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Rewriting systems for Coxeter groups

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Abstract

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A finite complete rewriting system for a group is a finite presentation which gives a solution to the word problem and a regular language of normal forms for the group. In this paper it is shown that the fundamental group of an orientable closed surface of genus g has a finite complete rewriting system, using the usual generators $a_1, \ldots, a_g, b_1, \ldots, b_g$ along with generators representing their inverses. Constructions of finite complete rewriting systems are also given for any Coxeter group G satisfying one of the following hypotheses. (1) G has three or fewer generators. (2) G does not contain a special subgroup of the form $\langle s_i, s_j, s_k \mid s_i^2 = s_j^2 = s_k^2 = (s_is_j)^2 = (s_is_k)^m = (s_js_k)^n = 1 \rangle$ with m and n both finite and not both equal to two.

1. Introduction

One of the fundamental questions in the study of group theory is the solvability of the word problem. In general the word problem for finitely presented groups is not solvable; that is, given two words in the generators of the group, there may be no algorithm to decide whether the words in fact represent the same element of the group. For groups presented by a rewriting system that is finite and complete (defined in Section 2), however, the word problem is solved in a way that is particularly easy to implement on a computer. A complete rewriting system for a group also gives a set of normal forms for elements of the group; that is, for each group element there is a unique word representing it which cannot be rewritten.

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In 1985 a computer scientist, Jantzen [9], asked whether a finitely presented monoid or group with a solvable word problem necessarily must have a finite complete rewriting system. A couple of years later a mathematician, Squier [14], showed that the answer to Jantzen's question is negative. In the process, Squier showed that a group with a finite complete rewriting system necessarily has the homological finiteness condition FP_3, and others [1, 4, 6] have extended this to show that having a finite complete rewriting system implies that a group has homological type FP_∞.

In his paper, Jantzen also showed that the existence of a finite complete rewriting system for a group may depend on the presentation that one starts with. That is, it may be that a group has no finite complete rewriting system based on one set of generators, while it does have such a rewriting system based on another set of generators. Recently Squier [15] has developed a topological criterion, known as finite derivation type, which is a necessary condition for a finitely presented group to satisfy in order to have a finite complete presentation based on some set of generators. A natural question to ask, then, is whether this criterion is sufficient to imply the existence of a finite complete presentation, and, if not, then what else is needed.

A starting point for understanding questions about finite derivation type may be the study of Coxeter groups. Tits [16] has proven that these groups satisfy a topological property that is similar, although not identical, to finite derivation type.

Recently, Brink and Howlett [2] have shown that Coxeter groups have automatic structures. Both automatic structures and finite complete rewriting systems involve a regular language of normal forms for the groups. However, at the moment it is not clear what connection, if any, exists between automatic structures and rewriting systems. In 1986 a computer scientist, Le Chenadec, published a survey [12] of complete rewriting systems for groups, which included Coxeter groups; when rewriting systems for Coxeter groups were given, however, they were finite in general.

In this paper constructions of finite complete presentations are given for many families of Coxeter groups, as well as for surface groups. The second section gives basic definitions and properties of rewriting systems. The third section contains a discussion of rewriting systems for surface groups, including a proof of the following proposition.

Proposition. *There is a finite complete rewriting system for the fundamental group of a closed orientable surface of genus g, using the alphabet*

$$S = \{a_1, \ldots, a_g, A_1, \ldots, A_g, b_1, \ldots, b_g, B_1, \ldots, B_g\}$$

of the usual generators and their inverses.

Finally, the last section is on rewriting systems for Coxeter groups, with a proof of the following theorem.

Theorem. *Let G be a Coxeter group. Suppose G satisfies one of the following two properties.*

(1) *G has three or fewer generators.*

(2) *G does not contain a special subgroup of the form*

$$\langle s_i, s_j, s_k \mid s_i^2 = s_j^2 = s_k^2 = (s_i s_j)^2 = (s_i s_k)^m = (s_j s_k)^n = 1 \rangle$$

with m and n both finite and not both equal to two.

Then G has a finite complete rewriting system.

2. Rewriting systems

Let S be a set (called an *alphabet*) and let S^* be the free monoid on S. S^* consists of all words in the letters of S; the empty word will be represented by 1. A *rewriting system* on S^* is a subset $R \subseteq S^* \times S^*$. An element $(u, v) \in R$, also written $u \to v$, is called a *rule* of R. The idea is that a rewriting system is an algorithm for substituting the right-hand side of a rule whenever the left-hand side appears in a word. Given a rewriting system R, write $x \to y$ for $x, y \in S^*$ if $x = uv_1w$, $y = uv_2w$ and $(v_1, v_2) \in R$. Write $x \xrightarrow{*} y$ if $x = y$ or $x \to x_1 \to x_2 \to \cdots \to y$ for some finite chain of arrows. An element x of S^* is *irreducible* with respect to R if there is no possible rewriting (or *reduction*) $x \to y$; otherwise x is called *reducible*. (S, R) is a rewriting system for a monoid M if

$$\langle S \mid v_1 = v_2 \text{ if } (v_1, v_2) \in R \rangle$$

is a presentation for M. A rewriting system for a group G is a rewriting system for G as a monoid; in particular, the alphabet must generate G as a monoid.

The rewriting system R is *Noetherian* if there is no infinite chain of rewritings $x \to x_1 \to x_2 \to \cdots$ for any word $x \in S^*$. R is *confluent* if whenever $x \xrightarrow{*} y_1$ and $x \xrightarrow{*} y_2$, there is a z so that $y_1 \xrightarrow{*} z$ and $y_2 \xrightarrow{*} z$. R is *complete* if R is Noetherian and confluent; a complete rewriting system for a group is also known as a complete presentation. Finally, a rewriting system is *finite* if both S and R are finite sets.

A group with a complete presentation has the property that there is exactly one irreducible word representing each of the group elements. So a finite complete rewriting system gives a solution to the word problem for the group. For examples and more information on rewriting systems for groups, see [8] or [12].

A *critical pair* of a rewriting system R is a pair of overlapping rules of one of the following forms, in which each r_i is a word in S^*.

(i) $(r_1 r_2, s) \in R$ and $(r_2 r_3, t) \in R$ with $r_2 \neq 1$.

(ii) $(r_1 r_2 r_3, s), (r_2, t) \in R$.

A critical pair is *resolved in R* if there is a word z such that $sr_3 \xrightarrow{*} z$ and $r_1 t \xrightarrow{*} z$ in the first case or $s \xrightarrow{*} z$ and $r_1 t r_3 \xrightarrow{*} z$ in the second. A Noetherian rewriting system is complete if and only if every critical pair is resolved [8, 13].

Knuth and Bendix [8, 11] have developed a procedure for creating complete rewriting systems; a simplified version is as follows. To begin the Knuth–Bendix procedure, one must start with a finite set S of generators and a finite set E of equations sufficient to present the group or monoid involved. Put a partial well-founded ordering on S^*, which is compatible with concatenation. That is, put an ordering on S^* so that for any $x \in S^*$, there is no infinite descending chain of words $x > x_1 > x_2 > \cdots$, and if $x > y$ then $axb > ayb$ for any $a, b \in S^*$. The set of rules R is initially defined by setting $x \to y$ for each equation $x = y$ in E with $x > y$. If there is an equation $x = y$ in E for which neither $x > y$ nor $y > x$ under the partial ordering, a different ordering must be used. Next check the rewriting rules in R for unresolved critical pairs. If there is an unresolved critical pair of either type, rewrite sr_3 and $r_1 t$ (or s and $r_1 t r_3$, respectively) to words x and y that are irreducible under the rules of R. Then add a rule $x \to y$ if $x > y$ or $y \to x$ if $y > x$ to R. Continue this process until there are no more unresolved critical pairs in R. Since each time that a rule is added to R more critical pairs may occur, this procedure may continue forever, creating infinitely many rules. If the procedure does stop, it will create a finite complete rewriting system.

In general the procedure for checking confluence by critical pairs can be very time consuming. There are several computer programs which can be used to check confluence for specific examples. In the course of the research for this paper, a program called RRL (Rewrite Rule Laboratory) [10] has been used on many examples.

If the Knuth–Bendix procedure does not produce a finite complete rewriting system for a group, there are two changes one can make which may produce a finite system under this procedure. One is to alter the ordering used in the procedure. The other is to change the alphabet; Jantzen's results [9] on the dependence of rewriting systems on generators show that the Knuth–Bendix procedure may stop with a finite rewriting system on one alphabet even though it does not with another alphabet. Both of these techniques have been used in constructing the rewriting systems in this paper.

3. Surface groups

Le Chenadec and Squier have constructed a finite complete presentation for the fundamental group of a closed orientable surface of genus g using generators from the presentation

$$\langle A_1, A_2, \ldots, A_{2g} \mid A_1 A_2 \ldots A_{2g} A_1^{-1} A_2^{-1} \ldots A_{2g}^{-1} = 1 \rangle,$$

along with extra letters to generate the group as a monoid. Since Jantzen [9] has shown that the existence of a finite complete rewriting system is dependent upon the presentation, these groups still may not have a finite complete rewriting system using generators from the usual presentation

$$\langle a_1, a_2, \ldots, a_g, b_1, b_2, \ldots, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = 1 \rangle.$$

In an effort to find such a system, we added letters to represent the inverses of the usual generators, creating the alphabet

$$S = \{a_i, A_i, b_i, B_i \text{ for } 1 \le i \le g\}.$$

In order to perform the Knuth–Bendix procedure on the rewriting system

$$R = \{a_i A_i \to 1,\ A_i a_i \to 1,\ b_i B_i \to 1,\ B_i b_i \to 1,\ 1 \le i \le g,$$
$$a_1 b_1 A_1 B_1 \ldots a_g b_g A_g B_g \to 1\},$$

a total ordering was defined on S^* by recursive path ordering.

Definition [5]. Let $>$ be a partial well-founded ordering on a set S. The *recursive path ordering* $>_{\mathrm{rpo}}$ on S^* is defined recursively from the ordering on S as follows. Given $s_1, \ldots, s_m, t_1, \ldots, t_n \in S$, $s_1 \ldots s_m >_{\mathrm{rpo}} t_1 \ldots t_n$ if and only if one of the following holds.

(1) $s_2 \ldots s_m \ge_{\mathrm{rpo}} t_1 \ldots t_n$.
(2) $s_1 > t_1$ and $s_1 \ldots s_m >_{\mathrm{rpo}} t_2 \ldots t_n$.
(3) $s_1 = t_1$ and $s_2 \ldots s_m >_{\mathrm{rpo}} t_2 \ldots t_n$.

The recursion is started from the ordering $>$ on S and from $s >_{\mathrm{rpo}} 1$ for all $s \in S$, where 1 is the empty word in S^*. Note that if $>$ is a total ordering on S, then $>_{\mathrm{rpo}}$ is a total ordering on S^*.

Theorem (Dershowitz [5]). *Recursive path ordering is a well-founded partial ordering which is compatible with concatenation.* $\square$

The Knuth–Bendix procedure on the rewriting system (S, R) above using recursive path ordering with $a_1 > A_1 > b_1 > B_1 > \cdots > a_g > A_g > b_g > B_g$ results in a finite complete presentation.

Proposition. *There is a finite rewriting system for the fundamental group of a closed orientable surface of genus g, using the alphabet*

$$S = \{a_1, \ldots, a_g, A_1, \ldots, A_g, b_1, \ldots, b_g, B_1, \ldots, B_g\}$$

of the usual generators and their inverses.

Proof. In order to make the notation easier, let $P = a_2b_2A_2B_2 \ldots a_gb_gA_gB_g$ and let $Q = b_ga_gB_gA_g \ldots b_2a_2B_2A_2$. The result of the Knuth–Bendix procedure described above is the rewriting system

$$R' = \{a_iA_i \to 1,\ A_ia_i \to 1,\ b_iB_i \to 1,\ B_ib_i \to 1,\ 1 \le i \le g,$$

$$a_1b_1 \to Qb_1a_1,\ A_1B_1 \to B_1A_1Q,$$

$$a_1B_1 \to B_1Pa_1,\ A_1Qb_1 \to b_1A_1\}$$

For all rules $v \to w$ in R', $v >_{\text{rpo}} w$, so R' is Noetherian. So all that is left is to check confluence, by checking that all critical pairs are resolved; details of this proof may be found in [7]. Since all of the critical pairs are resolved, (S, R') is a finite complete rewriting system for the surface group of genus g. $\square$

4. Coxeter groups

A Coxeter group G is a group with a presentation of the form

$$G = \langle s_1, \ldots, s_n \mid s_i^2 = (s_is_j)^{m_{ij}} = 1\rangle$$

with $2 \le m_{ij} \le \infty$ for $i \ne j$, where $m_{ij} = \infty$ denotes that there is no relation involving s_i and s_j. The set of letters

$$S = \{s_1, \ldots, s_n\}$$

generates G as a monoid, so it is natural to try to find rewriting systems for these groups using these generators.

Complete rewriting systems for Coxeter groups were first constructed by Le Chenadec [12], using the alphabet S. Le Chenadec performed the Knuth–Bendix procedure on these groups with a length-plus-lexicographic ordering or words in S^*. This ordering is defined using a total ordering on S. For any word $w \in S^*$ let $l(w)$ be the length of w, that is, the number of letters in w. Then in the length-plus-lexicographic ordering, two words $v, w \in S^*$ satisfy $v > w$ if either

(i) $l(v) > l(w)$ or

(ii) $l(v) = l(w)$ and if $v = v_1 \ldots v_n$, $w = w_1 \ldots w_n$, with $v_i, w_i \in S$, then the first letters v_i, w_i that are not equal satisfy $v_i > w_i$.

Le Chenadec found complete presentations in the case when none of the m_{ij} were equal to 2; that is, when none of the generators commute. However, his complete rewriting systems in general contain infinitely many rules, in families parametrized by the natural numbers. This was true no matter what lexicographic order was put on S in general. For example, the group

$$G = \langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = (ab)^4 = (ac)^4 = (ad)^3$$
$$= (bc)^3 = (bd)^4 = (cd)^4 = 1 \rangle$$

with any lexicographic ordering on $\{a, b, c, d\}$ has a complete rewriting system with infinitely many rules using Le Chenadec's procedure.

For Coxeter groups we changed both the alphabet and the ordering on words to produce finite complete presentations. Letters were added to the alphabet which represent longest length words of finite special subgroups. *A special subgroup* of a Coxeter group G is a subgroup generated by a subset of the generators $\{s_1, \ldots, s_n\}$ of G. A special subgroup of a Coxeter group is again a Coxeter group. Define the *length* of an element in a Coxeter group to be the length of a shortest possible (or *reduced*) word in S^* which represents the element. If a Coxeter group is finite, then there is a longest element of the group. For a discussion of these and other properties of Coxeter groups, see [3]. With this new alphabet, and a weight-plus-lexicographic ordering (defined below), we were able to construct finite complete rewriting systems for many Coxeter groups.

Theorem. *Let G be a Coxeter group. Suppose G satisfies one of the following two properties.*

(1) *G has three or fewer generators.*

(2) *G does not contain a special subgroup of the form*

$$\langle s_i, s_j, s_k \mid s_i^2 = s_j^2 = s_k^2 = (s_i s_j)^2 = (s_i s_k)^m = (s_j s_k)^n = 1 \rangle$$

with m and n both finite and not both equal to two.

Then G has a finite complete rewriting system.

Proof. The only groups in (1) not also covered in (2) are the triangle groups listed in (2). These are broken up into four cases.

Notation: $[ab]_k$ is the alternating product of k letters $abab\ldots$; $[ab]_1 = a$, $[ab]_0 = 1$. Let $G = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^2 = (ac)^m = (bc)^n = 1 \rangle$.

Case I: $m = 2$.

Alphabet: a, b, c

Rules: $a^2 \to 1, \quad b^2 \to 1, \quad c^2 \to 1,$

$ab \to ba, \quad ac \to ca, \quad [bc]_n \to [cb]_n$

Case II: $m \geq 4$, $n \geq 3$, m even.

Alphabet: a, b, c

Rules: $a^2 \to 1, \quad b^2 \to 1, \quad c^2 \to 1, \quad ab \to ba,$

$ab \to ba, \quad [ac]_m \to [ca]_m, \quad [bc]_n \to [cb]_n,$

$a[cb]_n \to ba[cb]_{n-1}, \quad [ac]_{m-2}ba[cb]_{n-1} \to [ca]_{m-1}ba[cb]_{n-2}$

Case III: $m \geq 3$, $n \geq 5$, m, n both odd.

Alphabet: a, b, c

Rules: $a^2 \to 1, \quad b^2 \to 1, \quad c^2 \to 1, \quad ab \to ba,$

$[ac]_m \to [ca]_m, \quad [bc]_n \to [cb]_n, \quad [ac]_{m-1}ba \to [ca]_m b,$

$a[cb]_n \to ba[cb]_{n-1}, \quad [ac]_{m-1}b[ca]_m \to [ca]_m b[ca]_{m-1},$

$[ac]_{m-1}b[ca]_{m-2}ba[cb]_{n-1} \to [ca]_m b[ca]_{m-2}ba[cb]_{n-2}$

Case IV: $m = n = 3$.

Alphabet: a, b, c

Rules: $a^2 \to 1, \quad b^2 \to 1, \quad c^2 \to 1, \quad ab \to ba,$

$aca \to cac, \quad bcb \to cbc, \quad acba \to cacb, \quad acbc \to bacb$

All other cases may be obtained from these by swapping a and b, and hence m and n.

In all four cases, put a total ordering on the words in the alphabet $S = \{a, b, c\}$ using a length-plus-lexicographic ordering with the ordering $a > b > c$ on S. With these orderings, all of the rules $v \to w$ in cases I–IV satisfy $v > w$; since for any word v the number of words w with $v > w$ is finite, this ordering is well-founded, and the rewriting systems are Noetherian. In order to show that these systems are confluent, then, it suffices to check that every critical pair is resolved; the details of this may be found in [7]. This concludes the proof of part (1) of the theorem.

The proof of part (2) will be done all together rather than in separate cases.

Alphabet: In this case, we will use words in the usual generators of G for our new alphabet S'. An expression surrounded by parentheses () or braces $\{\}$ will represent a letter in the new alphabet. For each generator s_i of G, associate a letter (i) in the new alphabet. For each longest length element $[s_i s_j]_{m_{ij}} = [s_j s_i]_{m_{ij}}$ (using the notation of part (1)) of the special subgroup generated by s_i and s_j when $2 < m_{ij} < \infty$, asssociate a letter $\{ij\}$; that is, $\{ij\}$ and $\{ji\}$ will represent the same letter of S'. Finally, for each product $s_i s_j s_k \ldots$ of two or more generators all of which commute with one another, associate a letter $(ijk\ldots)$. In other words, if $m_{ij} = 2$, then the expressions (ij) and (ji) will represent the same letter of S', and similarly for letters $(ijk\ldots)$ representing longer words in the usual generators.

Notation: As in the notation in part (1) above, $[ab]_k$ is the alternating product of k letters $(a)(b)(a)(b)\ldots$, and ${}_k[ab]$ is the alternating product of k letters $\ldots(a)(b)(a)(b)$.

The following conventions have been used in writing down the set R' of rules in this part. First, a rule occurs only when the letters exist; that is, if a symbol $(ij\ldots)$ occurs in a rule, s_i and s_j must commute, and if a symbol $\{ij\}$ appears, the order m_{ij} of the product s_is_j in G must satisfy $2 < m_{ij} < \infty$. On the left-hand side of a rule, $(ij\ldots)$ or $(i\ldots)$ will denote any letter of the new alphabet S' which is associated to a word of a finite special subgroup containing s_i, and possibly containing other generators $s_j,\ldots$ which commute with s_i. On the right-hand side of a rule, $(i\ldots)$ will denote either an empty expression $()$ or again a letter of S' associated to a finite commutative subgroup of G; an empty expression $()$ represents the trivial word. Finally, in each rule, the numbers i, j, and k are assumed to be distinct, with one exception: in rule (G), j and k may be the same.

Rules:

(A) $(i\ldots)^2 \to 1$

(B) $\{ij\}^2 \to 1$

(C) $(i\ldots)(jk\ldots) \to (i\ldots j)(k\ldots)$

(D) ${}_{m_{ij}-1}[ij](ik\ldots) \to \{ij\}(k\ldots)$

(E) $(i\ldots)\{jk\} \to (i\ldots j)[kj]_{m_{jk}-1}$

(F) ${}_{m_{ij}-1}[ij]\{ik\} \to \{ij\}[ki]_{m_{ik}-1}$

(G) $(ij\ldots)(ik\ldots) \to (j\ldots)(k\ldots)$

(H) $\{ij\}(ik\ldots) \to {}_{m_{ij}-1}[ij](k\ldots)$

(I) $(ij\ldots)\{ik\} \to (j\ldots)[ki]_{m_{ik}-1}$

(J) $\{ij\}\{ik\} \to {}_{m_{ij}-1}[ij][ki]_{m_{ik}-1}$

Example. Let G be the Coxeter group with presentation

$$G = \langle s_1, s_2, s_3, s_4 \mid s_1^2 = s_2^2 = s_3^2 = s_4^2 = (s_1s_2)^2 = (s_1s_3)^4 = (s_3s_4)^3 \rangle.$$

Then the rewriting system for G will have alphabet

$$S' = \{(1), (2), (3), (4), (12), \{13\}, \{34\}\}.$$

The rules are given by

$$\begin{aligned}
R' = \{ &(\mathrm{A}) \quad (1)^2 \to 1,\ (2)^2 \to 1,\ (3)^2 \to 1,\ (4)^2 \to 1,\ (12)^2 \to 1,\\
&(\mathrm{B}) \quad \{13\}^2 \to 1,\ \{34\}^2 \to 1,\\
&(\mathrm{C}) \quad (1)(2) \to (12),\ (2)(1) \to (12),\\
&(\mathrm{D}) \quad (1)(3)(1)(3) \to \{13\},\ (3)(1)(3)(1) \to \{13\},\\
&\qquad\ (3)(1)(3)(12) \to \{13\}(2),\\
&\qquad\ (3)(4)(3) \to \{34\},\ (4)(3)(4) \to \{34\},\\
&(\mathrm{E}) \quad (2)\{13\} \to (12)(3)(1)(3),\\
&(\mathrm{F}) \quad (1)(3)(1)\{34\} \to \{13\}(4)(3),\ (3)(4)\{13\} \to \{34\}(1)(3)(1),\\
&(\mathrm{G}) \quad (1)(12) \to (2),\ (2)(12) \to (1),\ (12)(1) \to (2),\ (12)(2) \to (1),\\
&(\mathrm{H}) \quad \{13\}(1) \to (3)(1)(3),\ \{13\}(3) \to (1)(3)(1),\ \{13\}(12) \to (3)(1)(3)(2),\\
&\qquad\ \{34\}(3) \to (3)(4),\ \{34\}(4) \to (4)(3),\\
&(\mathrm{I}) \quad (1)\{13\} \to (3)(1)(3),\ (3)\{13\} \to (1)(3)(1),\ (12)\{13\} \to (2)(3)(1)(3),\\
&\qquad\ (3)\{34\} \to (4)(3),\ (4)\{34\} \to (3)(4),\\
&(\mathrm{J}) \quad \{13\}\{34\} \to (1)(3)(1)(4)(3),\ \{34\}\{13\} \to (3)(4)(1)(3)(1)\}
\end{aligned}$$

Let M be the monoid presented by the rewriting system (S', R'). Rules (C) and (D) show that the generators $(ij\ldots)$ and $\{ij\}$ can be expressed as products of letters of the form (i). Considering each letter (i) as the usual generator s_i of $G = \langle s_1, \ldots, s_n \mid s_i^2 = (s_i s_j)^{m_{ij}} = 1 \rangle$, rules (A), (B), (C), and (D) give all of the relations in the usual (monoid) presentation of G. In the same way, all of the relations in M are implied by those in G. So this rewriting system gives a monoid presentation of G.

Put a weight-plus-lexicographic total ordering on the words of the alphabet S' as follows. Define a system of weights by $\mathrm{wt}((i)) = 1$, $\mathrm{wt}(\{ij\}) = m_{ij}$, and $\mathrm{wt}((ijk\ldots))$ = the number of generators in the expression. So for $s \in S'$, $\mathrm{wt}(s)$ = the length of s in the usual generators. Then for a word $w \in S'^*$, let $\mathrm{wt}(w)$ be the sum of the weights of the letters in w. Put a partial ordering on the generators by defining $t > u$ for two letters t and u of S' if $\mathrm{wt}(t) < \mathrm{wt}(u)$; so this lexicographic ordering and the weight ordering on S' are precisely opposite. Then define the weight-plus-lexicographic ordering on S'^* by $v > w$ for two words v and w if $\mathrm{wt}(v) > \mathrm{wt}(w)$ or if $\mathrm{wt}(v) = \mathrm{wt}(w)$ and v is lexicographically greater than w. Although this ordering does allow $v > w$ when v has a shorter word length on S' than w, the weight, or word length considered in S, of v may not be less than that of w. As with the other rewriting systems dealt with so far, all rules $v \to w$ in this rewriting system satisfy $v > w$. Since for every word v there are only finitely many words w with $v > w$, the ordering is well-founded, and the system is Noetherian.

As before, it suffices to check resolution of critical pairs in order to show this is confluent. It is important to note that the left-hand side of a rule may be completely contained within the left-hand side of another rule for this rewriting system; such overlapping critical pairs must be checked for confluence with the rest. The resolutions of these critical pairs may be found in [7]. □

Notes. (i) The alphabets used in cases I–IV of the proof of part (1) are not the same as those in the proof of part (2) of this theorem. In all cases except case III we have found a rewriting system using the same alphabet as in part (2), but some of the rules have arrows in the opposite direction.

(ii) In part (2) of this theorem, unresolved critical pairs may occur when the hypothesis on special subgroups which G may contain is omitted. If G contains a subgroup

$$\langle s_i, s_j, s_k \mid s_i^2 = s_j^2 = s_k^2 = (s_i s_j)^2 = (s_i s_k)^2 = (s_j s_k)^n = 1 \rangle,$$

with $2 < n < \infty$, there are two unresolved critical pairs. The first results from applications of rule (E):

$$\underline{(i)\{jk\}} \to (ij)[kj]_{n-1}, \qquad \underline{(i)\{jk\}} \to (ik)[jk]_{n-1},$$

where the portion of each word that is underlined is the portion being rewritten. This is a critical pair of type (i) with $r_1 = r_3 = 1$ and $r_2 = (i)\{jk\}$. The second involves applications of rules (D) and (C), respectively:

$$\underline{{}_{n-2}[jk](j)(ik)} \to \{jk\}(i), \qquad {}_{n-2}[jk]\underline{(j)(ik)} \to {}_{n-2}[jk](ji)(k).$$

This is also a critical pair of type (i), with $r_1 = {}_{n-2}[jk]$, $r_2 = (j)(ik)$, and $r_3 = 1$. If G contains a subgroup

$$\langle s_i, s_j, s_k \mid s_i^2 = s_j^2 = s_k^2 = (s_i s_j)^2 = (s_i s_k)^m = (s_j s_k)^n = 1 \rangle,$$

with $2 < m, n < \infty$, there is an unresolved critical pair given by applications of rules (F) and (E), respectively:

$$\underline{{}_{m-2}[ik](i)\{jk\}} \to \{ik\}[jk]_{n-1}, \qquad {}_{m-2}[ik]\underline{(i)\{jk\}} \to {}_{m-2}[ik](ij)[kj]_{n-1}.$$

This is a critical pair of type (i) with $r_1 = {}_{m-2}[ik]$, $r_2 = (i)\{jk\}$, and $r_3 = 1$. These critical pairs are not resolved without at least the addition of more rules, and, perhaps, more letters.

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