Abstract

Systems of weight functions and corresponding generalised derivatives are classically used to build extended Chebyshev spaces on a given interval. This is a well-known procedure. Conversely, if the interval is closed and bounded, it is known that a given extended Chebyshev space can always be associated with a system of weight functions via the latter procedure. In the present article we determine all such possibilities, that is, all systems of weight functions which can be used to define a given extended Chebyshev space on a closed bounded interval.

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1. Introduction

Extended Chebyshev spaces are classical tools in approximation theory [7,17], their importance being due to the fact that they are the spaces in which Hermite interpolation is always possible. More recently, they have been also intensively exploited for geometric design purposes because they permit shape preserving design like polynomial spaces, while providing users with interesting shape parameters enabling them to modify the curves that they create (see for instance [16,15,3,4]).

On any given real interval $I$ there exists a classical procedure for building extended Chebyshev spaces, which we recall below:

– choose any sequence of non-vanishing functions $w_0, w_1, \ldots, w_n$ such that $w_i \in C^{n-i}(I)$ for $0 \leq i \leq n$;

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– consider the associated generalised derivatives $L_0, \ldots, L_n$ defined on $C^n(I)$ by

$$
L_0 V := \frac{V}{w_0}, \quad L_k V := \frac{1}{w_k} DL_{k-1} V, \quad k = 1, \ldots, n,
$$

(1)

where $D$ denotes ordinary differentiation;

– define $\mathbb{E}$ as the set of all functions $V \in C^n(I)$ for which $L_n V$ is constant on $I$.

The set of all extended Chebyshev spaces on $I$ being stable under integration as well as under multiplication by sufficiently differentiable non-vanishing functions, one can easily check that $\mathbb{E}$ is indeed an $(n + 1)$-dimensional extended Chebyshev space on $I$ (see [7, 17]). In the latter situation, we say that $(w_0, \ldots, w_n)$ is a system of weight functions on $I$ and that $\mathbb{E}$ is its associated extended Chebyshev space, which we denote as

$$
\mathbb{E} = EC(w_0, \ldots, w_n).
$$

(2)

From now on we assume that the interval $I$ is a closed bounded interval $[a, b], a < b$. Then, it is known that the converse is true too: namely, given any $(n + 1)$-dimensional extended Chebyshev space $\mathbb{E}$ on $[a, b]$, it is always possible to find a system $(w_0, \ldots, w_n)$ of weight functions on $[a, b]$ ensuring the equality $\mathbb{E} = EC(w_0, \ldots, w_n)$. In previous papers we already described a systematic procedure for building infinitely many essentially different such systems associated with $\mathbb{E}$ via (2) (see, for instance, [13]). The purpose of the present article is to find all of them. To achieve the result, it is sufficient to determine all non-vanishing functions $w_0 \in \mathbb{E}$ such that the space $DL_0 \mathbb{E} := \left\{D\left(\frac{L}{w_0}\right) \mid F \in \mathbb{E}\right\}$ is an $n$-dimensional extended Chebyshev space on $[a, b]$ and then to iterate. As a matter of fact, we transform the latter problem into an equivalent one in the extended Chebyshev space $\hat{\mathbb{E}}$ obtained from $\mathbb{E}$ by integration, in which we can take advantage of the presence of blossoms. The new problem can be stated as follows: find all functions $\hat{w}_0$, meeting the following two requirements: firstly, $\hat{w}_0 := DL\hat{w}_0$ does not vanish on $[a, b]$, and secondly the space $\hat{L}_1 \hat{\mathbb{E}} = \left\{\hat{L}_1 \hat{F} := \frac{D\hat{F}}{\hat{w}_0} \mid \hat{F} \in \hat{\mathbb{E}}\right\}$ possesses blossoms.

The difficult and interesting part consists in proving that any function $\hat{w}_0$ of which the Bézier points relative to $(a, b)$ form a strictly monotone sequence does fulfill the latter requirements. It should be observed that the proof (Section 3) strongly relies on the three fundamental properties of blossoms. As a direct consequence we obtain in Section 4 the answer to our initial problem: the functions $\hat{w}_0 \in \hat{\mathbb{E}}$ which are convenient are exactly those all coordinates of which in any given Bernstein-like basis relative to $(a, b)$ have the same strict sign.

To make the description of the article complete, let us mention that Section 2 is devoted to a brief reminder about important properties of extended Chebyshev spaces, in particular how they are connected with Bernstein-type bases and blossoms, along with the link existing between weight functions and nested sequences of extended Chebyshev spaces. In our final section (Section 5) we consider the implications of the results described above on Bernstein operators as we recently introduced them in [14].

2. Background on extended Chebyshev spaces

In the present section we give the necessary background on extended Chebyshev spaces which will be needed in the next three sections. For further acquaintance with the subject we refer the reader either to the classical books [7, 17], or to more recent articles on Chebyshevian blossoms, e.g., [16, 8, 9, 11, 12, 10].
2.1. EC-spaces and Bernstein-type bases

An \((n + 1)\)-dimensional space \(\mathbb{E} \subset C^n([a, b])\) is said to be an extended Chebyshev space (for short, EC-space) on \([a, b]\) if any Hermite interpolation problem in \((n + 1)\) data has a unique solution in \(\mathbb{E}\). This means that, for any integer \(r, 1 \leq r \leq n + 1\), any positive integers \(\mu_1, \ldots, \mu_r\) such that \(\sum_{i=1}^{r} \mu_i = n + 1\), any pairwise distinct \(x_1, \ldots, x_r \in [a, b]\), and any real numbers \(\alpha_{i,j}, 0 \leq j \leq \mu_i - 1, 1 \leq i \leq r\), there exists a unique function \(F \in \mathbb{E}\) which satisfies

\[
F^{(j)}(x_i) = \alpha_{i,j}, \quad 0 \leq j \leq \mu_i - 1, \quad 1 \leq i \leq r.
\]

Equivalently, the space \(\mathbb{E}\) is an EC-space on \([a, b]\) if any non-zero element of \(\mathbb{E}\) vanishes at most \(n\) times in \([a, b]\), counting multiplicities up to \((n + 1)\).

EC-spaces can be characterised in terms of special bases of Bernstein type, according to the definition below.

**Definition 2.1.** Given \(c, d \in [a, b]\), \(c < d\), and given \(V_0, \ldots, V_n \in C^n([a, b])\), we say that \((V_0, \ldots, V_n)\) is a Bernstein-like basis relative to \((c, d)\) when the following two properties are satisfied:

1. for \(k = 0, \ldots, n\), \(V_k\) vanishes exactly \(k\) times at \(c\), and exactly \((n - k)\) times at \(d\);
2. for \(k = 0, \ldots, n\), \(V_k\) is positive on \([c, d]\).

A Bernstein-like basis \((V_0, \ldots, V_n)\) is automatically a basis of the space \(\mathbb{E}\) that it spans, each function \(V_i\) being uniquely determined in \(\mathbb{E}\) up to multiplication by a positive real number. If we start with an \((n + 1)\)-dimensional EC-space \(\mathbb{E}\) on \([a, b]\), Hermite interpolation being always possible, clearly \(\mathbb{E}\) possesses a basis \((V_0, \ldots, V_n)\) which satisfies (1) of **Definition 2.1**. Since each \(V_i\) keeps the same strict sign on \([a, b]\), we can assume this sign to be positive. Hence, the implication (i) \(\Rightarrow\) (ii) of **Proposition 2.2** readily follows from the definition of EC-spaces. The interesting part is thus the converse implication for which we can even omit part (2) of **Definition 2.1**. For the proof, we refer the reader to [11].

**Proposition 2.2** ([11]). Let \(\mathbb{E} \subset C^n([a, b])\) be an \((n + 1)\)-dimensional space. Then the following properties are equivalent:

1. \(\mathbb{E}\) is an EC-space on \([a, b]\);
2. \(\mathbb{E}\) possesses a Bernstein-like basis relative to any \((c, d)\) \(\in [a, b]^2, c < d\).

**Definition 2.3.** Given \(c, d \in [a, b]\), \(c < d\), and given \(B_0, \ldots, B_n \in C^n([a, b])\), we say that \((B_0, \ldots, B_n)\) is a Bernstein basis relative to \((c, d)\) if it is a Bernstein-like basis relative to \((c, d)\) which is normalised in the sense that \(\sum_{k=0}^{n} B_k(x) = 1\) for all \(x \in [a, b]\).

When \(\mathbb{E}\) is an EC-space on \([a, b]\) which contains constants, it is not always possible to choose non-zero constants \(\alpha_0, \ldots, \alpha_n\) so as to transform a given Bernstein-like basis \((V_0, \ldots, V_n)\) into a Bernstein basis \((\alpha_0 V_0, \ldots, \alpha_n V_n)\). For instance, for \(\pi \leq b < 2\pi\), the space \(\mathbb{E}\) spanned on \([0, b]\) by the functions \(1, \cos, \sin\) is an EC-space on \([0, b]\), but it possesses no Bernstein basis relative to \((0, b)\). See [11] and **Example 4.5**.

Let us recall that an \((n + 1)\)-dimensional space \(\mathbb{E} \subset C^n([a, b])\) is said to be a \(W\)-space on \([a, b]\) when the Wronskian of a basis \((U_0, \ldots, U_n)\) of \(\mathbb{E}\) never vanishes on \([a, b]\), i.e., when

\[
W(U_0, \ldots, U_n)(x) := \det(U_i^{(j)}(x))_{0 \leq i,j \leq n} \neq 0, \quad x \in [a, b].
\]
An EC-space on \([a, b]\) is automatically a \(W\)-space on \([a, b]\). The converse is not true as proved by the example of the space spanned by the two functions \(\cos, \sin\), the Wronskian of which is equal to 1 on the whole of \(\mathbb{R}\) and which is an EC-space on \([0, b]\) only when \(0 < b < \pi\). The following result, proved in [11], will be essential in the search for weight functions (for the proof of (i) \(\Rightarrow\) (ii) and (iii), see also [15]). Subsequently, for any \(x \in [a, b]\) and any non-negative integer \(k\), the notation \(x^{[k]}\) stands for \(x\) repeated \(k\) times.

**Proposition 2.4.** Let \(E \subset C^n([a, b])\) be an \((n + 1)\)-dimensional \(W\)-space on \([a, b]\) containing constants. Then the following three properties are equivalent:

(i) the space \(D E\) is an \((n\)-dimensional\) \(EC\)-space on \([a, b]\);

(ii) \(E\) possesses a Bernstein basis relative to any \((c, d) \in [a, b]_2, c < d\);

(iii) blossoms exist in the space \(E\), the blossom \(f\) of any function \(F \in E\) being a function of \(n\) variables defined on \([a, b]^n\).

Assume that any of the properties (i), (ii), (iii) above holds. Then, the Bernstein basis \((B_0, \ldots, B_n)\) relative to \((a, b)\) is the optimal normalised totally positive basis of the space \(E\). Moreover, any \(F \in E\) can be decomposed as

\[
F = \sum_{k=0}^{n} f(a^{[n-k]}, b^{[k]}) B_k,
\]

the real numbers \(f(a^{[n-k]}, b^{[k]}), 0 \leq k \leq n\), being called the Bézier points of \(F\) relative to \((a, b)\).

Roughly speaking, subject to existence, the optimal normalised totally positive basis is the best basis for design purposes in a given space of functions. For the precise meaning, see [9,4,6]. Let us recall that in the Chebyshevian framework, blossoms are defined in a geometrical way, with the help of osculating flats. We shall not say more on the precise definition of this powerful and elegant tool, limiting ourselves to listing below the three main properties resulting from their geometrical definition:

(B)_1 symmetry: \(f\) is symmetric on \([a, b]^n\);

(B)_2 diagonal property: for all \(x \in [a, b]\), \(f(x^{[n]}) = F(x)\);

(B)_3 pseudoaffinity property: given any \(y_1, \ldots, y_{n-1}, c, d \in [a, b]\), with \(c < d\), there exists a strictly increasing function \(\beta(y_1, \ldots, y_{n-1}; c, d); : [a, b] \rightarrow \mathbb{R}\) (independent of \(F\)) such that

\[
f(y_1, \ldots, y_{n-1}, x) = \left[1 - \beta(y_1, \ldots, y_{n-1}; c, d; x)\right] f(y_1, \ldots, y_{n-1}, c) + \beta(y_1, \ldots, y_{n-1}; c, d; x) f(y_1, \ldots, y_{n-1}, d), \quad x \in [a, b].
\]

The latter three properties make blossoming the ideal tool for developing all design algorithms. They are the underlying reason why the Bernstein basis relative to \((a, b)\) is totally positive; see [9]. Readers interested in blossoms can have a look at [16,15,8,11,12], for instance.

### 2.2. Weight functions and nested sequences of EC-spaces

In the introduction we recalled how to define EC-spaces by means of weight functions. As a matter of fact, a given system \((w_0, \ldots, w_n)\) of weight functions on \([a, b]\) provides us not only with one \((n + 1)\)-dimensional EC-space on \([a, b]\), but even with a nested sequence

\[
E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n \subset C^n([a, b]),
\]

...
in which each \( \mathbb{E}_k \) is the \((k + 1)\)-dimensional EC-space on \([a, b]\)

\[
\mathbb{E}_k := \text{EC}(w_0, \ldots, w_k), \quad 0 \leq k \leq n.
\]

With the notation introduced in (1), each \( \mathbb{E}_k \) can thus be described as the set all functions \( V \in C^n([a, b]) \) for which \( L_k V \) is constant on \([a, b]\). Consider two different systems \((w_0, \ldots, w_n)\) and \((\overline{w}_0, \ldots, \overline{w}_n)\) of weight functions on \([a, b]\) which are equivalent, in the sense that there exist (non-zero) real numbers \( \alpha_0, \ldots, \alpha_n \) such that

\[
\overline{w}_i = \alpha_i w_i, \quad 0 \leq i \leq n.
\]

Then, they clearly satisfy

\[
\text{EC}(w_0, \ldots, w_k) = \text{EC}(\overline{w}_0, \ldots, \overline{w}_k), \quad 0 \leq k \leq n.
\]

Let us now start with a nested sequence (5) in which we simply assume that each \( \mathbb{E}_k \) is a \((k + 1)\)-dimensional \( W \)-space on \([a, b]\). For \( k = 0, \ldots, n \), choose \( U_k \in \mathbb{E}_k \setminus \mathbb{E}_{k-1} \), with \( \mathbb{E}_{-1} := \{0\} \). For \( k = 0, \ldots, n \), the functions \( U_0, \ldots, U_k \) form a basis of the space \( \mathbb{E}_k \), and therefore their Wronskian does not vanish on \([a, b]\). For any \( V \in C^n([a, b]) \) one can thus consider the quantities

\[
L_k V := \frac{W(U_0, \ldots, U_{k-1}, V)}{W(U_0, \ldots, U_{k-1}, U_k)}, \quad 0 \leq k \leq n.
\]

(6)

For \( 0 \leq k \leq n \), \( L_k \) is a linear differential operator of order \( k \). Up to non-zero multiplicative constants, the operators \( L_0, \ldots, L_n \) are uniquely associated with the nested sequence of \( W \)-spaces (5). Clearly, each space \( \mathbb{E}_k \) can then be described as the set of all functions \( V \in C^n([a, b]) \) for which \( L_k V \) is constant on \([a, b]\). Actually, when setting (see [5,12])

\[
w_0 := U_0, \quad w_k := D(L_{k-1} U_k) = \frac{W(U_0, \ldots, U_{k-2}) W(U_0, \ldots, U_{k-1}, U_k)}{W(U_0, \ldots, U_{k-1})^2}
\]

for \( 1 \leq k \leq n \),

we obtain non-vanishing functions \( w_0, \ldots, w_n \), with \( w_k \in C^{n-k}([a, b]) \), that is, we obtain a system of weight functions on \([a, b]\). Moreover, the differential operators (6) are associated with the system \((w_0, \ldots, w_n)\) according to (1).

As a consequence, the equalities \( \mathbb{E}_k = \text{EC}(w_0, \ldots, w_k) \), \( 0 \leq k \leq n \), provide us with a one-to-one correspondence between all possible nested sequences (5) of \( W \)-spaces (or of EC-spaces as well) and all possible equivalence classes of systems of weight functions \((w_0, \ldots, w_n)\).

Let us now consider a given \((n + 1)\)-dimensional EC-space \( \mathbb{E} \) on \([a, b]\). We similarly have a one-to-one correspondence between all possible equivalence classes of systems of weight functions \((w_0, \ldots, w_n)\) such that \( \mathbb{E} = \text{EC}(w_0, \ldots, w_n) \) and all possible nested sequences (5) of EC-spaces contained in \( \mathbb{E} \), i.e., for which \( \mathbb{E}_n = \mathbb{E} \). Let us recall a systematic procedure for building such nested sequences. From the fact that we are dealing with a closed bounded interval, we can extend \( \mathbb{E} \) into an \((n + 1)\)-dimensional EC-space \( \tilde{\mathbb{E}} \) on an interval \([\tilde{a}, \tilde{b}]\) such that \( \tilde{a} < a, \tilde{b} > b \). Select any \( c \in [\tilde{a}, \tilde{b}] \setminus [a, b] \). For \( 0 \leq k \leq n \), let \( \mathbb{E}_k \) be the linear subspace of \( \tilde{\mathbb{E}} \) composed of all functions \( \tilde{F} \in \tilde{\mathbb{E}} \) which vanish (at least) \((n - k)\) times at \( c \), and let \( \mathbb{E}_k \) denote its restriction to \([a, b]\). This provides us with a nested sequence (5) in which each \( \mathbb{E}_k \) is an \((k + 1)\)-dimensional EC-space on \([a, b]\), and with \( \mathbb{E}_n = \mathbb{E} \) [13]. We can thus assert the existence of infinitely many nested sequences (5) composed of EC-spaces contained in \( \mathbb{E} \), that is, infinitely many equivalence classes of systems of weight functions such that \( \mathbb{E} = \text{EC}(w_0, \ldots, w_n) \).
Remark 2.5. One major difficulty encountered in the Chebyshevian framework is that, unlike the space of all polynomials, the set of all EC-spaces on a given interval \([a, b]\) is not stable under differentiation, even understood in a Chebyshevian way. If \(E\) is an EC-space on \([a, b]\), of dimension \((n + 1) \geq 2\), as soon as a function \(w_0 \in E\) does not vanish on \([a, b]\) one can define \(L_0\) by \(L_0 V := V / w_0\) and the space \(L_0 E\) is an \((n + 1)\)-dimensional EC-space on \([a, b]\) which contains constants. However the \(n\)-dimensional space \(DL_0 E\) is not necessarily an EC-space on \([a, b]\). The most famous counter-example is certainly the space spanned by the three functions \(\cos, \sin\) which is an EC-space on \([0, 2\pi]\) for which the space \(DE\) is an EC-space on \([0, \pi]\) but not on \([0, \pi]\). In contrast, if \(E\) is an \((n + 1)\)-dimensional \(W\)-space on \([a, b]\), and if \(w_0 \in E\) does not vanish on \([a, b]\), then one can assert that the space \(DL_0 E\) is in turn an \(n\)-dimensional \(W\)-space on \([a, b]\). Accordingly, whenever we start with a given nested sequence (5) in which each \(E_k\) is a \((k + 1)\)-dimensional \(W\)-space on \([a, b]\), and with corresponding operators \(L_0, \ldots, L_n\), not only is each \(E_k\) automatically an EC-space on \([a, b]\), but also under generalised differentiation it leads to other nested sequences of EC-spaces, on the one hand

\[I \in E_{k+1} \subset E_{k+2} \subset \cdots \subset E_{n-1} \subset E_n, \quad 0 \leq k \leq n, \quad (7)\]

and on the other

\[DL_k E_{k+1} \subset DL_k E_{k+2} \subset \cdots \subset DL_k E_{n-1} \subset DL_k E_n, \quad 0 \leq k \leq n - 1, \quad (8)\]

in which for \(k + 1 \leq i \leq n\), \(DL_k E_i\) is \((i - k)\)-dimensional.

On account of the observations above, it is easy to obtain (see [14] for details):

Proposition 2.6. Let \(E, E_k\) be two EC-spaces on \([a, b]\), of dimension \((n + 1)\) and \((k + 1)\), respectively, with \(n \geq k + 1\). Assume that \(E_k \subset E\). Choose any system of weight functions \((w_0, \ldots, w_k)\) such that \(E_k = EC(w_0, \ldots, w_k)\), and denote by \(L_0, \ldots, L_k\) the associated differential operators defined according to (1). The following three properties are equivalent:

(i) \(DL_k E\) is an \(((n - k)\)-dimensional) EC-space on \([a, b]\);
(ii) there exists a system \((w_{k+1}, \ldots, w_n)\) of weight functions on \([a, b]\) such that \(E = EC(w_0, \ldots, w_k, w_{k+1}, \ldots, w_n)\);
(iii) there exists a nested sequence

\[E_k \subset E_{k+1} \subset \cdots \subset E_n := E,\]

in which each \(E_i\) is an \((i + 1)\)-dimensional EC-space on \([a, b]\).

Under the same assumptions as in Proposition 2.6, assume (i) holds and that \(n \geq k + 2\). Then, there even exist infinitely many different equivalence classes of weight functions \((w_{k+1}, \ldots, w_n)\) so as to ensure the equality \(E = EC(w_0, \ldots, w_n)\), or, equivalently, infinitely many different ways to form a nested sequence of EC-spaces from \(E_k\) to \(E\).

3. The key result

Let \(E\) be a given \((n + 1)\)-dimensional EC-space on \([a, b]\). It is possible to find a nested sequence

\[I \in E_1 \subset E_2 \subset \cdots \subset E_{n-1} \subset E_n := E, \quad (9)\]

in which each \(E_k\) is a \((k + 1)\)-dimensional EC-space on \([a, b]\), if and only if \(E\) contains constants and the space \(DE\) is an EC-space on \([a, b]\) (see Proposition 2.6). Such a nested sequence is of
special interest for geometric design for several reasons. It implies the existence of blossoms not only in the space $E$ itself (blossoms in $n$ variables), but also in each space $E_k$, $1 \leq k \leq n - 1$ (blossoms in $k$ variables). This follows from Proposition 2.4 due to the fact that, for $k = 1, \ldots, n$, the space $D E_k$ is a $k$-dimensional EC-space on $[a, b]$ (see Remark 2.5). If $1 \leq k < n$, we can thus apply the so-called dimension elevation procedure to calculate the $(k + 2)$ Bézier points relative to $(a, b)$ of a given $F \in E_k$ considered as an element of $E_{k+1}$ from its $(k + 1)$ Bézier points in $E_k$ (see [14] and references therein). Remark 2.5 and Proposition 2.4 also guarantee the existence of blossoms in each space of the nested sequence (7). For $k = 0, \ldots, n - 2$, blossoms in $L_{k+1}E$ (in $(n - k - 1)$ variables) can be calculated from blossoms in $L_kE$ (in $(n - k)$ variables) [10].

Supposing that blossoms exist in $E$, we shall determine all two-dimensional EC-spaces $E_1$ leading to a nested sequence (9). This is the object of Theorem 3.2. Equivalently, Theorem 3.2 states all possible ways to differentiate (in a Chebyshevian sense) $E$ while maintaining the existence of blossoms. Beforehand we need to make the following observation on the two-dimensional case.

**Remark 3.1.** As already observed, if $E$ is an $(n + 1)$-dimensional $W$-space on $[a, b]$, it is not necessarily an EC-space on $[a, b]$. Nevertheless, if we consider the special case $n = 1$, and if we assume $E \subset C^1([a, b])$ to contain constants, then $E$ is an EC-space on $[a, b]$ if and only if it is a $W$-space on $[a, b]$, and also if and only if the space $DE$ is a one-dimensional EC-space (or $W$-space as well) on $[a, b]$. Equivalently, a two-dimensional space $E \subset C^1([a, b])$ containing constants is an EC-space on $[a, b]$ if and only if the first derivative of any non-constant $U \in E$ keeps the same strict sign on $[a, b]$.

**Theorem 3.2.** Given any integer $n \geq 2$, we assume that $E \subset C^n([a, b])$ contains constants and that the space $DE$ is an $n$-dimensional EC-space on $[a, b]$. Given $U \in E$, let $U$ denote the space spanned by the two functions $U, U$. The following eight properties are then equivalent:

(i) the Bézier points of $U$ relative to $(a, b)$ form a strictly increasing sequence;

(ii) for any $c, d \in [a, b], c < d$, the Bézier points of $U$ relative to $(c, d)$ form a strictly increasing sequence;

(iii) in the space $DE$, the coordinates of $DU$ in any Bernstein-like basis relative to $(a, b)$ are positive;

(iv) in the space $DE$, for any $c, d \in [a, b], c < d$, the coordinates of $DU$ in any Bernstein-like basis relative to $(c, d)$ are positive;

(v) the blossom $u$ of $U$ is strictly increasing in each variable on $[a, b]$;

(vi) the function $w_1 := DU$ is positive on $[a, b]$ and, if we define the first-order linear differential operator $L_1$ by $L_1V := (DV)/w_1$, then the set $DL_1E$ is an EC-space on $[a, b]$;

(vii) $U(a) < U(b)$, the space $U$ is a two-dimensional EC-space on $[a, b]$, and there exists a nested sequence

$$E_1 := U \subset E_2 \subset \cdots \subset E_{n-1} \subset E_n := E,$$

in which each $E_k$ is a $(k + 1)$-dimensional EC-space on $[a, b]$;

(viii) the function $w_1 := DU$ is positive on $[a, b]$ and there exists a system $(w_2, \ldots, w_n)$ of weight functions such that $E = EC(U, w_1, w_2, \ldots, w_n)$.

**Proof.** Clearly, there is nothing to prove concerning the statements (ii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (iii).

- (vii) $\Rightarrow$ (i): This is a consequence of dimension elevation. The Bézier points of $U$ in $E_1$ are just the two real numbers $U(a)$ and $U(b)$. If (vii) holds they do form a strictly increasing
sequence. By dimension elevation the Bézier points of \( U \) in any of the spaces \( E_k \) also form a strictly increasing sequence. For details, see [14] and earlier references therein.

- (vi) \( \Leftrightarrow \) (vii) \( \Leftrightarrow \) (viii): On account of Remark 3.1 this is a special case of Proposition 2.6.

- (i) \( \Rightarrow \) (v): Suppose that (i) holds. Given any \( x_1, \ldots, x_{n-1} \in [a, b] \), let us prove that the function \( u(x_1, \ldots, x_{n-1}) \) is strictly increasing on \([a, b]\). Due to the pseudoaffinity property \((B)_3\), it is sufficient to show that \( u(x_1, \ldots, x_{n-1}, a) < u(x_1, \ldots, x_{n-1}, b) \). For \( 0 \leq r \leq n-1 \), consider the real numbers

\[
p_i^r := u(x_1, \ldots, x_r, a^{n-r-i}, b^{i}), \quad 0 \leq i \leq n-r.
\]

Let us prove by induction on \( r \) that each sequence \((p_i^0, \ldots, p_i^{n-r})\) is strictly increasing. This clearly holds for \( r = 0 \) since the points \( p_i^0 = u(a^{n-i}, b^{i}), i = 0, \ldots, n \), are the Bézier points of \( U \) relative to \((a, b)\). Assume that the result holds for some \( r \leq n-2 \), and let us prove it for \( r + 1 \). If \( x_{r+1} = a \) or \( x_{r+1} = b \), there is nothing to prove since we have either \( p_i^{r+1} = p_i^r \) for \( 0 \leq i \leq n-r-1 \) or \( p_i^{r+1} = p_{i+1}^r \) for \( 0 \leq i \leq n-r-1 \). Assume that \( x_{r+1} \in [a, b] \). By application of the symmetry and pseudoaffinity properties \((B)_1\) and \((B)_3\) we have

\[
p_i^{r+1} = \left[ 1 - \beta(x_1, \ldots, x_r, a^{n-r-1-i}, b^{i}; a, b; x_{r+1}) \right] p_i^r + \beta(x_1, \ldots, x_r, a^{n-r-1-i}, b^{i}; a, b; x_{r+1}) p_{i+1}^r.
\]

Moreover, the fact that \( x_{r+1} \in [a, b] \) ensures that

\[
0 < \beta(x_1, \ldots, x_r, a^{n-r-1-i}, b^{i}; a, b; x_{r+1}) < 1.
\]

As a consequence, we have

\[
p_i^r < p_i^{r+1} < p_{i+1}^r, \quad 0 \leq i \leq n-r,
\]

which proves the expected result for \( r + 1 \). We thus have \( p_0^{n-1} = u(x_1, \ldots, x_{n-1}, a) < p_1^{n-1} = u(x_1, \ldots, x_{n-1}, b) \).

- (v) \( \Rightarrow \) (ii): Given \((c, d)\) in \([a, b]^2\), with \( c < d \), the Bézier points \((u_0, \ldots, u_n)\) of \( U \) relative to \((c, d)\) are defined by

\[
u_i := u(c^{n-i}, d^{i}), \quad 0 \leq i \leq n.
\]

According to (v), for \( 0 \leq i \leq n-1 \), the function \( u(c^{n-i-1}, d^{i}, c) \) is strictly increasing on \([a, b]\). In particular we thus have

\[
u_i = u(c^{n-i-1}, d^{i}, c) < u(c^{n-i-1}, d^{i}, d) = u_{i+1}, \quad 0 \leq i \leq n-1.
\]

- (i) \( \Leftrightarrow \) (iii) and (ii) \( \Leftrightarrow \) (iv): It clearly suffices to prove the equivalence between (i) and (iii). On the other hand, (iii) holds iff the coordinates of \( DU \) in one given Bernstein-like basis relative to \((a, b)\) are positive.

Let \((B_0, \ldots, B_n)\) denote the Bernstein basis relative to \((a, b)\) in the space \( E \). In [14] we proved that the sequence \((V_0, \ldots, V_{n-1})\) defined by

\[
V_k := \sum_{j=k+1}^{n} B_j' = -\sum_{j=0}^{k} B_j', \quad 0 \leq k \leq n-1
\]  
(11)
is a Bernstein-like basis relative to \((a, b)\) in the space \(D \mathbb{E}\). We also showed that if \(U = \sum_{i=0}^{n} u_i B_i\), then

\[
U' = \sum_{k=0}^{n-1} (u_{k+1} - u_k) V_k,
\]

(12)
due to \((B_0, \ldots, B_n)\) being normalised. Clearly, the Bézier points \(u_0, \ldots, u_n\) of \(U\) relative to \((a, b)\) form a strictly increasing sequence if and only if all coordinates \((u_{i+1} - u_i), 0 \leq i \leq n-1\), of \(DU\) in the Bernstein-like basis \((V_0, \ldots, V_{n-1})\) are positive. The equivalence between (i) and (iii) is thus achieved.

• (iv) \(\Rightarrow\) (vi): Let \((V_0, \ldots, V_{n-1})\) be any Bernstein-like basis relative to \((a, b)\) in the space \(D \mathbb{E}\). Assume that

\[
w_1 := DU = \sum_{i=0}^{n-1} \alpha_i V_i, \quad \text{with } \alpha_i > 0 \text{ for } 0 \leq i \leq n - 1.
\]

(13)

Given that each \(V_i\) is positive on \(]a, b[\), we can assert that \(w_1(x) > 0\) on \(]a, b[\). On the other hand, since \(V_0(a) \neq 0, V_{n-1}(b) \neq 0\), continuity arguments show that \(V_0(a)\) and \(V_{n-1}(b)\) are both positive. Accordingly \(w_1(a) = \alpha_0 V_0(a) > 0, w_1(b) = \alpha_{n-1} V_{n-1}(b) > 0\). Therefore, \(w_1\) is positive on the whole of \([a, b]\). After division by \(w_1\), equality (13) yields

\[
\mathbb{I} = \sum_{i=0}^{n-1} \bar{B}_i, \quad \text{with } \bar{B}_i := \frac{V_i}{w_1} \text{ for } 0 \leq i \leq n - 1.
\]

The sequence \((\bar{B}_0, \ldots, \bar{B}_{n-1})\) is thus normalised. Clearly, each \(\bar{B}_i\) has exactly the same zeros at \(a\) and \(b\) as \(V_i\). It follows that \((\bar{B}_0, \ldots, \bar{B}_{n-1})\) is a Bernstein basis relative to \((a, b)\) in the space \(L_{1 \mathbb{E}}\).

Starting with any \(c, d \in [a, b]\), \(c < d\), instead of \(a, b\), the same arguments would similarly prove the existence of a Bernstein basis relative to \((c, d)\) in the same space \(L_{1 \mathbb{E}}\). The \(W\)-space \(L_{1 \mathbb{E}}\) is thus shown to possess a Bernstein basis relative to any \((c, d), c < d\). According to Proposition 2.4, this means that the space \(DL_{1 \mathbb{E}}\) is an EC-space on \([a, b]\) (of dimension \(n - 1\)).

4. Blossoms and generalised differentiation

If we were considering an EC-space \(\mathbb{E}\) on an interval \(I\) which would not be supposed to be closed and bounded, we would have no guarantee of finding a function \(w_0 \in \mathbb{E}\) such that \(w_0(x) \neq 0\) for all \(x \in I\). For instance the space spanned by \(\cos, \sin\) possesses no non-vanishing functions on \([0, \pi]\) although it is an EC-space on \([0, \pi]\). The fact that we are working on the closed bounded interval \([a, b]\) makes things different. There is indeed always a very simple way to build a non-vanishing function \(w_0\) in an \((n + 1)\)-dimensional EC-space \(\mathbb{E}\) on \([a, b]\). Select a Bernstein-like basis \((V_0, \ldots, V_n)\), and take \(w_0 := \sum_{i=0}^{n} V_i\). This function is positive on \([a, b]\). When defining \(L_0\) by \(L_0 V := V/w_0\) as usual, the space \(L_0 \mathbb{E}\) does contain constants. For all that, can we guarantee existence of blossoms in \(L_0 \mathbb{E}\)? We shall obtain the answer to the latter question on the way to determining all possible weight functions associated with a given EC-space.

4.1. All weight functions associated with an EC-space

Finding all possible weight functions associated with any given \((n + 1)\)-dimensional EC-space on \([a, b]\) will be made possible by Theorem 4.1. The latter theorem itself will be obtained as a consequence of the key result (Theorem 3.2) established in the previous section.
**Theorem 4.1.** Let $\mathcal{E}$ be an $(n + 1)$-dimensional EC-space on $[a, b]$. Given any $w_0 \in \mathcal{E}$, the following six properties are equivalent:

(i) the coordinates of $w_0$ in a given Bernstein-like basis relative to $(a, b)$ all have the same strict sign;
(ii) the coordinates of $w_0$ in any Bernstein-like basis relative to any $(c, d) \in [a, b]^2$, $c < d$, all have the same strict sign;
(iii) $w_0$ does not vanish on $[a, b]$ and, setting $L_0 V := V/w_0$ for all functions $V$ defined on $[a, b]$, blossoms exist in the space $L_0 \mathcal{E}$;
(iv) $w_0$ does not vanish on $[a, b]$ and the space $DL_0 \mathcal{E}$ is an EC-space on $[a, b]$;
(v) $w_0$ does not vanish on $[a, b]$ and there exists a system $(w_1, \ldots, w_n)$ of weight functions on $[a, b]$ such that $\mathcal{E} = \text{EC}(w_0, w_1, \ldots, w_n)$;
(vi) there exists a nested sequence

$$E_0 := \text{span}(w_0) \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = \mathcal{E},$$

in which, for $0 \leq k \leq n - 1$, the space $E_k$ is a $(k + 1)$-dimensional EC-space on $[a, b]$.

**Proof.** The set $\hat{\mathcal{E}}$ composed of all functions $\hat{F} \in C^{n+1}([a, b])$ such that $D \hat{F} \in \mathcal{E}$ is an $(n + 2)$-dimensional EC-space on $[a, b]$ and it contains constants. The equivalence between the properties (i), (iii), (iv), and (v) above readily follows by applying Theorem 3.2 in $\hat{\mathcal{E}}$. On the other hand, if $w_0 \in \mathcal{E}$ does not vanish on $[a, b]$, the space $L_0 \mathcal{E}$ is a $W$-space on $[a, b]$ which contains constants. Therefore, according to Proposition 2.4, blossoms exist in $L_0 \mathcal{E}$ if and only if $DL_0 \mathcal{E}$ is an EC-space on $[a, b]$.

As usual let us denote by $L_0$ the division by a given non-vanishing function $w_0 \in C^n([a, b])$. Then, for any system $(w_1, \ldots, w_n)$ of weight functions on $[a, b]$, the following equivalence holds:

$$\mathcal{E} = \text{EC}(w_0, w_1, \ldots, w_n) \iff DL_0 \mathcal{E} = \text{EC}(w_1, \ldots, w_n).$$

(14)

Consider a given $(n + 1)$-dimensional EC-space $\mathcal{E}$ on $[a, b]$. Equivalence (14) means that, once we have chosen a non-vanishing function $w_0$ such that $DL_0 \mathcal{E}$ is in turn an EC-space, the search for all possible non-vanishing functions $w_1 \in C^{n-1}([a, b])$ leading to an equality $\mathcal{E} = (w_0, w_1, w_2, \ldots, w_n)$ amounts to the search for all non-vanishing functions $w_1 \in DL_0 \mathcal{E}$ leading to an equality $DL_0 \mathcal{E} = \text{EC}(w_1, w_2, \ldots, w_n)$. As a consequence, we just have to iterate Theorem 4.1 as explained in the following procedure:

- **first step:**
  1- select a Bernstein-like basis relative to $(a, b)$ in $\mathcal{E}^0 := \mathcal{E}$, say $(V_0^0, \ldots, V_n^0)$;
  2- choose any real numbers $\alpha_0^0, \ldots, \alpha_n^0$ all of the same strict sign, and set $w_0 := \sum_{i=0}^n \alpha_i^0 V_i^0$, and $L_0 V := V/w_0$ for any $V \in C^n([a, b])$;

- **$(k + 1)$th step:** given an integer $k$, $1 \leq k \leq n - 1$, assume that we have built $w_0, \ldots, w_{k-1}$, with $w_i \in C^{n-i}([a, b])$, so that $\mathcal{E}^k := DL_{k-1} \mathcal{E} \subset C^{n-k}([a, b])$ is an $(n - k + 1)$-dimensional EC-space on $[a, b]$, where $L_0, \ldots, L_{k-1}$ are defined from $w_0, \ldots, w_{k-1}$ according to (1);
  - select a Bernstein-like basis relative to $(a, b)$ in $\mathcal{E}^k$, say $(V_0^k, \ldots, V_{n-k}^k)$;
  - choose any real numbers $\alpha_0^k, \ldots, \alpha_{n-k}^k$ all of the same strict sign, and set $w_k := \sum_{i=0}^{n-k} \alpha_i^k V_i^k$, and $L_k V := (DL_{k-1} V)/w_k$ for any $V \in C^n([a, b])$;
  - the space $\mathcal{E}^{k+1} := DL_k \mathcal{E} \subset C^{n-k-1}([a, b])$: it is an $(n - k)$-dimensional EC-space on $[a, b]$. 

At the $n$th step we thus obtain $\mathbb{E}^n := DL_{n-1}\mathbb{E} \subset C^0([a, b])$ which is a one-dimensional EC-space on $[a, b]$. It can thus be written as $\mathbb{E}^n := \text{EC}(w_n)$ for any non-zero element $w_n \in \mathbb{E}^n$. Equivalence (14) shows that

$$\mathbb{E}^k = \text{EC}(w_k, \ldots, w_n), \quad 0 \leq k \leq n.$$  

Furthermore, it even proves the following result:

**Theorem 4.2.** Given any $(n + 1)$-dimensional EC-space on $[a, b]$, the sequence of functions $(w_0, \ldots, w_n)$ built according to the previous procedure is a system of weight functions on $[a, b]$ such that $\mathbb{E} = \text{EC}(w_0, \ldots, w_n)$. Moreover, this procedure provides us with all possible systems of weight functions on $[a, b]$ such that $\mathbb{E} = \text{EC}(w_0, \ldots, w_n)$.

Equivalently, one can say that the previous procedure provides us with all possible nested sequences (5) composed of EC-spaces (or of $W$-spaces) contained in $\mathbb{E}$. More precisely, at each step $k = 0, \ldots, n - 1$, it provides us with all infinitely many different possibilities for choosing $\mathbb{E}_k$, either to start such a nested sequence or to continue it.

### 4.2. Consequences and comments

As a result of Theorem 4.1 we can state the following corollary.

**Corollary 4.3.** Let $\mathbb{E}$ be an $(n + 1)$-dimensional EC-space on $[a, b]$ which contains constants. Then the following properties are equivalent:

(i) $\mathbb{E}$ possesses a Bernstein basis relative to $(a, b)$;

(ii) the space $D\mathbb{E}$ is an $n$-dimensional EC-space on $[a, b]$;

(iii) blossoms exist in the space $\mathbb{E}$.

**Proof.** If $\mathbb{E}$ possesses a Bernstein basis $(B_0, \ldots, B_n)$ relative to $(a, b)$, then $w_0 := 1 = \sum_{i=0}^{n} B_i$ has positive coordinates in $(B_0, \ldots, B_n)$. By (i) $\Rightarrow$ (ii) of Theorem 4.1 we can assert that $D\mathbb{E}$ is an $n$-dimensional EC-space on $[a, b]$. The converse follows from Proposition 2.4, as well as the equivalence between (ii) and (iii). \qed

Note that, on account of Proposition 2.4, one can also state:

**Corollary 4.4.** Let $\mathbb{E}$ be an $(n + 1)$-dimensional EC-space on $[a, b]$. If $\mathbb{E}$ possesses a Bernstein basis relative to $(a, b)$, then the latter basis is the optimal normalised totally positive basis of $\mathbb{E}$. Moreover, the space $\mathbb{E}$ also possesses a Bernstein basis relative to $(c, d)$ for any $(c, d) \in [a, b]^2, c < d$.

**Example 4.5.** Choose any $b$ such that $\pi < b < 2\pi$. The space $\mathbb{E}$ spanned by the three functions $1, \cos, \sin$ is then a three-dimensional EC-space on $[0, b]$. The three functions $U_0(x) := 1 + \cos x, U_1(x) := \sin x, \text{ and } U_2(x) := 1 - \cos x$ form a Bernstein-like basis relative to $(0, \pi)$. It is however impossible to find positive $\alpha_0, \alpha_1, \alpha_2$ such that $\sum_{i=0}^{2} \alpha_i U_i = 1$. Accordingly, although $\mathbb{E}$ contains constants, it does not possess a Bernstein basis relative to $(0, \pi)$. Corollary 4.4 ensures that it possesses no Bernstein basis relative to $(0, b)$ either.

**Remark 4.6.** In Proposition 2.4 we recalled the equivalence between existence of blossoms and existence of a Bernstein basis relative to any $(c, d) \in [a, b]^2, c < d$. Corollary 4.3 states the equivalence between existence of blossoms and existence of only one Bernstein basis, namely
the one relative to \((a, b)\). However it is important to stress that in Corollary 4.3 the space \(E\) is known in advance to be an EC-space on \([a, b]\), whereas in Proposition 2.4 it was only supposed to be a \(W\)-space on \([a, b]\).

**Remark 4.7.** To stress the fact that a function \(w_0 \in E\) may be positive without its coordinates in a given Bernstein-like basis being all positive—and therefore, due to Theorem 4.1, without the space \(DL_0E\) being an EC-space on \([a, b]\)—let us consider the simple case where \(E\) is the restriction to \([0, 1]\) of the polynomial space of degree 2, in which we denote the Bernstein basis by \((B_0^2, B_1^2, B_2^2)\).

1. Take \(w_0 := B_0^2 - 2B_1^2 + 5B_2^2\). Then \(w_0(x) = 10x^2 - 6x + 1\) is positive on the whole of \(\mathbb{R}\). The space \(L_0E\) can be described as the set of all functions \(x \in [0, 1] \mapsto \alpha + \frac{\beta x + \gamma}{10x^2 - 6x + 1}\), where \(\alpha, \beta, \gamma\) are any real numbers. As for the space \(DL_0E\), it is composed of all functions \(x \in [0, 1] \mapsto (-10\beta x^2 - 20\gamma x + \beta + 6\gamma)/(10x^2 - 6x + 1)^2\), \(\beta, \gamma \in \mathbb{R}\). Theorem 4.1 tells us that \(DL_0E\) is not an EC-space on \([a, b]\). Indeed, if we choose for instance \(\beta = -2, \gamma = 1\), the numerator is equal to \(20x^2 - 20x + 4\) and it vanishes twice in \([0, 1]\).

2. Take \(w_0 := B_0^2 + B_2^2\). Then \(w_0(x) = 2x^2 - 2x + 1\) is positive on \(\mathbb{R}\). Similarly the space \(DL_0E\) is composed of all functions \(x \in [0, 1] \mapsto (-2\beta x^2 - 4\gamma x + \beta + 2\gamma)/(2x^2 - 2x + 1)^2\), \(\beta, \gamma \in \mathbb{R}\). For \(\beta = -2, \gamma = 1\), the numerator vanishes again twice in \([a, b]\) since it is equal to \(4x(x - 1)\).

**Remark 4.8.** To conclude the present section, let us come back to the question that we mentioned at its very beginning. Assume that \(E\) is an EC-space on \([a, b]\). Then, from Theorem 4.1 we can assert the following. Not only does choosing \(w_0\) as the sum of all elements of a Bernstein-like basis relative to \((a, b)\) guarantee the existence of blossoms in the space \(\frac{1}{w_0}E\), but also there is no alternative for guaranteeing it. Indeed, if \((V_0, \ldots, V_n)\) is a Bernstein-like basis relative to \((a, b)\) in the space \(E\), then, for any positive real numbers \(\alpha_0, \ldots, \alpha_n\), we know that \((\alpha_0V_0, \ldots, \alpha_nV_n)\) is also a Bernstein-like basis relative to \((a, b)\).

### 5. Application to Bernstein-type operators

Bernstein-type operators were first considered in the context of exponential polynomials [1] (see also [2]).

Consider a fixed space \(E \subset C^n([a, b])\). We assume that \(E\) is an EC-space on \([a, b]\) and that it possesses a Bernstein basis relative to \((a, b)\), which we denote as \((B_0, \ldots, B_n)\). According to Corollary 4.3, this amounts to assuming that \(E\) contains constants and the space \(DE\) is an \(n\)-dimensional EC-space on \([a, b]\). We are therefore in the precise situation in which we introduced Bernstein operators in [14]. We say that an operator \(B_n\) on \(C^0([a, b])\) is a Bernstein operator based on \(E\) if it is of the form

\[
B_n F := \sum_{k=0}^{n} F(t_k) B_k.
\]

where \(a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b\), and if it reproduces a two-dimensional EC-space \(E_1\), in the sense that it reproduces each element of \(E_1\), i.e.,

\[
B_n F = F \quad \text{for any } F \in E_1.
\]

We showed in particular the following results [14].
As soon as $n \geq 2$, there exist infinitely many Bernstein operators based on $E$. They are characterised by the two-dimensional EC-spaces that they reproduce which must all contain constants.

2- A function $U \in E$ being given, the space spanned by $(\mathbb{I}, U)$ is a two-dimensional EC-space on $[a, b]$ and it is reproduced by a (unique) Bernstein operator based on $E$ if and only if the Bézier points of $U$ relative to $(a, b)$ form a strictly monotone sequence.

3- As a special case, whenever we have a sequence of EC-spaces (9) contained in $E$, the two-dimensional EC-space $E_1$ is reproduced by a (unique) Bernstein operator based on $E$.

On account of Theorem 3.2 we now assert that all Bernstein operators based on $E$ are of the latter kind. Let us state this as follows:

**Theorem 5.1.** Let $E$ be an $(n + 1)$-dimensional EC-space on $[a, b]$ supposed to possess a Bernstein basis relative to $(a, b)$. Let $U \subset E$ be a two-dimensional EC-space containing constants. Then, the following three properties are equivalent:

(i) $U$ is reproduced by a (unique) Bernstein operator based on $E$;

(ii) there exists a nested sequence of EC-spaces (9) with $E_1 := U$;

(iii) if $U = \text{EC}(\mathbb{I}, w_1)$ and if $L_1$ denotes the associated generalised derivative $L_1V := (DV)/w_1$, then $DL_1$ is an EC-space on $[a, b]$ (of dimension $(n - 1)$).

When (ii) is satisfied we know that if $n \geq 3$ there exist infinitely many nested sequences (9) such that $E_1 = U$. Actually, according to Theorem 5.1 and Remark 2.5, a given nested sequence (9) provides us not only with a Bernstein operator $B_n$ based on the space $E$, but even with two sequences of Bernstein operators:

1. the sequence $(B_1, B_2, \ldots, B_{n-1}, B_n)$, where, for $1 \leq k \leq n$, $B_k$ is the unique Bernstein operator based on the space $E_k$ which reproduces $E_1$;

2. the sequence $(B^{[0]} := B_n, B^{[1]}, \ldots, B^{[n-1]})$, where, for $1 \leq k \leq n$, $B^{[k]}$ is the unique Bernstein operator based on the space $L_kE_k$ which reproduces the two-dimensional EC-space $L_kE_{k+1}$.

**Corollary 5.2.** The assumptions are the same as in Theorem 5.1. Then, if $U$ is reproduced by a Bernstein operator based on $E$, it is also reproduced by a Bernstein operator based on the restriction of $E$ to $[c, d]$, for any given $c, d \in [a, b]$, $c < d$.

**Proof.** This is due to (i) $\Rightarrow$ (ii) of Theorem 3.2. □

If $E$ does not possess a Bernstein basis relative to $(a, b)$, obviously there exists no Bernstein operator based on $E$. Nevertheless, one can similarly define Bernstein-like operators based on $E$ by simply replacing in (15) the Bernstein basis relative to $(a, b)$ by any possible Bernstein-like basis relative to $(a, b)$ [14]. On account of Theorem 4.1, results similar to Theorem 5.1 and Corollary 5.2 can be stated for Bernstein-like operators. We leave this to the reader.

**References**

