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Comparison results between Jacobi and other iterative methods☆

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Abstract

Some comparison results between Jacobi iterative method with the modified preconditioned simultaneous displacement (MPSD) iteration and other iterations, for solving nonsingular linear systems, are presented. It is showed that spectral radius of Jacobi iteration matrix B is less than that of several iteration matrices introduced in Liu (J. Numer. Methods Comput. Appl. 1 (1992) 58) under some conditions.

MSC: 65F10

Keywords: Comparison results; Jacobi iteration; MPSD iteration; Spectral radius

1. Introduction

For solving linear system

$$Ax = b, \tag{1.1}$$

where $A \in C^{n\times n}$ is nonsingular, $b \in C^n$ is given and $x \in C^n$ is unknown. Iterative methods are effective and practical when the matrix A is large and sparse and are studied by many authors (cf., [1–14], etc.).

In order to solve system (1.1) with iterative methods, the coefficient matrix A is split into

$$A = M - N,$$

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where M is nonsingular, then a linear stationary iterative formula for solving the system (1.1) can be described as follows:

$$x^{(i+1)} = M^{-1}Nx^{(i)} + M^{-1}b, \quad i = 0, 1, \ldots,$$

where $M^{-1}N$ is the iterative matrix. It is well-known that the iterative method converges iff the spectral radius of iterative matrix is less than 1.

Let $A = D - C_L - C_U$ where $D = \mathrm{diag}(A)$ is nonsingular, $-C_L$ and $-C_U$ are strictly lower and upper triangular matrices obtained from A, respectively. We also let $L = D^{-1}C_L$, $U = D^{-1}C_U$. The Jacobi iterative matrix is

$$B = L + U = I - D^{-1}A. \tag{1.2}$$

The modified preconditioned simultaneous displacement (MPSD) method is studied in [2,7,10] and the MPSD iterative matrix is

$$S_{\tau,\omega_1,\omega_2} = M^{-1}N, \tag{1.3}$$

where $M = D(I - \omega_2 U)(I - \omega_1 L)$, $N = M - \tau A$; $\omega_1, \omega_2, \tau \in R$ and $\tau \neq 0$. $S_{\tau,\omega_1,\omega_2}$ can be specifically written as follows:

$$S_{\tau,\omega_1,\omega_2} = (I - \omega_1 L)^{-1}(I - \omega_2 U)^{-1}[(1-\tau)I + (\tau - \omega_1)L + (\tau - \omega_2)U + \omega_1\omega_2 UL]. \tag{1.4}$$

Some special cases of MPSD method are studied in [4,5,9,14]. With special values of ω_1, ω_2 and τ, the corresponding iterative methods are given in the following table:

When $0 < \omega_i < \tau \leqslant 1$, $i = 1, 2$, the following theorem is presented in [2].

Theorem 1 (Chen [2]). *Let A be irreducible, $B = L + U \geqslant 0$. Then, for $0 < \omega_i < \tau \leqslant 1$, $i = 1, 2$, we have*

(1) $\rho(B) > 0$, $\rho(S_{\tau,\omega_1,\omega_2}) > 1 - \tau$,
(2) $0 < \rho(B) < 1 \Leftrightarrow 1 - \tau < \rho(S_{\tau,\omega_1,\omega_2}) < 1$,
(3) $\rho(B) = 1 \Leftrightarrow \rho(S_{\tau,\omega_1,\omega_2}) = 1$,
(4) $\rho(B) > 1 \Leftrightarrow \rho(S_{\tau,\omega_1,\omega_2}) > 1$.

Theorem 1 shows that the Jacobi iterative method and the MPSD iterative method are either both convergent, or both divergent. But, for the case that two iterative methods are both convergent, which one is better? Theorem 1 does not give the answer. In this paper, we obtain the comparison results between the iterative methods given in Table 1 with Jacobi iterative method when $\rho(B) < 1$, and the results obtained show that Jacobi iterative method is better under some conditions.

Table 1

ω_1	ω_2	τ	Matrix M	Iterative method
ω	ω	$\omega(2-\omega)$	$D(I - \omega U)(I - \omega L)$	SSOR
ω	ω	ω	$D(I - \omega U)(I - \omega L)$	EMA
ω	0	ω	$D(I - \omega L)$	SOR
0	0	ω	D	JOR

2. Preliminary results

In the following, we need the following theorems:

Theorem 2 (Berman and Plemmons [1]). *If A is a nonnegative square matrix, then*

(1) $\rho(A)$, *the spectral radius of A, is an eigenvalue,*
(2) *A has a nonnegative eigenvector corresponding to $\rho(A)$,*
(3) *A^{t} has a nonnegative eigenvector corresponding to $\rho(A)$.*

Theorem 3 (Berman and Plemmons [1]). *For $A \geqslant 0$,*

$$\alpha x \leqslant Ax,\ x \geqslant 0,\ x \neq 0 \text{ implies } \alpha \leqslant \rho(A)$$

and

$$Ax \leqslant \beta x, \quad x > 0 \text{ implies } \rho(A) \leqslant \beta.$$

3. Main results

Let A be the coefficient matrix in (1.1), B be Jacobi iterative matrix in (1.2) and nonnegative. Then by Theorem 2, we know that B has an eigenvalue $\lambda = \rho(B) \geqslant 0$.

For the λ above and $x \geqslant 0$, $x \neq 0$, we have

$$\begin{aligned} S_{\tau,\omega_1,\omega_2}x - \lambda x &= (I-\omega_1 L)^{-1}(I-\omega_2 U)^{-1}[(1-\tau)I+(\tau-\omega_1)L + (\tau-\omega_2)U + \omega_1\omega_2 UL]x - \lambda x \\ &= (I-\omega_1 L)^{-1}(I-\omega_2 U)^{-1}[(1-\tau)I + (\tau-\omega_1)L + (\tau-\omega_2)U \\ &\quad + \omega_1\omega_2 UL - \lambda(I-\omega_2 U)(I-\omega_1 L)]x \\ &= (I-\omega_1 L)^{-1}(I-\omega_2 U)^{-1}[(1-\tau-\lambda)I + (\tau-\omega_1+\lambda\omega_1)L \\ &\quad + (\tau-\omega_2+\lambda\omega_2)U + (1-\lambda)\omega_1\omega_2 UL]x. \end{aligned} \tag{3.1}$$

When $0 \leqslant \omega_i \leqslant \tau \leqslant 1$, $i = 1,2$, by (1.4), we know

$$\begin{aligned} S_{\tau,\omega_1,\omega_2} &= (I-\omega_1 L)^{-1}(I-\omega_2 U)^{-1}[(1-\tau)I + (\tau-\omega_1)L + (\tau-\omega_2)U + \omega_1\omega_2 UL] \\ &= (I + \omega_1 L + \omega_1^2 L^2 + \cdots)(I + \omega_2 U + \omega_2^2 U^2 + \cdots)[(1-\tau)I + (\tau-\omega_1)L \\ &\quad + (\tau-\omega_2)U + \omega_1\omega_2 UL] \\ &\geqslant (1-\tau)I + (\tau-\omega_1)L + (\tau-\omega_2)U + \omega_1\omega_2 UL \\ &\geqslant 0. \end{aligned} \tag{3.2}$$

(a) When $\omega_1 = \omega_2 = \omega$ and $\tau = \omega(2-\omega)$, (1.4) becomes the SSOR iteration matrix

$$S_{\omega(2-\omega),\omega,\omega} = (I-\omega L)^{-1}(I-\omega U)^{-1}[(\omega-1)^2 I + (\omega-\omega^2)B + \omega^2 UL].$$

Theorem 4. *Let A be a nonsingular matrix. $B \geqslant 0$ the Jacobi iteration matrix in (1.2). If $0 \leqslant \omega \leqslant 1$ and $\rho(B) \leqslant (1-\omega)^2$, we have that*

$$\rho(B) \leqslant \rho(S_{\omega(2-\omega),\omega,\omega}),$$

where $S_{\omega(2-\omega),\omega,\omega}$ is the iteration matrix of SSOR *iterative method.*

Proof. Since $\tau = \omega(2-\omega) = 2\omega - \omega^2$, we have $\tau_{\max} = (4\times(-1)\times 0 - 2^2)/(4\times(-1)) = 1$. Thus $\tau \leqslant 1$. We also know that

$$\tau \geqslant 0 \quad \text{when } 0 \leqslant \omega \leqslant 2.$$

$$\tau - \omega = (2\omega - \omega^2) - \omega = \omega - \omega^2 \geqslant 0 \quad \text{for } 0 \leqslant \omega \leqslant 1,$$

i.e.,

$$\tau \geqslant \omega \quad \text{for } 0 \leqslant \omega \leqslant 1.$$

So we obtain, by (3.2), that $S_{\omega(2-\omega),\omega,\omega} \geqslant 0$.

Now, we consider $S_{\omega(2-\omega),\omega,\omega}x - \lambda x$ (where $\lambda = \rho(B)$ and $x \geqslant 0, x \neq 0$). From (3.1), we know

$$\begin{aligned} S_{\omega(2-\omega),\omega,\omega}x - \lambda x = {} & (I-\omega L)^{-1}(I-\omega U)^{-1}\{[1-\omega(2-\omega)-\lambda]I \\ & + [\omega(2-\omega)-\omega+\lambda\omega]B + (1-\lambda)\omega^2 UL\}x. \end{aligned}$$

It is obvious that

$$\omega(2-\omega) - \omega + \lambda\omega = 2\omega - \omega^2 - \omega + \lambda\omega = (1+\lambda)\omega - \omega^2 \geqslant 0 \quad \text{when } 0 \leqslant \omega \leqslant 1$$

and

$$1 - \omega(2-\omega) - \lambda \geqslant 0 \text{ if and only if } \lambda \leqslant (1-\omega)^2.$$

So, when $\lambda \leqslant (1-\omega)^2$, we have

$$\begin{aligned} S_{\omega(2-\omega),\omega,\omega}x - \lambda x &= (I + \omega L + \omega^2 L^2 + \cdots)(I + \omega U + \omega^2 U^2 + \cdots)\{[1-\omega(2-\omega)-\lambda]I \\ &\quad + [\omega(2-\omega)-\omega+\lambda\omega]B + (1-\lambda)\omega^2 UL\}x \\ &\geqslant 0 \end{aligned}$$

i.e.,

$$S_{\omega(2-\omega),\omega,\omega}x \geqslant \lambda x.$$

Thus, by Theorem 3, we have

$$\rho(S_{\omega(2-\omega),\omega,\omega}) \geqslant \rho(B).$$

(b) When $\omega_1 = \omega_2 = \tau = \omega$, (1.4) becomes the EMA iteration matrix

$$S_{\omega,\omega,\omega} = (I - \omega L)^{-1}(I - \omega U)^{-1}[(1-\omega)I + \omega^2 UL]. \qquad \square$$

Theorem 5. *Let A be a nonsingular matrix. $B \geqslant 0$ the Jacobi iteration matrix in (1.2). If $0 \leqslant \omega \leqslant 1$ and $\rho(B) \leqslant 1 - \omega$, we have that*

$$\rho(B) \leqslant \rho(S_{\omega,\omega,\omega}),$$

where $S_{\omega,\omega,\omega}$ is the iteration matrix of EMA *iterative method.*

Proof.

$$\begin{aligned} S_{\omega,\omega,\omega} &= (I - \omega L)^{-1}(I - \omega U)^{-1}[(1-\omega)I + \omega^2 UL] \\ &= (I + \omega L + \omega^2 L^2 + \cdots)(I + \omega U + \omega^2 U^2 + \cdots)[(1-\omega)I + \omega^2 UL], \end{aligned}$$

and $S_{\omega,\omega,\omega} \geqslant 0$ when $0 \leqslant \omega \leqslant 1$.

Considering $S_{\omega,\omega,\omega}x - \lambda x$, where λ and x are the same as that in Theorem 4. By (3.1), we have

$$\begin{aligned} S_{\omega,\omega,\omega}x - \lambda x &= (I - \omega L)^{-1}(I - \omega U)^{-1}[(1-\omega-\lambda)I + \lambda\omega B + (1-\lambda)\omega^2 UL]x \\ &= (I + \omega L + \omega^2 L^2 + \cdots)(I + \omega U + \omega^2 U^2 + \cdots) \\ &\quad \times [(1-\omega-\lambda)I + \lambda\omega B + (1-\lambda)\omega^2 UL]x. \end{aligned}$$

It is obvious that

$$S_{\omega,\omega,\omega}x - \lambda x \geqslant 0 \quad \text{when } 0 \leqslant \omega \leqslant 1 \text{ and } \lambda \leqslant 1 - \omega.$$

Or, equivalently

$$S_{\omega,\omega,\omega}x \geqslant \lambda x.$$

From Theorem 3, we obtain

$$\rho(B) \leqslant \rho(S_{\omega,\omega,\omega}).$$

(c) When $\omega_1 = \omega_2 = 0$ and $\tau = \omega$, (1.4) becomes the JOR iteration matrix

$$S_{\omega,0,0} = (1-\omega)I + \omega B. \qquad \square$$

Theorem 6. *Let A be a nonsingular matrix. $B \geqslant 0$ the Jacobi iteration matrix in (1.2). If $0 \leqslant \omega \leqslant 1$ and $\rho(B) \leqslant 1 - \omega$, we have that*

$$\rho(B) \leqslant \rho(S_{\omega,0,0}),$$

where $S_{\omega,0,0}$ is the iteration matrix of JOR *iterative method.*

Proof. $S_{\omega,0,0} = (1-\omega)I + \omega B$ and $S_{\omega,0,0} \geqslant 0$ when $0 \leqslant \omega \leqslant 1$.

Considering $S_{\omega,0,0}x - \lambda x$, where λ and x are the same as that in Theorem 4.

$$S_{\omega,0,0}x - \lambda x = [(1-\omega)I + \omega B]x - \lambda x = [(1-\omega-\lambda)I + \omega B]x.$$

So we know that

$$S_{\omega,0,0}x - \lambda x \geqslant 0 \quad \text{when } 0 \leqslant \omega \leqslant 1 \text{ and } \lambda \leqslant 1-\omega$$

or

$$S_{\omega,0,0}x \geqslant \lambda x.$$

From Theorem 3, we have

$$\rho(B) \leqslant \rho(S_{\omega,0,0}).$$

(d) When $\omega_1 = \tau = \omega$ and $\omega_2 = 0$, (1.4) becomes the SOR iteration matrix

$$S_{\omega,\omega,0} = (I - \omega L)^{-1}[(1-\omega)I + \omega U]. \qquad \square$$

Theorem 7. *Let A be a nonsingular matrix. $B \geqslant 0$ the Jacobi iteration matrix in* (1.2). *If $0 \leqslant \omega \leqslant 1$ and $\rho(B) \leqslant 1-\omega$, we have that*

$$\rho(B) \leqslant \rho(S_{\omega,\omega,0}),$$

where $S_{\omega,\omega,0}$ is the iteration matrix of SOR *iterative method.*

Proof. $S_{\omega,\omega,0} = (I-\omega L)^{-1}[(1-\omega)I + \omega U] = (I + \omega L + \omega^2 L^2 + \cdots)[(1-\omega)I + \omega U]$ and $S_{\omega,\omega,0} \geqslant 0$ when $0 \leqslant \omega \leqslant 1$.

For $S_{\omega,\omega,0}x - \lambda x = (I-\omega L)^{-1}[(1-\omega-\lambda)I + \omega U + \lambda\omega L]x$, we know that

$$S_{\omega,\omega,0}x \geqslant \lambda x \quad \text{when } 0 \leqslant \omega \leqslant 1 \text{ and } \lambda \leqslant 1-\omega.$$

So, by Theorem 3, we obtain

$$\rho(B) \leqslant \rho(S_{\omega,\omega,0}). \qquad \square$$

Remark. For iterative methods, the case that its spectral radius is smaller than 1 is worth noting. Although [2] presents a comparison result of the convergent and divergent relationship between Jacobi and the MPSD iterative method, it does not show which one is better when both methods converge. In this paper, we discuss several special cases of the MPSD method, and show that SSOR (SOR, EMA, JOR) iterative method is not better than Jacobi iterative method under some conditions. So, Jacobi iterative method is simple and effective under these conditions.

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