A Pattern of Asymptotic Vertex Valency Distributions in Planar Maps

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Received December 10, 1997

Let a vertex be selected at random in a set of \( n \)-edged rooted planar maps and \( p_k \) denote the limit probability (as \( n \to \infty \)) of this vertex to be of valency \( k \). For diverse classes of maps including Eulerian, arbitrary, polyhedral, and loopless maps as well as 2- and 3-connected triangulations, it is shown that non-zero \( p_k \) behave asymptotically in a uniform manner: 
\[
p_k \sim c (nk)^{-1/2} \left( \frac{\pi}{\xi} \right)^{\xi-1} R \quad \text{as} \quad k \to \infty
\]
with some constants \( c \) and \( \xi \) depending on the class. This distribution pattern can be reformulated in terms of the root vertex valency. By contrast, \( p_2 = 2^{-k} \) for the class of arbitrary plane trees and \( p_3 = (k-1) 2^{-k} \) for triangular dissections of convex polygons.

1. INTRODUCTION

1.1. Cardinality Pattern. In recent years, impressive results have been obtained in asymptotic enumeration of planar and, more general, topological maps. One of the most striking achievements is the discovery of a uniform asymptotic behavior pattern of the number of rooted maps. It can be presented in the following form (cf. [6])

\[
\left| \mathcal{X}_n \right| \sim C (\pi^{1/2}) n^{-5/2} R^n, \quad n \to \infty, \quad n \in \text{Dom}(\mathcal{X}), \quad (C)
\]

with algebraic constants \( C = C(\mathcal{X}) \) and \( R = R(\mathcal{X}) \) depending on the type of maps. Here \( \mathcal{X} \) stands for a class of rooted planar maps, \( \mathcal{X}_n \) denotes the set of maps in \( \mathcal{X} \) having \( n \) edges and \( \text{Dom}(\mathcal{X}) \subseteq \mathbb{N} \) denotes the set of all \( n \) for which \( \mathcal{X}_n \) is not empty (it is assumed that \( \text{Dom}(\mathcal{X}) \) is infinite).

More generally,

\[
\left| \mathcal{X}_n^{\xi} \right| \sim \frac{C}{1(g/4)} (\pi^{(g-1)/2}) n^{-5/2} R^n, \quad n \to \infty, \quad n \in \text{Dom}(\mathcal{X}^{\xi}), \quad (C^\xi)
\]

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where $\mathcal{X}^g$ stands for a class of rooted maps on the orientable surface of genus $g$. There exists a similar expression for non-orientable surfaces as well. A remarkable feature of these general formulas is that $R$ depends only on the class of maps (provided the latter is defined irrespective of the surface) while the exponent $5(g-1)/2$ does not depend even on the class of maps (see [12]).

We will call (C) and (C$^g$) the (basic asymptotic map) cardinality pattern and general cardinality pattern, respectively.

The cardinality pattern (usually together with the general cardinality pattern) proved to be valid for numerous natural classes of maps, such as all maps, 2- and 3-connected maps, triangulations and Eulerian maps (see the above-mentioned papers as well as [2, 4, 5]). On the other hand, the pattern (C) is known to be false for various classes of outer-planar maps, where the similar uniform asymptotics takes place with the exponent $-3/2$ instead of $-5/2$.

1.2. Higher Models. In general, a uniform asymptotic behavior is characteristic for combinatorial-topological objects, such as polyominoes and self-avoiding walks. For them, the general asymptotics of the form

$$cn^an^b, \quad n \to \infty,$$

is known or, more often, supposed to hold for the number of $n$-sized objects of various classes. In particular for polyominoes, the value of the critical exponent $a$ is typically (conjectured to be) equal to $-1/2$ (cf. [15]). For self-avoiding walks, the well-known universality hypothesis (see, e.g., [18, 16]) asserts that the value of the critical exponent $a$ does not depend on the detailed structure of the lattice though it may depend on the dimension of the space. In both theories, the existence of the connective constant $\rho$ is often provable but its exact value is known only in several exceptional cases (and is not assumed to be algebraic).

By analogy, we call $R = R(\mathcal{X})$ in (C) and (C$^g$) the connective constant. The power $a = -5/2$ in (C) and $a = 5(g-1)/2$ in (C$^g$) can be called the critical exponent.

1.3. Limit Valency Distribution. In the present paper, the following property of maps will be considered. Let a vertex be chosen at random and uniformly in all maps in $\mathcal{X}_n$ and $p_{k,n} = p_{k,n}(\mathcal{X})$ denote the probability that this vertex is of valency $k$. We are interested in the asymptotics of the variables $p_{k,n}$ as $n \to \infty$, i.e., the limit values (whenever exist) $p_k = p_k(\mathcal{X}) = \lim_{n \to \infty} p_{k,n}(\mathcal{X})$, $n \in \text{Dom}(\mathcal{X})$. We study the limit average value of the vertex valencies $\mu = \mu(\mathcal{X})$, the initial values $p_1, p_2, \ldots$ and the behavior of $p_k$ as $k \to \infty$. In the latter question we assume $\text{Val} = \text{Val}(\mathcal{X})$ to be infinite where $\text{Val}(\mathcal{X})$ denotes the set of valencies $k$ for which $p_k(\mathcal{X}) \neq 0$. 


For several important classes of maps, such results have been obtained earlier [1, 7, 13, 17] but in some cases only implicitly or in another form. An explicit uniform presentation and comparison of these results made it possible to reveal a hidden and previously unknown behavior uniformity. Namely,

\[ p_k(\mathcal{X}) \sim c(\pi k)^{-1/2} r^k, \quad k \to \infty, \quad k \in \text{Val}(\mathcal{X}), \quad (V) \]

with some algebraic constants \( c \) and \( r \) depending on the type of maps. We will call \((V)\) the (basic asymptotic) valency distribution pattern in (planar) maps. \( r = r(\mathcal{X}) \) and the power \( a = -1/2 \) in \((V)\) will be referred to as the connective valency constant and the critical valency exponent, respectively.

It is essential and characteristic that this valency distribution pattern does not hold for (at least) several natural classes of outer-planar (tree-like) maps. For them, just as in the cardinality pattern case, similar asymptotics are valid but with different values of the critical valency exponent \( a \).

Instead of vertex valencies, we could consider the limit distribution of the face size probabilities \( p_k^* \). Given a class of maps, they are, generally speaking, independent quantities with a similar expected asymptotic behavior,

\[ p_k^*(\mathcal{X}) \sim c_*(\pi k)^{-1/2} r_*^k, \quad k \to \infty, \quad k \in \text{Val}^*(\mathcal{X}), \quad (V^*) \]

where \( \text{Val}^*(\mathcal{X}) \) is the set of face sizes occurring with positive limit probabilities. But this is simply the vertex valency distribution of the dual maps: \( p_k^*(\mathcal{X}) = p_k(\mathcal{X}^*) \) where \( \mathcal{X}^* \) consists of the maps topologically dual to the maps in \( \mathcal{X} \).

Following the previously published results, we consider, in fact, one more random variable: the valency of the root (or any given) vertex. At first glance, the corresponding probabilities \( q_k, n \) should coincide with \( p_k, n \). However, this is not the case since the root vertex is not merely a vertex selected at random. Instead, we select first a root edge at random and then one of its ends. Thus, the more valent a vertex, the more probable it is selected as the root vertex. As a matter of fact, the two variables are closely connected: usually,

\[ q_k = k p_k / \mu, \quad k = 1, 2, \ldots \]

Thus, the root vertex valency distribution pattern appears as

\[ q_k(\mathcal{X}) \sim c'(k/\pi)^{1/2} r^k, \quad k \to \infty, \quad k \in \text{Val}(\mathcal{X}), \quad (RV) \]

with \( c' = c/\mu \) and the same connective valency constant \( r \) as in \((V)\).

To summarize, our aim here is mainly to reinterpret and present uniformly some known or easily deducible results on the subject and to establish new
TABLE I
Distribution Parameters of Limit Vertex Valency Probabilities

<table>
<thead>
<tr>
<th>No.</th>
<th>( \mathcal{X} )</th>
<th>( a )</th>
<th>( r )</th>
<th>( R )</th>
<th>( \mu )</th>
<th>( p_1 )</th>
<th>( p_2 )</th>
<th>( q_1 )</th>
<th>( q_2 )</th>
<th>( c )</th>
<th>Maps</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>( \mathcal{E} )</td>
<td>(-1/2)</td>
<td>( \sqrt{3}/2 )</td>
<td>8</td>
<td>6</td>
<td>0</td>
<td>3/8</td>
<td>0</td>
<td>1/R</td>
<td>( \sqrt{3}/2 )</td>
<td>Eulerian*</td>
</tr>
<tr>
<td>3.2</td>
<td>( \mathcal{P} )</td>
<td>(-1/2)</td>
<td>1/2</td>
<td>4</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>9 ( \sqrt{5}/2 )</td>
<td>Polyhedral</td>
</tr>
<tr>
<td>3.3</td>
<td>( \mathcal{Q} )</td>
<td>(-1/2)</td>
<td>3/2</td>
<td>4</td>
<td>4</td>
<td>0</td>
<td>1/3</td>
<td>1/6</td>
<td>1/R</td>
<td>1/R</td>
<td>( \sqrt{3}/2 )</td>
</tr>
<tr>
<td>3.4</td>
<td>( \mathcal{L} )</td>
<td>(-1/2)</td>
<td>3/4</td>
<td>4</td>
<td>6</td>
<td>0</td>
<td>2/3</td>
<td>0</td>
<td>1/R</td>
<td>( 2 \sqrt{3}/2 )</td>
<td>2-connected triang.</td>
</tr>
<tr>
<td>3.5</td>
<td>( \mathcal{A} )</td>
<td>(-1/2)</td>
<td>3/4</td>
<td>4 ( \sqrt{3}/3 )</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>3-connected triang.</td>
</tr>
<tr>
<td>3.6</td>
<td>( \mathcal{L} )</td>
<td>(-1/2)</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>1/4</td>
<td>1/R</td>
<td>( 1/R )</td>
<td>1</td>
</tr>
<tr>
<td>4.1</td>
<td>( \mathcal{F} )</td>
<td>0</td>
<td>1/2</td>
<td>4</td>
<td>2</td>
<td>1/2</td>
<td>1/4</td>
<td>1/R</td>
<td>1/R</td>
<td>1</td>
<td>Triang. dissect</td>
</tr>
</tbody>
</table>

* The same critical valency exponent \( a = -1/2 \) is valid for the maps with an arbitrary infinite \( \text{Val} \subseteq 2\N \).

interrelations among them. The quintessence of this paper can be presented in the following form:

1.4. Theorem. The basic (root) vertex valency distribution pattern (V) (resp., (RV)) is valid for planar maps of all classes considered in Section 3 and is not valid for the classes considered in Section 4. The main numerical values are given in Table I.

For completeness, Table I contains also the values of the connective constant \( R \) of the pattern (C) and reflects the property (quite predictable, in fact, as we will see below) that the nonzero limit probabilities of the root vertex to be of valency 1 or 2 are often equal to \( 1/R \).

Some related questions in a general setting are considered in [14].

2. PRELIMINARIES

2.1. Vertex versus Root Vertex Valency. The set of rooted maps \( \mathcal{X} \) is always supposed to be closed with respect to relabeling. In other words, \( \mathcal{X}_n \) is obtained from a certain set of unrooted \( n \)-edged planar maps by rooting them in all possible ways. Recall that rooting a map (by W. T. Tutte), which deprives it all non-trivial symmetries, means distinguishing an arbitrary edge-end, i.e., an incidence pair (edge, vertex) in it, provided that both sides of the sphere are distinguishable. These edge and vertex as well as the edge-end and, finally, the face incident to them and lying to the left are called the root elements of the map.

\[ M_n = M_n(\mathcal{X}) = |\mathcal{X}_n| \] denotes the number of maps in \( \mathcal{X}_n \) while \( m_{k,n} = m_{k,n}(\mathcal{X}) \) denotes the number of maps with the root vertex of valency \( k \).
\[ H_n = H(\mathcal{X}_n) \] stands for the overall number of vertices in \( \mathcal{X}_n \) and \( h_{k,n} = h_{k,n}(\mathcal{X}) \) for the overall number of vertices of valency \( k \) in them. Thus, \( M_n = \sum_k m_{k,n} \) and \( H_n = \sum_k h_{k,n} \).

Now, \( q_{k,n} = m_{k,n}/M_n \) is the probability of the event that the root vertex of a map chosen in \( \mathcal{X}_n \) at random is of valency \( k \). Likewise, \( p_{k,n} = h_{k,n}/H_n \) is the probability that a vertex chosen in \( \mathcal{X}_n \) at random (where all vertices have equal probabilities to be chosen) is of valency \( k \).

\[ m_{k,n}^*, h_{k,n}^*, H_n^*, q_{k,n}^* \text{ and } p_{k,n}^* \text{ denote the corresponding variables concerning face sizes. If } \mathcal{X}^* \text{ denotes the set of maps dual to } \mathcal{X}, \text{ then } m_{k,n}^*(\mathcal{X}) = m_{k,n}(\mathcal{X}^*), \text{ etc.} \]

\[ +n = \sum_{k=1}^{\infty} k p_{k,n} \] denotes the mean vertex valency (mean face size, resp.); so that \( +n = \sum_k k p_{k,n} \).

\[ q_{k,n}^* \text{ and } p_{k,n}^* \text{ denote the limit values of the corresponding variables, whenever exist, as } n \to \infty, n \in \text{Dom}(\mathcal{X}). \]

The probabilities \( q_{k,n} \) and \( p_{k,n} \) are closely connected with each other (cf. the note [19] devoted to a similar formula for plane trees):

**2.2. Lemma.** \( h_{k,n} = 2n \cdot m_{k,n}/k. \)

**Proof.** By definition, \( \mathcal{X}_n = \{(\Gamma, b)\} \) where \( \Gamma \) ranges over the set of the corresponding unrooted maps and \( b = (e, x) \) is an arbitrary edge-end selected as a root. In these terms, \( h_{k,n} \) enumerates the triples \((\Gamma, b, x')\) where \( x' \) is an arbitrary vertex of valency \( k \) in \( \Gamma \). Now, \( k h_{k,n} \) enumerates the quadruples \((\Gamma, b, x', b')\) with the same \( b \) and \( x' \) as above and an arbitrary edge-end \( b' \) incident to \( x' \). But they are equinumerous with the triples \((\Gamma, b', b)\) where \( b' \) is a (root) edge-end with the end of valency \( k \), whereas \( b \) is an arbitrary edge-end. The number of the latter triples is equal to \( m_{k,n} \cdot 2n \).

**2.3. Corollary.** \( H_n = 2n \sum_k (m_{k,n}/k), \quad \mu_n = 2n M_n/H_n, \quad \text{and } p_{k,n} = (\mu_n/k) q_{k,n}. \) If \( \text{Dom}(\mathcal{X}) \) is infinite and \( \mu = \lim_{n \to \infty} \mu_n \) exists for \( n \in \text{Dom}(\mathcal{X}), \) then

\[ p_k = \frac{\mu}{k} q_k, \quad k = 1, 2, ..., \quad (2.3.1) \]

and

\[ \sum_{k=1}^{\infty} \frac{q_k}{k} = \frac{1}{\mu}, \quad (2.3.2) \]

**Proof.** Straightforward.

**2.4. Third Scheme.** There is one more related variable, the number \( d_{v,n} = d_{v,n}(\mathcal{X}) \) of maps having \( v \) vertices. And another natural randomized
procedure corresponds to it: a map is first selected at random in $X_n$ and then a vertex in it (again distributed uniformly). What is the probability $s_{k,n}$ of the event that this vertex is of valency $k$? Generally speaking, this probability is independent of the ones introduced above. But for some interesting types of maps (such as, e.g., triangulations), the number of vertices is a function of $n$, and then $s_{k,n}$ is easily expressed via $q_{k,n}$. Moreover, the same reduction is possible asymptotically (as $n \to \infty$) if “almost all” maps have almost the mean number of vertices. The latter property is known, in particular, for all planar maps, non-separable maps, polyhedral (i.e., 3-connected) maps and Eulerian maps (cf. [3]). It is reasonable to expect that all “properly” behaving types of maps considered previously and known to satisfy the cardinality pattern (C) possess the same property to concentrate near the mean size.

Note also that $M_n, H_n$ and, accordingly, $\mu_n$ are expressed directly in terms of $d_{v,n}$:

$$M_n = \sum_v d_{v,n} \quad \text{and} \quad H_n = \sum_v v d_{v,n}.$$  \hfill (2.4.1)

2.5. Total Vertex Enumeration. Finding $M = M_n$ is a classical enumerative problem initiated by W. T. Tutte, and it has been resolved for many interesting types of rooted maps. But what about $H_n$? Of course by Corollary 2.3, this is only a technical problem if $m_{k,n}$ are known. Besides, $H_n$ is often easily expressed via $M_n$. By (2.4.1), this is trivial when all maps in $X_n$ possess a unique number of vertices: if $M_n = d_{v_0,n}$ (with $v_0$ depending on $n$), then $H_n = v_0 d_{v_0,n} = v_0 M_n$. It follows that

$$p_{k,n} = \frac{2n q_{k,n}}{v_0 k}.$$  \hfill (2.5.1)

For instance, a planar $n$-edged triangulation contains $2n/3$ faces ($3|n$). Therefore, it contains $v_0 = 2 + n/3$ vertices. So that, for triangulations,

$$p_{k,n} = \frac{6n q_{k,n}}{n + 6 k}.$$  \hfill (2.5.2)

Thus, $p_{k,n} \approx 6q_{k,n}/k$ for large $n$ and any fixed $k$. Similarly for planar quadrangulations, $v_0 = 2 + n/2$ and

$$p_{k,n} = \frac{4n q_{k,n}}{n + 4 k} \approx \frac{4 q_{k,n}}{k}.$$  \hfill (2.5.3)
2.6. Proposition. If \( \mathcal{X} \) is a self-dual class of maps, i.e., \( \mathcal{X}_n^* = \mathcal{X}_n \), then \( H_n = ((n+2)/2) M_n \). Thus, for infinite \( \text{Dom}(\mathcal{X}) \), \( \mu_n = 4n/(n+2) \sim \mu = 4 \) and \( p_k = 4q_k/k \).

Proof. Let with any map \( \Gamma \), \( \mathcal{X}_n \) contains its dual map \( \Gamma^* \). Then by Euler's formula, \( d_v, n = d_v^*, n \) where \( v^* = n + 2 - v \). Now, by (2.4.1),

\[
H_n = \sum_{v^* = 1}^{n+1} v^* d_v^*, n = (n+2-v) d_v, n = (n+2) \sum_{v = 1}^{n+1} d_v, n - \sum_{v = 1}^{n+1} vd_v, n
\]

whence \( H_n = ((n+2)/2) M_n \), and we are done by Corollary 2.3.

All (planar) maps, non-separable maps, polyhedral maps and the maps without both loops and isthmuses provide examples of self-dual classes. For other types of maps, there is no direct interconnection between the two quantities, valid a fortiori, and the problem of finding closed formulas for \( H_n \) (weighted enumerators) may be of a certain combinatorial interest. In particular, simple formulas do exist for loopless and Eulerian maps; the corresponding results will be published elsewhere.

In general, there is a simple connection between the mean vertex valency and the mean face size.

2.7. Lemma. For any \( \mathcal{X} \), \( 2n|\mu_n| + 2n|\mu^*_n| = n + 2 \). Moreover, if \( \mathcal{X} \) is a class of triangulations and \( \text{Dom}(\mathcal{X}) \) is infinite, then \( \mu_n \to \mu = 6 \).

Really, \( 2n|\mu_n| \) and \( 2n|\mu^*_n| \) are the mean numbers of vertices and faces, resp. Hence, the first formula is obtained by summing Euler's formula over all maps in \( \mathcal{X}_n \). Finally, for triangulations \( \mu^*_n = 3 \) for all admissible \( n \).

2.8. Corollary. If \( \text{Dom}(\mathcal{X}) \) is infinite and \( \mu = \lim_{n \to \infty} \mu_n \) for \( n \in \text{Dom}(\mathcal{X}) \) exists, then \( \mu^* = \lim_{n \to \infty} \mu^*_n \) for \( n \in \text{Dom}(\mathcal{X}) \) exists too, and

\[
\frac{2}{\mu} + \frac{2}{\mu^*} = 1. \quad (2.8.1)
\]

2.9. Observations Concerning Mono- and Bivalent Root Vertices. Let a class of maps \( \mathcal{X} \) admit endpoints. Take a map with the root vertex of valency 1, remove it together with the root edge and assign the new root properly (e.g., to the edge-end lying immediately on the left of the second end of the old root edge). If this map also belongs to \( \mathcal{X} \) and this operation turns out reversible, we obtain the equality \( m_{1,n} = M_{n-1} \). Similarly, if the root vertex of a map is homeomorphically reducible (i.e., 2-valent), we may erase it adjoining its second incident edge to the root one (and declaring
the other end of the latter as the root). If this operation is reversible, then \( m_{2,n} = M_{n - 1} \). Besides, for triangulation, the same idea can result in the equality \( m_{2,n} = M_{n - 3} \): after erasing the root vertex, one needs also to remove two digons that arose and can do this by joining three corresponding parallel edges into one new root edge.

In view of the asymptotics (C), these equalities yield values \( q_1 = 1/R \) and, resp., \( q_2 = 1/R \) or \( q_2 = 1/R^3 \). Such reasonings are in fact applicable to almost all classes of maps considered below which possess 1- or 2-valent vertices (cf. Table I). Instead, we will obtain these values simply as particular cases of general formulas for \( q_n \).

3. ORDINARY CLASSES OF MAPS

3.1. Eulerian Maps: \( \mathcal{E} \)

In a sense, Eulerian maps are the most appropriate ones for the question under consideration since there exists a simple sum-free formula for any contribution specified by the set of (even) vertex valencies, and it is a simple matter to find and estimate the greatest contribution [17]. Therefore, instead of one class of maps, multiparameter classes can be analyzed.

3.1.1. Valency Restricted Maps: \( \mathcal{E}(2\mathcal{K}) \). Let \( \mathcal{E}(2\mathcal{K}) \) be the set of planar (Eulerian) maps with all half-valencies belonging to a set \( \mathcal{K} \subseteq \mathbb{N} \). That is, \( \text{Val}(\mathcal{E}(2\mathcal{K})) = 2\mathcal{K} \). In particular, \( \mathcal{E} = \mathcal{E}(\mathbb{N}) \) contains all Eulerian maps without restrictions.

As we showed in [17, formulas (13) and (14)], the limit fraction of \( 2k \)-valent vertices in Eulerian maps under consideration is the following:

\[
x_k = x_k(\mathcal{E}(2\mathcal{K})) = a_k R_k^{-1} \lambda_k^{-1}, \quad k \in \mathcal{K}.
\]

Here \( a_k = (2k!/(2 \cdot (k!)^2)) \), \( k \geq 1 \),

\[
R_k = A_k(\lambda_k) \quad (3.1.1)
\]

is the connective constant of the cardinality pattern for \( \mathcal{E}(2\mathcal{K}) \) and \( \lambda_k \) is the real positive root of the equation

\[
z A_k'(z) - A_k(z) = 1, \quad (3.1.2)
\]

where \( A_k(z) = \sum_{k \in \mathcal{K}} a_k z^k \). In particular, for \( \mathcal{K} = \mathbb{N} \),

\[
A(z) = \sum_{k \geq 1} a_k z^k = \frac{(1 - 4z)^{-1/2} - 1}{2}. \quad (3.1.3)
\]
Furthermore (implicit in [17]),
\[ \mu_K = \mu(\partial(2K)) = \frac{2R_K^2}{R_K^2 - 1} \quad \text{and} \quad \mu_K^* = 2R_K^2. \]  
(3.1.4)

Then it is clear from formulas (3.1.1) and (3.1.2) that
\[ K = \frac{E(2K)}{2R_K^2} \, K_R \, K_R^* \text{ and } K^* = 2R_K^2 K_R. \] (3.1.4)

Then it is clear from formulas (3.1.1) and (3.1.2) that
\[ K = \frac{E(2K)}{2R_K^2} \, K_R \, K_R^* \text{ and } K^* = 2R_K^2 K_R. \] (3.1.4)

Moreover (loc.cit.), almost all \( n \)-edged Eulerian maps under consideration possess almost \( 2n/K \) vertices. Therefore we may use equivalently the third distribution scheme pointed out in 2.4. The probabilities \( p_{2k} \) are the appropriate ratios with respect to the total number of vertices. Thus, they are merely obtained from \( x_k \) by the suitable renormalization:
\[ p_{2k} = \mu_K x_k/2. \]
Hence,
\[ p_{2k} = p_{2k}(\partial(2K)) = (1/2) \, \mu_K K_R^{-1} x_k^{k-1}, \quad k \in K. \]

If \( 1 \in K \), it follows that \( p_{2k}(\partial(2K)) = \mu_K/(2R_K) \), whence by formula (2.3.1),
\[ q_z(\partial(2K)) = 1/R_K. \]

Since \( a_k \sim (1/2) \cdot (\pi K)^{-1/2} 4^k \), we conclude that the required asymptotic behavior pattern (V) is valid for any infinite \( K \):
\[ p_{2k} \sim c \cdot (\pi \cdot 2k)^{-1/2} (2 \sqrt{\lambda_k})^{2k}, \quad k \to \infty, \quad k \in K, \] (3.1.5)

where \( c = c_K = \sqrt{2}/(2R_K^2 - 1) = (\sqrt{2}/2) \, \mu_K / \mu_K^* = \sqrt{2}/(\mu_K^* - 2) \). Accordingly, the connective valency constant \( r_k = r(\partial(2K)) = 2 \sqrt{\lambda_k}^* \).

For instance, in the case of all Eulerian maps, \( r = r(\partial) = \sqrt{3}/2 \), \( R = R(\partial) = 8 \), \( \mu = \mu(\partial) = 6 \), \( \mu^* = \mu^*(\partial) = 3 \),
\[ p_{2k} = p_{2k}(\partial) \sim \sqrt{2} \cdot (\pi \cdot 2k)^{-1/2} (\sqrt{3}/2)^{2k}, \quad k \to \infty, \] (3.1.6)
\[ p_2 = 3/8, \quad q_2 = 2p_2/\mu = 1/8, \quad p_4 = 127/128, \quad \text{and} \quad q_4 = 9/64. \]

Remark. The above considerations are valid for arbitrary infinite \( K \). But in order to ensure that all parameters are algebraic numbers, as we required in the definition of the pattern (V) in 1.3, only algebraic sets \( K \) should be considered, i.e., ones for which the generating function \( A_K(z) \) is algebraic.

Numerical results obtained with the help of Maple are presented in Table II. Some interesting observations in it can be made. For instance, among the sets \( K = \mathbb{N} [n] \), \( \mu_K \) reaches its minimum value when \( n = 9 \), and \( r_K \) reaches its maximum value when \( n = 4 \).

Several useful estimates follow directly from the above formulas. Suppose \( K' \subseteq K \). Then by (3.1.2), \( \lambda_K \leq \lambda_{K'} \) with equality if and only if \( K = K' \cup \{1\} \). Hence, \( r_K \leq r_{K'} \) with equality if and only if \( K = K' \cup \{1\} \). Therefore, \( r_K \geq r_{K_n} = \sqrt{3}/2 = 0.8660 \ldots \) for any infinite \( K \). Also, \( R_K \geq R_{K_n} \) by (3.1.2).
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**VALENCY DISTRIBUTION IN PLANAR MAPS**

**TABLE II**

Numerical Values for Various Sets K of Admissible Hall-Valencies
Moreover, $R_K = R_K - 1$ and $\mu^{*}_K = \mu^{*}_K - 2z_K$ if $K = K' \cup \{1\}$, $1 \notin K'$. In particular, $\mu^{*}(\delta') = 3 - 2 \cdot 3/16 = 21/8$ where $\delta' = \delta(\mathcal{N} \setminus \{1\})$. Now, for any $K \subseteq \mathcal{N}$, we have $\lambda_K \geq \lambda_0 = 3/16$ and $\mu^{*}_K \geq 2$. It follows that, whenever $1 \in K$, $\mu^{*}_K = \mu^{*}(1) + 2z_K \geq 2 + 2 \cdot 3/16 = 19/8$. Hence, by Corollary 2.8, $\mu_K \leq 38/3$.

In contrast to $\mu^{*}_K$, $R_K$ and $r_K$, the value of $\mu^{*}_K$ may not be monotonic with respect to set inclusion. For example, as the data in Table II show for $K_v = \{2, v, v + 1, v + 2, \ldots\}$, $v = 3, 4, \ldots$, the values of $\mu^{*}_K$ first increase till $v = 37$ and then decrease. The behavior of $\mu^{*}_K$ depends heavily on that of $k_0 = \min\{k \in K\}$. If $k_0 \to \infty$ when $K$ runs through a descending chain of sets, then of course $\mu^{*}_K \to \infty$. If $k_0$ is bounded then it contributes more and more significantly, and $\mu^{*}_K$ is bounded too.

#### 3.1.2. Eulerian Maps with a Specified Mean Valency: $\delta^{(e)}$

This somewhat artificial subclass of Eulerian maps can be analyzed in a similar way and also turns out to satisfy the pattern (V). For brevity, we restrict ourselves with Eulerian maps having no prohibited (even) valencies, that is, with $K = \mathcal{N}$.

Given $x < 1$, let $\delta^{(e)}_n$ denote the number of planar Eulerian maps with $n$ edges and $v = nx(1 + o(1))$ vertices and, thus, with the limit mean valency $\mu^{*}_n = 2/\alpha$. According to the last formula in [17], normalized by the factor $\mu^{*}_n$, just as in Subsection 3.1.1, we have

$$p^{(e)}_{2k} = p_{2k}(\delta^{(e)}_n) = \frac{d_\alpha \lambda_k}{A(\lambda_\alpha)},$$

where $\lambda_\alpha$ is the real positive root of the equation

$$A(z) = \alpha z A'(z). \tag{3.1.2'}$$

Hence,

$$p^{(e)}_{2k} \sim c(\pi \cdot 2k)^{-1/2} (2 \sqrt{\lambda_\alpha})^{2k}, \quad k \to \infty, \tag{3.1.5'}$$

where $c = \sqrt{2/(2A(\lambda_\alpha))}$. Again, the connective valency constant $r_\alpha = r(\delta^{(e)}_n) = 2/\sqrt{\lambda_\alpha}$. Now due to formula (3.1.3), $\lambda_\alpha$ can be easily expressed as a function of $\alpha$:

$$\lambda_\alpha = \frac{1}{4} \left(1 - \frac{\pi}{2} \left(1 + \frac{\pi + \sqrt{\pi^2 + 8\pi}}{4}\right)^2\right).$$

After elementary transformations, one can obtain the following expression for $p^{(e)}_{2k}$:

$$p^{(e)}_{2k} = \frac{(\pi + \sqrt{\pi^2 + 8\pi})^3}{64 \alpha}.$$
For example, $\lambda_{1/3} = 3/16$ and $p_{1/3} = 3/8$, the same values as in the class of unrestricted Eulerian maps considered in 3.1.1 (naturally, since $\mu_{1/3} = 6 = \mu_{1/6}; \lambda_{1/6} = 2/9, p_2 = 2/9, q_2 = q_{1/6} = 1/27; \lambda_{1/10} = 15/64, p_{1/10} = 2/25, q_{1/10} = 1/125$.

The connective constant $R_\alpha = R(\mathcal{E}(\alpha))$ is expressed [17] as:

$$R_\alpha = \frac{A^*(\lambda_\alpha)}{\alpha'(1-\alpha)^{1-\alpha} \lambda_\alpha},$$

whence, for example, $R_{1/3} = 8$ and $R_{1/6} = 27 \sqrt[3]{3}/5$.

3.2. Polyhedral Maps: $\mathcal{P}$

For polyhedral maps (polytopal or 3-connected, in other terms), everything we need has been obtained by E. A. Bender and E. R. Canfield [1, Theorem 2]. Namely, the limit probabilities $q_\alpha^*$ exist and are determined by the equation

$$y^k = 1 + \frac{20 + y - y^2}{8(4 + y)} - \frac{(2 - y)^3(50 - y) + 27y(12 - 20y + y^2)}{8(4 + y)(2 - y)^3(50 - y)^{1/2}} \equiv 3^y \left( \frac{1}{2} y^2 + \frac{9}{25} y^3 + \frac{23}{1000} y^4 + \frac{16207}{125000} y^5 + \cdots \right),$$

whence

$$q_\alpha^* \sim 9 \sqrt[3]{6/8} \cdot (k/\pi)^{1/2} 2^{-k}, \quad k \to \infty.$$

By self-duality, $\mu = 4$ (Proposition 2.6). Hence, by Corollary 2.3, $p_k$ meet the pattern (V) with $r = r(\mathcal{P}) = 1/2$:

$$p_k = p_k(\mathcal{P}) \sim 9 \sqrt[3]{6/2} \cdot (\pi k)^{-1/2} 2^{-k}, \quad k \to \infty, \quad (3.2.1)$$

and $p_1 = p_2 = 0, p_3 = 243/500$.

Moreover, “almost all” polyhedral maps have almost the mean number of vertices $v = n/2$. Hence, one can equivalently use the third valency distribution scheme pointed out in 2.4 (see [1, Theorem 3]). Finally, as is well known, $R(\mathcal{P}) = 4$.

3.3. Arbitrary Maps: $\mathcal{A}$

According to Zh. Gao and L. B. Richmond [13, Theorem 1], for the class of all maps, the limit probabilities $q_\alpha$ exist and are determined by the equation

$$y^k = 1 + \frac{20 + y - y^2}{8(4 + y)} - \frac{(2 - y)^3(50 - y) + 27y(12 - 20y + y^2)}{8(4 + y)(2 - y)^3(50 - y)^{1/2}} \equiv 3^y \left( \frac{1}{2} y^2 + \frac{9}{25} y^3 + \frac{23}{1000} y^4 + \frac{16207}{125000} y^5 + \cdots \right),$$

whence

$$q_\alpha^* \sim 9 \sqrt[3]{6/8} \cdot (k/\pi)^{1/2} 2^{-k}, \quad k \to \infty.$$
\[
\sum_{k=1}^{\infty} q_k y^k = \left(\frac{y}{12}\right)\left(1 + \frac{y}{2}\right)^{-1/2} \left(1 - \frac{5y}{6}\right)^{-3/2} \\
= \left(\frac{1}{12}\right) y + \left(\frac{1}{12}\right) y^2 + \left(\frac{13}{144}\right) y^3 + \left(\frac{55}{648}\right) y^4 + \ldots,
\]
this being valid for the maps on every surface. Hence,

\[q_k \sim \sqrt{10}/20 \cdot (k/\pi)^{1/2} (5/6)^k, \quad k \to \infty.\]

Again by self-duality, \(\mu = 4\), and we can easily express these results in terms of the variables \(p_k\). The latter satisfy the pattern (V) with \(r(\mathcal{A}) = 5/6\),

\[p_k = p_1(\mathcal{A}) \sim \sqrt{10}/5 \cdot \pi k)^{-1/2} (5/6)^k, \quad k \to \infty, \quad (3.3.1)\]

and \(p_1 = 1/3, p_2 = 1/6, p_3 = 13/108, R = R(\mathcal{A}) = 12\), therefore \(q_1 = q_2 = 1/R\).

### 3.4. Non-Separable Triangulations: \(\mathcal{A}\)

The valency distribution of non-separable (i.e., 2-connected) triangulations turned out close to that of arbitrary maps. According to [13, Theorem 3 and Lemma 1], in this case,

\[ y \sum_{k=1}^{\infty} q_k y^k - 1/2 = \left(\frac{1}{24}\right) y^2 + 12y - 12(1 + \frac{y}{2})^{-1/2} \left(1 - \frac{5y}{6}\right)^{-3/2} \\
= \left(\frac{2}{27}\right) y^3 + \left(\frac{2}{27}\right) y^4 + \left(\frac{7}{81}\right) y^5 + \left(\frac{20}{243}\right) y^6 + \ldots
\]

(in fact, \(q_0(n,k)\) in [13] are \(q_{k,3n-k}\) in our terms; hence, the limit probabilities \(q_k\) are the same). Again, this is valid regardless of the surface. Now,

\[q_k \sim \sqrt{10}/15 \cdot (k/\pi)^{1/2} (5/6)^k, \quad k \to \infty.\]

By Lemma 2.7, we can express these in terms of \(p_k\) with \(\mu = 6\), thus obtaining a particular case of the pattern (V) with \(r(\mathcal{A}) = 5/6\);

\[p_k = p_1(\mathcal{A}) \sim 2 \sqrt{10}/5 \cdot (\pi k)^{-1/2} (5/6)^k, \quad k \to \infty, \quad (3.4.1)\]

\(p_1 = 0, p_2 = 2/9, p_3 = 4/27\). Finally, \(R(\mathcal{A}) = 3 \cdot \sqrt[3]{4}/2\) (with respect to the number of edges \(n\) as adopted throughout the paper), therefore \(q_2 = 1/R^3\).
3.5. Loopless Maps: $\mathcal{L}$

Evidently, the asymptotics of vertex valencies for loopless planar maps has not been considered before. But there is a simple exact formula for the number of loopless maps with $n$ edges and root vertex valency $k$ obtained by E. A. Bender and N. C. Wormald [7]:

$$L_{k,n} = \frac{2k(4n-2k-1)!}{(n-k)!(3n-k+1)!} \cdot \binom{2k+1}{k}.$$

3.5.1. Proposition.

$$q_k = q_k(\mathcal{L}) = \frac{k}{2} \cdot \left(\frac{3}{16}\right)^{k+1}, \quad k = 1, 2, \ldots \tag{3.5.1}$$

This follows directly from the above formula and the well-known formula for $n$-edged rooted loopless planar maps (e.g., loc.cit.):

$$L_n = \frac{6(4n+1)!}{n!(3n+3)!}.$$

In particular, $q_1 = 3(3/16)^2$, $q_2 = 20(3/16)^3$, and $q_3 = 105(3/16)^4$. Also the connective constant $R = R(\mathcal{L}) = 256/27$, so that, $q_1 = 1/R$ and (unlike other classes) $q_2 = 5/(4R)$.

3.5.2. Corollary. The loopless planar maps satisfy the valency distribution pattern $(V)$ with $r = r(\mathcal{L}) = 3/4$:

$$p_k = p_k(\mathcal{L}) \sim 6/5 \cdot (\pi k)^{-1/2} (3/4)^k, \quad k \to \infty. \tag{3.5.2}$$

Proof. From expression (3.5.1) by Stirling’s formula we obtain immediately $q_k \sim 3/8 \cdot (k/\pi)^{1/2} (3/4)^k$ as $k \to \infty$. Moreover, it is clear that $\mu(\mathcal{L})$ exists, and by formulas (2.3.2) and (3.1.3) we calculate $\mu(\mathcal{L}) = (\sum_{k=1}^{\infty} q_k/k)^{-1} = 16/5$.

3.6. Polyhedral Triangulations: $\mathcal{F}$

The valency distribution of 3-connected planar triangulations turned out close to that of loopless maps: according to [7], the number of rooted 3-connected planar triangulations with $3n+3$ edges and root vertex valency $k+2$ is equal to the number $L_{k,n}$ of loopless maps. Now, the overall number of rooted 3-connected planar triangulations with $3n+3$ edges is equal to $L_n$. Since $\mu = 6$ (Lemma 2.7), we obtain
3.6.1. Proposition. The 3-connected planar triangulations satisfy the pattern (V) with \( r = r(\mathcal{F}) = 3/4 \):

\[
p_k = p_k(\mathcal{F}) \sim 4 (\pi k)^{-1/2} \left( 3/4 \right)^k, \quad k \to \infty.
\] (3.6.1)

By the above-mentioned equality, \( q_1(\mathcal{F}) = q_1 = 2 \), thus, \( q_1(\mathcal{F}) = q_1 = 2 \), \( q_3(\mathcal{F}) = 3(3/16)^2 \) and \( p_3(\mathcal{F}) = 6(3/16)^2 \). Finally, \( R = R(\mathcal{F}) = 4 \sqrt[4]{3} \).

An equivalent formula for \( p_k(\mathcal{F}) \) is also obtained in [8] (formula (3.14); in physical terms, the valency of a point is known as its coordination number).

4. OUTER-PLANAR MAPS

In the previous section we aimed at establishing the validity of the basic valency distribution pattern on as many classes of ordinary maps as we are capable at the moment. Unlike that, here we restrict ourselves to only two famous classes of tree-like maps.

4.1. Arbitrary Plane Trees: \( \mathcal{T} \). The number of rooted plane trees is equal to the \( n \)th Catalan number: \( M_n = |T_n| = 1/(n+1)(2n) \). Since an \( n \)-edged tree contains \( n+1 \) vertices, the overall number of vertices \( H_n = H(T_n) = (2n) \). Besides, for the number of vertices of valency \( k \), N. Dershowitz and S. Zaks [9] obtained the formula \( h_k, n(T) = (2n-k-1) \). For a fixed \( k \), we deduce at once

\[
p_k = p_k(T) = \lim_{n \to \infty} \frac{h_k, n(\mathcal{T})}{H_n} = 2^{-k}, \quad (4.1.1)
\]

a purely geometrical decrease, which differs from the pattern (V): the critical valency exponent \( a \) is equal to 0. In particular, \( p_1 = 1/2, p_2 = 1/4 \), so that (since \( \mu(\mathcal{T}) = 2 \) and \( R = R(\mathcal{T}) = 4 \)) we have \( q_1 = q_2 = 1/4 = 1/R \).


4.2. Triangular Dissections of Polygons: \( \mathcal{D} \). Let \( \mathcal{A} \) be a convex polygon with \( N \) sides dissected into triangles by \( N - 3 \) non-crossing (open) diagonals. This may be considered as an outer-planar map (near-triangulation) with \( n = 2N - 3 \) edges. As is well known, the number \( M'(\mathcal{D}) \) of such maps rooted at the external face is the \( (N-2) \)nd Catalan number \( 1/(N-1)(N-2) \).
The number of these maps with the root vertex of valency $k$, provided
that the two boundary links are taken into account and one of them
together with the external face serve as the root, is

$$m_{k, n}(\mathcal{D}) = \frac{k - 1}{2N - k - 3} \binom{2N - k - 3}{N - k - 1}, \quad k \geq 2, \quad n = 2N - 3. \quad (4.2.1)$$

I could find only one published (and short) proof of this simple formula:
in the paper [10], where it is presented in somewhat different form. Since
for arbitrary rootings (as adopted throughout the present paper), any edge-
and either of its sides may be chosen, we have $m_{k, n}(\mathcal{D}) = k \cdot m_{k, n}(\mathcal{D})$.
Likewise, $M_n = (2(2N - 3)/N)(1/(N - 1))(2N - 3)= (1/N)(2N - 3)$, the $(N - 1)$st
Catalan number. Moreover $R = R(\mathcal{D}) = 2$ (with respect to $n$).

Now, one can easily deduce the asymptotics of $q_{k, n}$ as $n \to \infty$, $n \in \text{Dom}(\mathcal{D})$
$= \{3, 5, 7, ...\}$: $q_k = q_k(\mathcal{D}) = k(k - 1)2^{-k}/4$, $k \geq 1$. Clearly, $\mu_n = 2(2N - 3)/N$,
hence $\mu = 4$ and

$$p_k = p_k(\mathcal{D}) = (k - 1) \cdot 2^{-k} \quad (4.2.2)$$
(with $p_1 = 0$, $p_2 = p_3 = 1/4$). This again differs from the pattern (V).

5. CONCLUDING OBSERVATIONS AND OPEN QUESTIONS

To the best of our knowledge, valency distributions for non-planar maps
have so far been investigated only in two cases, $\mathcal{A}$ and $\mathcal{S}$. In both, as
mentioned in Subsections 3.3 and 3.4, the asymptotic distribution pattern
does not depend on the surface genus at all. Is this a general phenomenon?
What can be said about face sizes for the classes of maps studied in
Section 3? The answer is clear for self-dual classes and is degenerate for
triangulations. The question seems particularly interesting for Eulerian
maps (that is, by duality, the question about vertex valencies of bipartite
maps) and loopless maps.

Plane trees are in general better tractable than general maps. Possibly
the following question based on (4.1.1) and additional observations can be
answered easily: Is there a “naturally” defined class of plane trees which
does not meet the pattern

$$p_k \sim c r^k, \quad k \to \infty, \quad k \in \text{Val}, \quad (TV)$$

with some algebraic constants $c$ and $r$? More specifically, which conditions
on a class of plane trees ensure the validity of the distribution pattern
(TV)! The same question for 2-connected outer-planar maps and the pattern \( p_k \sim c k^b \) (cf. (4.2.2)).

It is known that the set of self-dual maps (do not confuse with a self-dual set of maps!) does not satisfy the basic cardinality pattern (C) (cf. [17]). Do these maps satisfy the basic valency distribution pattern (V)?

ACKNOWLEDGMENTS

The idea of the present research appeared during my visit in Israel. I am deeply indebted to the Department of Mathematics and Computer Science of the Ben-Gurion University of Beer-Sheva and especially to Mikhail Klin for hospitality. I am also thankful to Edward Bender for valuable recommendations and to Bruce Richmond and Helmut Prodinger for their kind help with references.

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