Applied Mathematics Letters 22 (2009) 943–945

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

q-integral representation of the Al-Salam–Carlitz polynomials

Mingjin Wang

Department of Applied Mathematics, Jiangsu Polytechnic University, Changzhou city 213164, Jiangsu province, PR China

ARTICLE INFO

Article history:
Received 16 November 2006
Received in revised form 29 October 2008
Accepted 6 January 2009

Keywords:
q-integral representation
The Al-Salam–Carlitz polynomials
The Rogers–Szegö polynomials
The Hermite polynomials
The Andrews–Askey integral
The Leibniz rule for the q-difference operator

ABSTRACT

We use the Andrews–Askey integral and the Leibniz rule for the q-difference operator to give the q-integral representation of the Al-Salam–Carlitz polynomials, which includes the q-integral representation of the Rogers–Szegö polynomials and the q-integral representation of the q-Hermite polynomials as special cases.

1. Introduction and statement of result

The following are the well-known Rogers–Szegö polynomials:

$$h_n(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k. \tag{1.1}$$

Other hypergeometric polynomials related to the Rogers–Szegö polynomials are the q-Hermite polynomials $H_n(x|q)$, which are often defined via the generating function [1]

$$\sum_{n=0}^{\infty} H_n(x|q) \frac{t^n}{(q;q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-2xtq^n+t^2q^{2n})}.$$

The Rogers–Szegö polynomials and the q-Hermite polynomials have the following relationship:

$$H_n(\cos\theta|q) = \mathrm{e}^{\mathrm{i}n\theta} h_n(\mathrm{e}^{2\mathrm{i}\theta}|q). \tag{1.2}$$

The Rogers–Szegö polynomials play an important role in the theory of orthogonal polynomials, particularly in the study of the Askey–Wilson integral [2–4]. The Rogers–Szegö polynomials are the $a = 0$ case of the Al-Salam–Carlitz polynomials $\varphi_n^{(a)}(x|q)$, which are defined as [5]

$$\varphi_n^{(a)}(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k (a;q)_k. \tag{1.3}$$

E-mail addresses: wang197913@126.com, wmj@jpu.edu.cn.

0893-9659/$ – see front matter
doi:10.1016/j.aml.2009.01.002

In [6], some integral representations of hypergeometric polynomials are given. In this work, we give the q-integral representation of the Al-Salam–Carlitz polynomials, which includes the q-integral representation of the Rogers–Szegö polynomials and the q-integral representation of the q-Hermite polynomials. The main result of the work is the following q-integral representation of the Al-Salam–Carlitz polynomials:

Theorem 1.1. *We have*

$$\varphi_n^{(a)}(x|q) = \frac{(ax, a; q)_\infty}{(1-q)(q, q/x, x; q)_\infty} \int_x^1 \frac{(qt/x, qt; q)_\infty t^n}{(at; q)_\infty} \mathrm{d}_q t. \tag{1.4}$$

provided that no zero factors occur in the denominator.

2. Notation and known results

Before the proof of the theorem, we recall some definitions, notation and known results in [11,12] which will be used in the proof. Throughout this work, it is supposed that $0 < |q| < 1$. The q-shifted factorials are defined as

$$(a; q)_0 = 1, \qquad (a; q)_n = \prod_{k=0}^{n-1}(1 - aq^k), \qquad (a; q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k). \tag{2.1}$$

We also adopt the following compact notation for multiple q-shifted factorials:

$$(a_1, a_2, \ldots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n, \tag{2.2}$$

where n is an integer or ∞. The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}. \tag{2.3}$$

The q-difference operator D_q is defined by [7]

$$\mathrm{D}_q\{f(a)\} = \frac{1}{a}[f(a) - f(aq)]. \tag{2.4}$$

The following property of D_q is straightforward:

$$\mathrm{D}_q^n\left\{\frac{(at; q)_\infty}{(as; q)_\infty}\right\} = s^n (t/s; q)_n \frac{(atq^n; q)_\infty}{(as; q)_\infty}. \tag{2.5}$$

We also have the following Leibniz rule for D_q [8]:

$$\mathrm{D}_q^n\{f(a)g(a)\} = \sum_{k=0}^{n} q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix} \mathrm{D}_q^k\{f(a)\} \mathrm{D}_q^{n-k}\{g(q^k a)\}. \tag{2.6}$$

F.H. Jackson defined the q-integral via [9]

$$\int_0^d f(t)\mathrm{d}_q t = d(1-q)\sum_{n=0}^{\infty} f(\mathrm{d}q^n)q^n, \tag{2.7}$$

and

$$\int_c^d f(t)\mathrm{d}_q t = \int_0^d f(t)\mathrm{d}_q t - \int_0^c f(t)\mathrm{d}_q t. \tag{2.8}$$

The following is the Andrews–Askey integral [10] which can be derived from Ramanujan's ${}_1\psi_1$ summation:

$$\int_c^d \frac{(qt/c, qt/d; q)_\infty}{(at, bt; q)_\infty} \mathrm{d}_q t = \frac{d(1-q)(q, dq/c, c/d, abcd; q)_\infty}{(ac, ad, bc, bd; q)_\infty}, \tag{2.9}$$

provided that no zero factors occur in the denominator of the integrals.

3. The proof of Theorem 1.1

Using the Andrews–Askey integral and the Leibniz rule for the q-difference operator, the q-integral representation of the Al-Salam–Carlitz polynomials can be easily derived. Throughout this section, whenever D_q is applied, the argument should be viewed as a function of a.

Proof. Applying D_q^n to both sides of (2.9), using

$$\mathrm{D}_q^n\left\{\int_c^d f(a;t)\mathrm{d}_q t\right\} = \int_c^d \mathrm{D}_q^n\{f(a;t)\}\mathrm{d}_q t \tag{3.1}$$

and

$$\mathrm{D}_q^n\left\{\frac{(qt/c, qt/d;q)_\infty}{(at, bt;q)_\infty}\right\} = \frac{(qt/c, qt/d;q)_\infty t^n}{(at, bt;q)_\infty}, \tag{3.2}$$

we get

$$\begin{aligned}\int_c^d \frac{(qt/c, qt/d;q)_\infty t^n}{(at, bt;q)_\infty}\mathrm{d}_q t &= \mathrm{D}_q^n\left\{\frac{d(1-q)(q, dq/c, c/d, abcd;q)_\infty}{(ac, ad, bc, bd;q)_\infty}\right\} \\ &= \frac{d(1-q)(q, dq/c, c/d;q)_\infty}{(bc, bd;q)_\infty} D_q^n\left\{\frac{(abcd;q)_\infty}{(ac;q)_\infty}\cdot\frac{1}{(ad;q)_\infty}\right\} \\ &= \frac{d(1-q)(q, dq/c, c/d;q)_\infty}{(bc, bd;q)_\infty} \\ &\quad\times\sum_{k=0}^n q^{k(k-n)}\begin{bmatrix} n \\ k\end{bmatrix}\mathrm{D}_q^k\left\{\frac{(abcd;q)_\infty}{(ac;q)_\infty}\right\}D_q^{n-k}\left\{\frac{1}{(adq^k;q)_\infty}\right\}.\end{aligned} \tag{3.3}$$

Employing (2.5), we finally obtain

$$\int_c^d \frac{(qt/c, qt/d;q)_\infty t^n}{(at, bt;q)_\infty}\mathrm{d}_q t = \frac{d(1-q)(q, dq/c, c/d, abcd;q)_\infty}{(ac, ad, bc, bd;q)_\infty}\sum_{k=0}^n \begin{bmatrix} n \\ k\end{bmatrix}\frac{(ad, bd;q)_k}{(abcd;q)_k}c^k d^{n-k}. \tag{3.4}$$

If we let $b = 0$, $c = x$ and $d = 1$ in (3.4), we get (1.4). □

Acknowledgements

The author would like to express deep appreciation to the referees for their helpful suggestions. This work was supported by STF of Jiangsu Polytechnic University and Innovation Program of Shanghai Municipal Education Commission.

References

[1] D.M. Bressoud, A simple proof of Mehler's formula for q-Hermite polynomials, Indiana Univ. Math. J. 29 (1980) 577–580.
[2] R. Askey, M.E.H. Ismail, A generalization of ultraspherical polynomials, in: P. Erdös (Ed.), Studies in Pure Mathematics, Birkhäuser, Boston, MA, 1983, pp. 55–78.
[3] M.E.H. Ismail, D. Stanton, G. Viennot, The combinatorics of q-Hermite polynomials and the Askey–Wilson integral, European J. Combin. 8 (1987) 379–392.
[4] M.E.H. Ismail, D. Stanton, On the Askey–Wilson and Rogers polynomials, Canad. J. Math. 40 (1988) 1025–1045.
[5] H.M. Srivastava, V.K. Jain, Some multilinear generating functions for q-Hermite polynomials, J. Math. Anal. Appl. 144 (1989) 147–157.
[6] Shy-Der Lin, Yi-Shan Chao, H.M. Srivastava, Some families of hypergeometric polynomials and associated integral representations, J. Math. Anal. Appl. 294 (2004) 399–411.
[7] W.Y.C. Chen, Z.-G. Liu, Parameter augmentation for basic hypergeometric series, II, J. Combin. Theory Ser. A 80 (1997) 175–195.
[8] S. Roman, More on the umbral calculus, with emphasis on the q-umbral calculus, J. Math. Anal. Appl. 107 (1985) 222–254.
[9] F.H. Jackson, On q-definite integrals, Quart. J. Pure Appl. Math., 50, 101–112.
[10] G.E. Andrews, R. Askey, Another q-extension of the beta function, Proc. Amer. Math. Soc. 81 (1981) 97–100.
[11] G.E. Andrews, q-Series: Their development and applications in analysis, in: Number Theory, Combinatorics, Physics and Computer Algebra, in: CBMS Regional Conference Lecture Series, Vol. 66, Amer. Math., Providence, RI, 1986.
[12] G. Gasper, M. Rahman, Basic Hypergeometric Series, Cambridge Univ. Press, Cambridge, MA, 1990.