algebra is of polynomial growth, that is, there is a natural number $m$ such that the indecomposable modules occur, in each dimension $d \geq 1$, in a finite number of discrete and at most $d^m$ one-parameter families. These results extend the corresponding results proved for multicoil algebras in [2]. Moreover, we prove that, for a cycle-finite algebra $A$, the following conditions are equivalent: (i) $A$ is domestic, (ii) $A$ does not contain a tubular algebra as a full convex subcategory, (iii) the infinite radical $\text{rad}^\infty(\text{mod} A)$ is nilpotent, and (iv) all but a finite number of components in $\Gamma_A$ are stable tubes of rank 1.

The paper is organized as follows. In Section 1 we fix the notions and recall the needed definitions. Section 2 is devoted to semi-regular tubes. In Section 3 we describe the Auslander–Reiten components of cycle-finite algebras having sincere indecomposable modules lying in stable tubes. Sections 4 and 5 contain the proofs of our main results.

1. Preliminaries

Throughout this paper, $K$ will denote a fixed algebraically closed field. By an algebra $A$ is meant an associative finite-dimensional $K$-algebra with an identity, which we shall moreover assume to be basic and connected. In this case, there exists a connected bound quiver $(Q_A, I)$ and an isomorphism $A \simeq KQ_A/I$. Also, $A = KQ_A/I$ can equivalently be considered as a $K$-category, of which the object class is the set $(Q_A)_0$ of vertices of $Q_A$, and the set of morphisms $A(x, y)$ from $x$ to $y$ is the quotient of the $K$-vector space $KQ_A(x, y)$ having as a basis the set of paths in $Q_A$ from $x$ to $y$ by the subspace $I(x, y) = I \cap KQ_A(x, y)$, see [6]. If $Q_A$ has no oriented cycle, then $A$ is said to be triangular. A full subcategory $C$ of $A$ is said to be convex if any path in $Q_A$ with source and target in $Q_C$ lies entirely in $Q_C$.

Let $A$ be an algebra. By an $A$-module is meant a finitely generated right $A$-module. We shall denote by mod $A$ the category of $A$-modules, by $\text{rad}(\text{mod} A)$ the Jacobson radical of $\text{mod} A$, and by $\text{rad}^\infty(\text{mod} A)$ the intersection of all powers $\text{rad}^i(\text{mod} A)$, $i \geq 0$, of $\text{rad}(\text{mod} A)$. A path in $\text{mod} A$ is a sequence of non-zero non-isomorphisms $M_0 \to M_1 \to \cdots \to M_n$, where the modules $M_i$ are indecomposable. If $M_0 \cong M_n$, such a path is said to be a cycle. A cycle in $\text{mod} A$ is said to be finite if no morphism on this cycle lies in $\text{rad}^\infty(\text{mod} A)$. If all cycles in $\text{mod} A$ are finite then $A$ is said to be cycle-finite [1]. An indecomposable $A$-module $M$ is said to be directing if it lies on no cycle in $\text{mod} A$. For $i \in (Q_A)_0$, we denote by $S_A(i)$ the corresponding simple $A$-module and by $P_A(i)$ the projective cover of $S_A(i)$. For an $A$-module $M$, its support $\text{supp} M$ is the full subcategory of $A$ consisting of all objects $i \in (Q_A)_0$ such that $\text{Hom}_A(P_A(i), M) \neq 0$. If $\text{supp} M = A$ then the module $M$ is said to be sincere.
For an algebra $A$, we shall denote by $\Gamma_A$ the Auslander–Reiten quiver of $A$, and by $\tau_A = D\text{Tr}$ and $\tau_A^\perp = \text{Tr}D$ the Auslander–Reiten translations in $\Gamma_A$. We shall agree to identify the vertices of $\Gamma_A$ with the corresponding indecomposable $A$-modules. By a component of $\Gamma_A$ we mean a connected component in $\Gamma_A$. Let $\mathcal{C}$ be a component of $\Gamma_A$. Then $\mathcal{C}$ is said to be regular if $\mathcal{C}$ contains neither a projective module nor an injective module, and semi-regular if $\mathcal{C}$ does not contain both a projective and an injective module [9]. Following [14], $\mathcal{C}$ is said to be convex if, for any path $M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_t$ in mod $A$ with $M_0$ and $M_t$ from $\mathcal{C}$, all modules $M_i$ belong to $\mathcal{C}$. We shall say that $\mathcal{C}$ is standard if the full subcategory of mod $A$ formed by the vertices of $\mathcal{C}$ is equivalent to the mesh-category $K(\mathcal{C})$ of $\mathcal{C}$ [14]. Finally, $\mathcal{C}$ is said to be generalized standard if $\text{rad}^\infty(X, Y) = 0$ for all modules $X$ and $Y$ from $\mathcal{C}$ [17]. It is known [10] that every standard component is generalized standard. Examples of convex and (generalized) standard components are provided by preprojective components and preinjective components (see [19]). Recall that a component $\mathcal{C}$ is called preprojective (respectively, preinjective) if $\mathcal{C}$ contains no oriented cycle and each module in $\mathcal{C}$ belongs to the $\tau_A$-orbit of a projective module (respectively, an injective module). Finally, the support $\text{supp}\mathcal{C}$ of $\mathcal{C}$ is the full subcategory of $A$ formed by all objects $x$ such that $\text{Hom}_A(P_A(x), M) \neq 0$ for some $M \in \mathcal{C}$. An algebra $A$ is said to be tame if, for any dimension $d$, there is a finite number of $K[x]$-$A$-bimodules $M_i$ which are finitely generated and free as left $K[x]$-modules, and satisfy the following condition: all but a finite number of isomorphism classes of indecomposable $A$-modules of dimension $d$ are of the form $K[x]/(x - \lambda) \otimes_{K[x]} M_i$ for some $\lambda \in K$ and for some $i$. Let $\mu_A(d)$ be the least number of bimodules $M_i$ satisfying the above condition. Then $A$ is said to be of polynomial growth (respectively, domestic) if there is a natural number $m$ such that $\mu_A(d) \leq d^m$ (respectively, $\mu_A(d) \leq m$) for all $d \geq 1$. It was shown in [5] that this concept of a domestic algebra coincides with that introduced in [13]. Well-known examples of domestic algebras are tame concealed algebras [14]. On the other hand, the tubular algebras [14] are non-domestic of polynomial growth (see [15]). Moreover, it is known [1, (1.4)], that any cycle-finite algebra is tame.

For more details on the above notions we refer the reader to [14, 15].

2. Semi-regular tubes

A stable tube of rank $r$ ($r \geq 1$) is a translation quiver of the form $\mathbb{Z}A_\infty/(\tau')$. By a coray tube (respectively, ray tube) is meant a translation quiver which can be obtained from a stable tube by a finite number of coray insertions (respectively, ray insertions). Recall that a coray in a translation quiver $\Gamma$ is an infinite sectional path

$$\cdots \rightarrow X_{n+1} \rightarrow X_n \rightarrow \cdots \rightarrow X_2 \rightarrow X_1$$

in $\Gamma$ with pairwise different vertices such that for each integer $i \geq 1$, the path $X_{i+1} \rightarrow X_i \rightarrow \cdots \rightarrow X_2 \rightarrow X_1$ is the unique sectional path of length $i$ in $\Gamma$ which ends.
at $X_1$. A ray of a translation quiver is defined dually. We shall agree, by abuse of the language, to consider a stable tube as a coray tube and as a ray tube. For more details on tubes we refer to [14]. It is known that a coray tube (respectively, ray tube) of an Auslander–Reiten quiver $\Gamma_A$ is standard if and only if it is generalized standard (see [10, 20]).

We have the following theorem proved in [9, (2.6)].

**Theorem 2.1.** Let $A$ be an algebra and $\mathcal{C}$ be a semi-regular component of $\Gamma_A$ containing an oriented cycle. Then $\mathcal{C}$ is either a coray tube or a ray tube.

We shall need also the following facts.

**Proposition 2.2.** Let $A$ be an algebra and $B$ be a full convex subcategory of $A$. Assume that $\mathcal{T}$ is a non-regular coray tube in $\Gamma_B$ and $\mathcal{C}$ a standard semi-regular component of $\Gamma_A$ containing an oriented cycle and all non-directing modules of $\mathcal{T}$. Then $\mathcal{C}$ is a non-regular coray tube and all corays of $\mathcal{T}$ are complete corays of $\mathcal{C}$.

**Proof.** From the above theorem we know that $\mathcal{C}$ is either a coray tube or a ray tube. Suppose that $\mathcal{C}$ is a ray tube. Then all irreducible maps in mod $A$ corresponding to the arrows of rays in $\mathcal{C}$ are monomorphisms. Clearly, we may consider the $B$-modules as $A$-modules. Since $\mathcal{T}$ is a coray tube of $\Gamma_B$ containing an injective module, there exist two irreducible epimorphisms $I \xrightarrow{u} Z$ and $Y \xrightarrow{v} Z$ in mod $B$ such that $I, Y, Z$ are non-directing modules in $\mathcal{T}$ and $I$ is injective. Consider now the minimal right almost split map $M \oplus N \xrightarrow{u,v} Z$ in mod $A$ ending at $Z$, where $M$ is indecomposable, and $N$ is indecomposable or zero. If $N \neq 0$, then one of the maps $u$ or $v$, say $v$, is a monomorphism, because $\mathcal{C}$ is a ray tube. But then $N$ is a $B$-module and $v$ is an irreducible monomorphism in mod $B$, a contradiction with our choice of $Z$; hence $N = 0$. Since $\mathcal{C}$ is a standard ray tube, this implies that $I$ and $Y$ lie on one sectional path in $\mathcal{C}$ with target $Z$, and so either $f = gh$ for some $h \in \text{Hom}_B(I, Y)$ or $g = fp$ for some $p \in \text{Hom}_B(Y, I)$. We have again a contradiction because $f$ and $g$ are irreducible maps corresponding to different arrows of $\mathcal{T}$. Therefore, $\mathcal{C}$ is a coray tube and contains an injective module. Moreover, all irreducible maps in mod $A$ corresponding to the arrows of corays in $\mathcal{C}$ are epimorphisms. We then infer that every coray of $\mathcal{T}$ is a complete coray of $\mathcal{C}$. This finishes our proof.

Dually, we have the following:

**Proposition 2.3.** Let $A$ be an algebra and $B$ be a full convex subcategory of $A$. Assume that $\mathcal{T}$ is a non-regular ray tube in $\Gamma_B$ and $\mathcal{C}$ a standard semi-regular component of $\Gamma_A$ containing all non-directing modules of $\mathcal{T}$. Then $\mathcal{C}$ is a non-regular ray tube and all rays of $\mathcal{T}$ are complete rays of $\mathcal{C}$.
3. Auslander–Reiten components of cycle-finite algebras

We shall need the following lemma proved in [2, (2.7)].

**Lemma 3.1.** Let $A$ be a cycle-finite algebra and $X$ be an indecomposable module lying in a stable tube of $\Gamma_A$ such that, for all $m \geq 0$, $\tau^m X$ is sincere. Then

(i) If $P$ is an indecomposable projective $A$-module, then for any $t \geq 0$, $\text{Hom}_A(\tau^{-t} P, \tau^{-t} X) \neq 0$.

(ii) If $I$ is an indecomposable projective $A$-module, then for any $s \geq 0$, $\text{Hom}_A(\tau^s X, \tau^s I) \neq 0$.

The following proposition will play a crucial role in our investigations.

**Proposition 3.2.** Let $A$ be a cycle-finite algebra having a sincere indecomposable module lying in a stable tube of $\Gamma_A$. Then every component in $\Gamma_A$ is semi-regular.

**Proof.** Let $\mathcal{F}$ be a stable tube of $\Gamma_A$ which contains a sincere indecomposable module, say $X$. Then all but finitely many modules in $\mathcal{F}$ are sincere. Therefore, we may assume that $\tau^m X$ is sincere for all $m \geq 0$. Then, by Lemma 2.1, we get $\tau^{-t} P \neq 0$ for any indecomposable projective $A$-module $P$ and $t \geq 0$. Similarly, $\tau^s I \neq 0$ for any indecomposable injective $A$-module $I$ and $s \geq 0$. Let $\mathcal{C}$ be a component in $\Gamma_A$. Suppose that $\mathcal{C}$ is not semi-regular. Consider the right stable part $\mathcal{C}_r$ of $\mathcal{C}$ obtained from $\mathcal{C}$ by removing the $A$-orbits of injective modules. Then $\mathcal{C}_r$ admits a connected component $\mathcal{D}$ containing a projective module $P$. Since $\mathcal{C}_r$ is connected and not semi-regular, there is in $\mathcal{C}$ an arrow $I \rightarrow Z$ with $I$ injective and $Z$ from $\mathcal{D}$. Observe that $\mathcal{D}$ contains a path from $Z$ to some $\tau^{-r} P$, $r \geq 0$. Indeed, since $\mathcal{D}$ is connected, there is a walk $Z = Y_m \rightarrow \cdots \rightarrow Y_1 - Y_0 = P$, $m \geq 1$, in $\mathcal{D}$, where $Y_i - Y_{i+1}$ means $Y_i \rightarrow Y_{i+1}$ or $Y_{i+1} \rightarrow Y_i$. Then we prove our claim by induction on $m$, using the right stability of $\mathcal{D}$. If $m = 1$, then either $Z \rightarrow P$ or $Z \rightarrow \tau_A^{-1} P$ is an arrow in $\mathcal{D}$. Assume that $m > 1$ and there is in $\mathcal{D}$ a path from $Y_{m-1}$ to $\tau_A^{-r} P$ for some $r \geq 0$. Then, either $Z \rightarrow Y_{m-1}$ or $Y_{m-1} \rightarrow Z$ is an arrow of $\mathcal{D}$. In the first case there is in $\mathcal{D}$ a path from $Z$ to $\tau_A^{-r} P$, and in the second case a path from $Z$ to $\tau_A^{-r} P$. Therefore, $\mathcal{C}$ contains a path from $I$ to $\tau_A^{-r} P$. But, by Lemma 2.1, we get $\text{Hom}_A(\tau_A^{-r} P, \tau_A^{-r} X) \neq 0$, and $\text{Hom}_A(\tau_A^{-r} P, \tau_A^{-r} I) \neq 0$ because $\tau_A^{-r} X$ is sincere. Since $\mathcal{C} \notin \mathcal{F}$ and $\tau_A^{-r} X \in \mathcal{F}$, there is an infinite cycle $I \rightarrow \cdots \rightarrow \tau_A^{-r} P \rightarrow \tau_A^{-r} X \rightarrow I$, which gives the required contradiction.

**Proposition 3.3.** Let $A$ be a cycle-finite algebra, and $\mathcal{C}$ be a semi-regular component. The $\mathcal{C}$ is generalized standard, convex, and one of the following forms: preprojective of Euclidean type, preinjective of Euclidean type, ray tube, or coray tube.

**Proof.** Without loss of generality, we may assume that $\mathcal{C}$ has no injective modules. Consider first the case when $\mathcal{C}$ contains an oriented cycle. Then, by Theorem 2.1, $\mathcal{C}$ is a ray tube. In this case, there is a cofinite full translation subquiver $\mathcal{D}$ of $\mathcal{C}$, formed by
all non-directing modules in $\mathcal{C}$, such that for any modules $M$ and $N$ in $\mathcal{D}$ we have paths from $M$ to $N$. Suppose that $\text{rad}^*(X, Y) \neq 0$ for some $X$ and $Y$ in $\mathcal{C}$. Then $Y$ belongs to $\mathcal{D}$ and there is a module $Z$ in $\mathcal{D}$ such that $\text{rad}^*(Z, Y) \neq 0$. Hence, there is an infinite path $Y \rightarrow \cdots \rightarrow Z \rightarrow Y$, a contradiction because $A$ is cycle-finite. Consequently, $\mathcal{C}$ is generalized standard. This implies that $\mathcal{C}$ is also convex. Indeed, if this is not the case, then there is a path in $\text{mod } A$ of the form $X \rightarrow U \rightarrow \cdots \rightarrow V \rightarrow Y$ with $X, Y$ in $\mathcal{C}$ and $U, V$ not in $\mathcal{C}$. Then $\text{rad}^*(X, U) \neq 0$, $\text{rad}^*(V, Y) \neq 0$, and so, as above, there is an infinite cycle $M \rightarrow U \rightarrow \cdots \rightarrow V \rightarrow N \rightarrow \cdots \rightarrow M$, for some $M$ and $N$ in $\mathcal{D}$.

Assume that $\mathcal{C}$ does not contain oriented cycle. Then, by [9,(3.7)], there exists a quiver $A$ containing no oriented cycle such that $\mathcal{C}$ is isomorphic to a full translation subquiver of $ZA$ which is closed under successors. Since $A$ is cycle-finite, we infer that $\mathcal{C}$ consists entirely of directing modules. Hence, by [21], $A$ is finite. Moreover, then $\mathcal{C}$ is generalized standard (see [16, Lemma 5]). Then, by [17,(3.8)], $\mathcal{C}$ is a preprojective component of Euclidean type, because $A$ is tame as a cycle-finite algebra. Clearly, for any path $X \rightarrow \cdots \rightarrow Y$ in $\text{mod } A$ with $Y$ in $\mathcal{C}$, the module $X$ also belongs to $\mathcal{C}$, and so $\mathcal{C}$ is convex.

**Corollary 3.4.** Let $A$ be a cycle-finite algebra such that every component of $A$ is semi-regular. Then $A$ is triangular.

**Proof.** Suppose that $A$ is not triangular. Then there is a cycle in $\text{mod } A$ $P_0 \rightarrow P_1 \rightarrow \cdots \rightarrow P_m = P_0$ with all $P_i$ projective. Since $A$ is cycle-finite, the modules $P_1, \ldots, P_m$ belong to one ray tube $\mathcal{T}$. From Proposition 2.3 we deduce that $\mathcal{T}$ is (generalized) standard. But then there is no cycle in $\text{mod } A$ formed by projective modules from $\mathcal{T}$, a contradiction. Therefore, $A$ is triangular.

**4. Cycle-finite algebras with sincere stable tubes**

We shall prove the following characterization of tame concealed and tubular algebras.

**Theorem 4.1.** Let $A$ be an algebra. The following conditions are equivalent:

(i) $A$ is cycle-finite and admits a sincere indecomposable module lying in a stable tube of $\Gamma_A$.

(ii) $A$ is either tame concealed or tubular.

**Proof.** The implication (ii) $\Rightarrow$ (i) follows from [14, (4.3) and (5.2)]. We shall prove that (i) implies (ii). The proof will be done in several steps.

Let $A$ be cycle-finite and admits a sincere indecomposable module lying in a stable tube of $\Gamma_A$. Then, by Corollary 3.4, $A$ is triangular. Moreover, by Propositions 3.2 and 3.3, we infer that every component in $\Gamma_A$ is (generalized) standard, convex, and one of
the following forms: preprojective of Euclidean type, preinjective of Euclidean type, ray tube, or coray tube. We may assume that A is not tame concealed.

(1) We claim that $\Gamma_A$ admits a preprojective component and a preinjective component. Let $\Sigma_A$ be the set of all components in $\Gamma_A$. Since the components in $\Gamma_A$ are generalized standard and convex, we may endow $\Sigma_A$ with the partial order $\leq$ being the transitive closure of: for $\mathcal{C}$ and $\mathcal{D}$ in $\Sigma_A$ define

$$\mathcal{C} \leq \mathcal{D} \iff \text{Hom}_A(X, Y) \neq 0 \text{ for some } X \in \mathcal{C} \text{ and } Y \in \mathcal{D}. $$

We claim that a component $\mathcal{C}$ of $\Gamma_A$ is a minimal element of $\Sigma_A$, with respect to $\leq$, if and only if $\mathcal{C}$ is preprojective. The sufficiency is clear because if $\mathcal{C}$ is preprojective and $\text{Hom}_A(X, Y) \neq 0$ for some $Y \in \mathcal{C}$ then also $X$ belongs to $\mathcal{C}$. Assume now that $\mathcal{C}$ is a (ray or coray) tube. We shall show that there is an indecomposable projective $A$-module $P \not\in \mathcal{C}$ such that $\text{Hom}_A(P, Z) \neq 0$ for some $Z \in \mathcal{C}$. This will imply that $\mathcal{C}$ is not minimal in $\Sigma_A$. It is enough to consider only the case when $\mathcal{C}$ is a ray tube containing a projective module. In this case, there is an indecomposable projective module $P'$ in $\mathcal{C}$ such that a non-directing non-projective indecomposable direct summand $Z$. Since $\mathcal{C}$ is a standard ray tube we get that $\text{Hom}_A(P'', Z) = 0$ for any projective module $P''$ from $\mathcal{C}$. Hence, $\text{Hom}_A(P, Z) \neq 0$ for some indecomposable projective module $P$ which is not in $\mathcal{C}$. Similarly, we prove a component $\mathcal{D}$ is a maximal element of $\Sigma_A$ if and only if $\mathcal{D}$ is preinjective. Clearly, $\Sigma_A$ admits both a minimal and a maximal element because the number of non-regular components is finite.

(2) Let $\mathcal{P}$ be a preprojective component of $\Gamma_A$. Since $\mathcal{P}$ contains no injective module, there exists a hereditary algebra $H$ of Euclidean type and a tilting $H$-module $T$ without preprojective direct summands such that the tilted algebra $B = \text{End}_H(T)$ is a full convex subcategory of $A$ and $\mathcal{P}$ is the preprojective component of $\Gamma_B$ (see dual of (2.5) in [21]). Moreover, then there is a full convex tame concealed subcategory $C$ of $B$ such that $B$ is a domestic tubular coextension of $C$ (see [14, (4.7) and (4.9)]). We shall show that $B = C$. Suppose this is not the case. Then $\Gamma_B$ admits at least one coray tube $\mathcal{O}$ containing an injective module. The non-directing modules in $\mathcal{O}$ form a cofinite full translation subquiver $\Omega$ of $\mathcal{O}$, and for any two modules $U$ and $V$ in $\Omega$ there is a cycle in $\mathcal{O}$ passing through $U$ and $V$. Since $B$ is a full convex subcategory of $A$, we may consider the $B$-modules as $A$-modules. Using now the fact that $A$ is cycle-finite, we infer that all modules of $\mathcal{O}$ belong to one component, say $\mathcal{C}$, of $\Gamma_A$. We known that $\mathcal{C}$ is semi-regular and standard, and so, by Proposition 2.2, $\mathcal{C}$ is a non-regular coray tube and all corays of $\mathcal{O}$ are complete corays of $\mathcal{C}$. We claim that in fact $\mathcal{C} = \mathcal{O}$. Observe that $A$ does not contain full subcategory which is a one-point coextension $[N]B$ of $B$ by some non-zero $B$-module $N$. This follows from the duals of [14, p. 88] and the fact that $\mathcal{P}$ is a full component of $\Gamma_A$ containing all projective $B$-modules. By the same reason, $A$ does not contain full subcategory which is a one-point extension $B[M]$ of $B$ by a $B$-module $M$ which has a direct summand from $\mathcal{P}$. Suppose now that $\mathcal{C} \neq \mathcal{O}$. Then, applying [14, p. 88] again, we deduce that there is inside $A$ a one-point extension $B[L]$ of $B$ by a module $L$ which has a summand from $\mathcal{O}$, and hence
C contains a projective module. This contradiction shows that $C = \emptyset$. Therefore, we proved that every non-regular coray tube of $\Gamma_b$ is a full component of $\Gamma_a$. As a consequence we get that, for any one-point extension $B[W]$ of $B$ inside $A$, the module $W$ has no indecomposable direct summand which belongs to a non-regular coray tube of $\Gamma_b$. Moreover, this implies that if $M$ is an indecomposable $A$-module and $\text{Hom}_A(P, M) \neq 0$ for each indecomposable projective $A$-module $P$ from $\mathcal{P}$, then $M$ belongs to $\mathcal{P}$ or to one of the non-regular coray tubes in $\Gamma_b$. But this contradicts the fact that $A$ admits a sincere indecomposable $A$-module lying in a stable tube of $\Gamma_a$. Consequently, all tubes of $\Gamma_b$ are stable, and so $B = C$.

(3) We claim now that there is a full convex subcategory $D$ of $A$ which is a non-dominant tubular extension of $C$. Then $D$ will be a tubular algebra, because $A$ is tame (see [11, (2.1)]). First observe that if $\mathcal{T}_x, x \in \mathbb{P}_1(k)$, is a stable tube in $\Gamma_a$ then, by the above remarks and Propositions 2.2 and 2.3, we conclude that there is a ray tube $\mathcal{T}_x'$ in $\Gamma_a$ such that all rays of $\mathcal{T}_x$ are complete rays in $\mathcal{T}_x'$. Moreover, if $\lambda$ and $\rho$ are different elements of $\mathbb{P}_1(k)$, then $\mathcal{T}_x' \neq \mathcal{T}_x$ since the tubes $\mathcal{T}_x$ and $\mathcal{T}_x'$ are orthogonal. Hence, there is a full convex subcategory $D$ of $A$ such that $D$ is a tubular extension of $C$ and all tubes $\mathcal{T}_x', x \in \mathbb{P}_1(k)$, are components of $\Gamma_a$. We shall show now that $D$ is non-domestic. Suppose this is not the case. Then $D$ is a tilted algebra of Euclidean type which admits a complete slice $\mathcal{A}$ in its preinjective components $\mathcal{I}$. Applying again the duals of [14, p. 88], we get that $A$ does not contain full subcategory which is a one-point coextension $[F]D$ of $D$, because $\mathcal{P}$ and the tubes $\mathcal{T}_x$ are components of $\Gamma_a$. Since $A \neq C$ and $A$ admits an indecomposable sincere module lying in a stable tube, we have also that $D \neq A$. Then there is an integer $m$ such that the full translation subquiver $\mathcal{S}$ of $\mathcal{I}$ formed by all predecessors of $\tau^mA$ is a full translation subquiver of $\Gamma_a$ and there is an arrow in $\Gamma_a$ from a module in $\tau^mA$ to a projective module. Hence, there is a component $\mathcal{D}$ in $\Gamma_a$ such that $\mathcal{S}$ is a left stable full translation subquiver of $\mathcal{D}$ which is closed under predecessors, and so $\mathcal{D}$ is a preinjective component. On the other hand, $\mathcal{D}$ is semi-regular and so $\mathcal{D}$ is semi-regular and contains a projective module. This contradiction shows that $D$ is a tubular algebra.

(4) We shall prove now that $A = D$. Recall from [14, (5.2)] that $\Gamma_D$ is of the form

$$\mathcal{P}^0 \vee \mathcal{I}^0 \vee \left( \bigvee_{q \in \mathbb{Q}^+} \mathcal{I}^q \right) \vee \mathcal{I}^\infty \vee \mathcal{D}^\infty,$$

where $\mathcal{P}^0$ is a preprojective component, $\mathcal{I}^\infty$ is a preinjective component, $\mathcal{I}^0$ is a $\mathbb{P}_1(k)$-family of ray tubes, $\mathcal{I}^\infty$ is a $\mathbb{P}_1(k)$-family of coray tubes and each $\mathcal{I}^q$, $q \in \mathbb{Q}^+$, is a $\mathbb{P}_1(k)$-family of stable tubes. The ordering from the left to the right indicates that there are non-zero morphisms only from any of $\mathcal{P}^0, \mathcal{I}^q, q \in \mathbb{Q} \cup \{ \infty \}, \mathcal{D}^\infty$ to itself and to the families on its right. Further, the tubes from the same family are pairwise orthogonal. In our case, $\mathcal{P}^0 = \mathcal{P}$ and $\mathcal{I}^0$ is the family $\mathcal{T}_x, x \in \mathbb{P}_1(k)$. Moreover, there is a full convex tame concealed subcategory $C'$ of $D$, different from $C$, such that $D$ is a tubular coextension of $C'$ and $\mathcal{D}^\infty$ is the preinjective component of $\Gamma_c$. Since $C \neq C'$, we have also that $\mathcal{I}^0$ admits a non-regular ray tube and $\mathcal{I}^\infty$ admits a non-regular coray tube. Finally, since $A$ is tame, we get by [2, (3.2)] that, if $D[M]$
(respectively, $[M]D$) is a one-point extension (respectively, coextension) of $D$ inside $A$, then all indecomposable direct summands of $M$ belong to $\mathcal{F}^\infty \cup Q^\infty$ (respectively, $\mathcal{P}^0 \cup \mathcal{F}^0$). But then there is no one-point coextension of $D$ inside $A$, because $\mathcal{P}$ and the tubes $\mathcal{F}_\lambda, \lambda \in \mathcal{P}_1(k)$, are full components of $\Gamma_A$. In particular, we infer that all tubes from the families $\mathcal{F}^q, q \in \mathbb{Q}^+$, are also full components of $\Gamma_A$. Let now $\Gamma$ be a non-regular coray tube from $\mathcal{F}^\infty$. Then, by Proposition 2.2 and the cycle-finitness of $A$, there is a non-regular coray tube $\Phi$ of $\Gamma_A$ such that all corays of $\Gamma$ are complete corays of $\Phi$. Observe that then $\Gamma = \Phi$ and hence $\Gamma$ is a full component of $\Gamma_A$. Indeed, otherwise there is a one-point coextension $[N]D$ of $D$ inside $A$ by an indecomposable module $N$ from $\Gamma$, which contradicts the above remarks. Since one of the ray tubes $\mathcal{F}_\lambda$ in $\mathcal{F}^0$ is also non-regular, we then infer that every stable tube of $\Gamma_A$ containing an indecomposable sincere $A$-module belongs to one of the families $\mathcal{F}^q, q \in \mathbb{Q}^+$. Therefore, $A = D$, and this finishes the proof.

**Corollary 4.2.** Let $A$ be a cycle-finite algebra, and $\mathcal{F}$ be a stable tube of $\Gamma_A$. Then the support $B$ of $\mathcal{F}$ is a full convex subcategory of $A$ which is tame concealed or tubular, and $\mathcal{F}$ is a component of $\Gamma_B$.

**Proof.** For the convexity of $B$ inside $A$ we may repeat the proof of Proposition 3.1 in [1]. Clearly, $B$ is a cycle-finite algebra and admits a sincere indecomposable module lying in the stable tube $\mathcal{F}$ of $\Gamma_B$. Hence, by Theorem 4.1, $B$ is either tame concealed or tubular.

**Theorem 4.3.** Let $A$ be a cycle-finite algebra. Then $A$ is of a polynomial growth.

**Proof.** We may clearly assume that $A$ is representation-infinite. Since $A$ is cycle-finite, it is tame. Then, by [4, Corollary E], for any dimension $d$, all but a finite number of isomorphism classes of indecomposable $A$-modules of dimension $d$ lie in stable tubes of rank 1. Let $\mathcal{F}$ be a stable tube of rank 1 in $\Gamma_A$. Then, by Corollary 4.2, $B = \text{supp} \mathcal{F}$ is a full convex subcategory of $A$ which is tame concealed or tubular, and hence is of polynomial growth. Since $A$ admits only finitely many full convex subcategories we infer that $A$ is of polynomial growth.

We get also the following characterization of minimal representation-infinite cycle-finite algebras.

**Corollary 4.4.** For an algebra $A$ the following conditions are equivalent:

(i) $A$ is tame concealed.

(ii) $A$ is cycle-finite, representation-infinite, and every full convex subcategory of $A$ is representation-finite.
Proof. The implication (i) ⇒ (ii) follows from [14, (4.3)]. Assume that (ii) holds. Then \( A \) is tame, and therefore, by [4], \( \Gamma_A \) admits a stable tube \( \mathcal{T} \). From Corollary 4.2 the support \( B \) of \( \mathcal{T} \) is a full convex subcategory of \( A \) which is tame concealed or tubular. Then \( A = B \) because \( B \) is representation-infinite. Moreover, by [14, (5.1)], every tubular algebra admits a proper full convex tame concealed subcategory. Hence, (ii) implies that \( A \) is tame concealed.

5. Domestic cycle-finite algebras

The aim of this section is to prove the following characterization of domestic cycle-finite algebras.

Theorem 5.1. Let \( A \) be a cycle-finite algebra. The following conditions are equivalent:

(i) \( A \) is domestic.
(ii) \( A \) does not contain a tubular algebra as a full convex subcategory.
(iii) \( \text{rad}^\infty(\text{mod} A) \) is nilpotent.
(iv) All but finitely many components of \( \Gamma_A \) are stable tubes of rank one.

Proof. It is known that every tubular algebra is non-domestic (see [15, (3.6)]) and hence (i) implies (ii). Conversely, if (ii) holds, then, by Corollary 4.2, the support of every stable tube in \( \Gamma_A \) is a tame concealed full convex subcategory of \( A \). Hence, as in the proof of Theorem 4.3, we get that \( A \) is domestic. Moreover, if \( C \) is a tame concealed algebra, then all but finitely many components in \( \Gamma_A \) are stable tubes of rank one. Hence, the implication (ii) ⇒ (iv) is a direct consequence of Proposition 3.3 and Theorem 4.1. Further, it was shown in [8, (1.5)] that, if \( B \) is a tubular algebra, then \( \text{rad}^\infty(\text{mod} B) \) is not nilpotent, and so (iii) implies (ii). Therefore, it remains to prove that (iv) implies (iii). We shall prove this implication in several steps.

1. Let \( \mathcal{C} \) be a component of \( \Gamma_A \). Consider the left stable part \( \mathcal{C}' \) of \( \mathcal{C} \) obtained from \( \mathcal{C} \) by removing the \( \tau_A \)-orbits of the projective modules and the arrows attached to them. Let \( \mathcal{D} \) be a connected component of \( \mathcal{C}' \). If \( \mathcal{D} \) contains no oriented cycle then, by [9, (3.4)], there exists a quiver \( A \) without oriented cycles such that \( \mathcal{D} \) is a full translation subquiver of \( \mathbb{Z}A \) which is closed under predecessors. Then \( \mathcal{D} \) admits a full translation subquiver \( \mathcal{D}' \) isomorphic to the translation quiver \( \mathbb{N}A \) and which is closed under predecessors in \( \mathcal{C} \). Since \( \mathcal{D}' \) contains no oriented cycle and every cycle in \( \text{mod} A \) is finite, we infer that all modules in \( \mathcal{D}' \) are directing. Hence, by [21, (2.4)], there is a representation-infinite hereditary algebra \( H \) and a tilting \( H \)-module \( T \) without preinjective direct summands such that the tilted algebra \( B = \text{End}_H(T) \) is a full convex subcategory of \( A \) and \( \mathcal{D}' \) is a full translation subquiver of the connecting component of \( \Gamma_B \) which is closed under predecessors. In particular, we get \( \text{rad}^\infty(X, Y) = 0 \) for all \( X \) and \( Y \) from \( \mathcal{D}' \), that is, \( \mathcal{D}' \) is generalized standard. Assume now that there is an oriented cycle in \( \mathcal{D} \). If \( \mathcal{D} \) contains a \( \tau_A \)-periodic module then \( \mathcal{D} \) is...
a stable tube. If $D$ does not contain a $\tau_A$-periodic module, then, by [9, (2.2) and (2.3)],
there is an infinite sectional path

$$\cdots \rightarrow \tau_A^r X_1 \rightarrow \tau_A^s X_2 \rightarrow \cdots \rightarrow \tau_A^s X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_2 \rightarrow X_1$$

in $D$ with $r > s$ such that $\{X_1, \ldots, X_s\}$ is a complete set of representatives of
the $\tau_A$-orbits in $D$. We may choose then $m \geq 0$ such that the full translation subquiver $D'$ of $D$ given by
the modules $\tau_A^i X_i$, $1 \leq i \leq s, j \geq mr$, has the following properties:

(a) For any two modules $Y$ and $Z$ in $D'$ there is a path in $D$ from $Y$ to $Z$,

(b) No module in $D'$ is a direct predecessor of a projective module in $C$.

Since every cycle in $\text{mod } A$ is finite, we infer from (a) that $D'$ is generalized standard.
Repeating this procedure to any connected component of $C$, we get a finite family $D_1', \ldots, D_t'$
of left stable generalized standard full translation subquivers of $C$ such that, for every $f \in \text{rad}(X, Y)$
with $Y$ in $C$, we have $f = hg$ for some $g \in \text{rad}(X, Z), h \in \text{Hom}(Z, Y)$ and $Z$ being a direct sum
of modules from $D_1', \ldots, D_t'$. Similarly, there exists a finite family $E_1', \ldots, E_p'$ of right stable
generalized standard full translation subquivers of $C$ such that, for every $u \in \text{rad}(X, Y)$ with $X$ in $C$, we have $u = uv$
for some $v \in \text{Hom}(X, Z), w \in \text{rad}(Z, Y)$ and $Z$ being a direct sum of modules from
$E_1', \ldots, E_p'$. Moreover, all but finitely many modules in $C$ belong to the union of
$D_1', \ldots, D_t', E_1', \ldots, E_p'$.

(2) We claim now that $A$ does not contain a tubular algebra as a full convex subcategory.
Suppose that $B$ is a full convex tubular subcategory of $A$. Clearly, we may consider the $B$-modules as $A$-modules.
We know from [14, (5.2)] that $\Gamma_B$ admits a family $\mathcal{T}_q, q \in \mathbb{Q}^+$, of stable tubes of ranks $> 1$ such that $\text{Hom}_B(\mathcal{T}_q, \mathcal{T}_q') \neq 0$ if and
only if $q - q'$. Observe that, for any fixed $q \in \mathbb{Q}^+$ and $U, V \in \mathcal{T}_q$, there exists a cycle in
$\text{mod } B$ passing through $U$ and $V$. Since $A$ is a cycle-finite, this implies that all modules
of any $\mathcal{T}_q$ belong to one component of $\Gamma_A$. Moreover, if $\tau_A M \cong M$ for an indecomposable $B$-module $M$ then $\tau_B M \cong M$, and so $M$ lies in a stable tube of rank 1 in $\Gamma_B$.
Therefore, since all but finitely many components of $\Gamma_A$ are stable tubes of rank 1, we
conclude that there is a component $C$ in $\Gamma_A$ which contains all modules from infinitely
many tubes $\mathcal{T}_q$. Then, in notation of (1), one of the translation quivers $\mathcal{D}_1, \ldots, \mathcal{D}_t, \mathcal{E}_1, \ldots, \mathcal{E}_p$
contains all modules of infinitely many tubes $\mathcal{T}_q$. Clearly, such a translation quiver consists of non-directing modules lying on finitely many infinite sectional paths. Then there exist $p, q \in \mathbb{Q}^+, p < q$, and a sectional path in $\Gamma_A$ with the source in
a module $N \in \mathcal{T}_q$ and the target in a module $L \in \mathcal{T}_p$. Then $\text{Hom}_B(N, L) \neq 0$, and this is
a contradiction because $\text{Hom}_B(\mathcal{T}_q, \mathcal{T}_p) = 0$.

(3) From (2) and Corollary 4.2 we conclude that the support of any stable tube of
rank one in $\Gamma_A$ is a tame concealed full convex subcategory of $A$. Since $A$ admits only
finitely many full convex subcategories, we may divide the set $\Sigma$ of all stable tubes of rank 1 in $\Gamma_A$
into a finite number of disjoint families $\Sigma_1, \ldots, \Sigma_r$ of pairwise orthogonal tubes.
Namely, we put two tubes $\mathcal{F}$ and $\mathcal{F}'$ from $\Sigma$ to the same family $\Sigma_i$ if and only if
there are $X \in \mathcal{F}$ and $X' \in \mathcal{F}'$ with the same dimension-vectors $\dim X = \dim X'$.
Then, by [18, (4.9)], we get that, if \( T \in \Sigma_i \), \( \Gamma \in \Sigma_j \) and \( \text{Hom}_A(T, \Gamma) \neq 0 \), then \( \text{Hom}_A(T', \Gamma') \neq 0 \) for any tubes \( T' \in \Sigma_i \) and \( \Gamma' \in \Sigma_j \).

(4) Since \( A \) is a cycle-finite and all but finitely many components in \( \Gamma_A \) are stable tubes of rank 1, combining (1) and (3) we easily infer that \( (\text{rad}^\infty(\text{mod} A))^m = 0 \) for sufficiently large \( m \). This finishes our proof. \( \square \)

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References