On the size of edge-coloring critical graphs with maximum degree 4

Lianying Miao, Shiyou Pang

School of Science, China University of Mining and Technology, Xuzhou, 280001, PR China

Received 29 May 2006; received in revised form 10 September 2007; accepted 4 October 2007

Available online 4 March 2008

Abstract

In 1968, Vizing proposed the following conjecture: If $G = (V, E)$ is a $\Delta$-critical graph of order $n$ and size $m$, then $m \geq \frac{1}{2}[(\Delta - 1)n + 3]$. This conjecture has been verified for the cases of $\Delta \leq 5$. In this paper, we prove that $m \geq \frac{7}{4}n$ when $\Delta = 4$. It improves the known bound for $\Delta = 4$ when $n > 6$.

Keywords: Conjecture; Edge-coloring; Edge-coloring critical graphs

1. Introduction

In this paper, all graphs $G = (V, E)$ are finite, simple and undirected. Throughout, $G$ is assumed to have $n$ vertices and $m$ edges. The chromatic index $\chi'(G)$ of a graph $G$ is the minimum number of colors required to color the edges of $G$ so that two adjacent edges receive different colors. In 1965, Vizing [5] proved that if $G$ is a graph of maximum degree $\Delta$, then the chromatic $\chi'(G)$ is either $\Delta$ or $\Delta + 1$. A graph $G$ is said to be of Class one if $\chi'(G) = \Delta$, and it is said to be of Class two if $\chi'(G) = \Delta + 1$. A $\Delta$-critical graph $G$ is a connected graph of maximum degree $\Delta$ such that $G$ is of Class two and $G - e$ is of Class one for each edge $e$ of $G$. The following is a well known conjecture of Vizing proposed in 1968.

Conjecture (Vizing [6]). If $G = (V, E)$ is a $\Delta$-critical graph, then $m \geq \frac{1}{2}[(\Delta - 1)n + 3]$.

The conjecture has been proved for the case $\Delta \leq 5$ [1,6].

In [4], Sanders and Zhao proved that if $G = (V, E)$ is a $\Delta$-critical graph, then

$$m \leq \frac{1}{4}n(\Delta + \sqrt{2\Delta - 1}).$$

For $\Delta \in \{6, 7, 8, 9, 10, 11\}$, Yue Zhao [7] also proved that if $G = (V, E)$ is a $\Delta$-critical graph, then $m \geq \frac{nd\Delta}{2}$, where

---

*E-mail address: miaolianying@cumt.edu.cn (L. Miao).*

0012-365X/S - see front matter © 2008 Published by Elsevier B.V.
doi:10.1016/j.disc.2007.10.013
By contradiction, we assume that there exists a vertex \( L \). Let \( G \) be a \( \Delta \)-critical graph and \( x \) be a \( 2 \)-vertex of \( G \). Let \( N(x) \) be the set of vertices adjacent to \( x \), and has degree \( \Delta \). When \( \Delta = 4 \) and \( n = 5 \), the two bounds are the same.

Before proceeding, we introduce some notation. For \( x \in V \), \( N(x) \) is the set of vertices adjacent to \( x \), and the degree of \( x \), denoted by \( d(x) \), is \( |N(x)| \). A \( k \)-vertex, \( \geq k \)-vertex, or \( \leq k \)-vertex is a vertex of degree \( k \), at least \( k \) or at most \( k \) respectively. We define \( N_k(x) \), \( N_{\geq k}(x) \), or \( N_{\leq k}(x) \) to be the set of \( k \)-vertices, \( \geq k \)-vertices, or \( \leq k \)-vertices adjacent to \( x \) respectively, and \( d_k(x) \), \( d_{\geq k}(x) \), or \( d_{\leq k}(x) \) to be the number of \( k \)-vertices, \( \geq k \)-vertices, or \( \leq k \)-vertices adjacent to \( x \) respectively. In an edge-coloring \( \varphi \) of \( G \), if an edge incident with \( u \in V \) is colored \( k \), we say that \( u \) sees \( k \). An \((i, j)\)-chain of \( \varphi \) is a two-colored path in which the colors \( i \) and \( j \) alternate under \( \varphi \). We denote the maximal \((i, j)\)-chain starting from \( u \) by \( L_{i,j}(u) \).

2. Lemmas

Lemma 1 (Vizing Adjacency Lemma [6]). Let \( x \) be a vertex of a \( \Delta \)-critical graph; then (i) if \( d_k(x) \geq 1 \), then \( d_{\Delta}(x) \geq \Delta - k + 1 \); (ii) \( d_{\Delta}(x) \geq 2 \).

Lemma 2 ([8]). Let \( G \) be a \( \Delta \)-critical graph. If \( xy \in E(G) \) and \( d(x) + d(y) = \Delta + 2 \), then every vertex at distance \( 2 \) from \( x \) or \( y \) has degree at least \( \Delta - 1 \), and has degree \( \Delta \) if \( d(x), d(y) < \Delta \).

Lemma 3. Let \( G \) be a \( \Delta \)-critical graph and \( x \) be a 2-vertex of \( G \). Let \( N(x) = \{y, z\} \). If \( yz \in E \), then each of the \( \Delta \)-vertices of \( \{N_\Delta(y) \cup N_\Delta(z)\} \setminus \{y, z\} \) is not adjacent to any \((\Delta - 1)\)-vertices.

Proof. By contradiction, we assume that there exists a vertex \( v \in \{N_\Delta(y) \cup N_\Delta(z)\} \setminus \{y, z\} \) that is adjacent to a \((\Delta - 1)\)-vertex \( t \). We may assume that \( v \in N_\Delta(y) \setminus \{z\} \).

Let \( G' = G - xy \); then \( G' \) has a \( \Delta \)-edge-coloring \( \varphi : E(G') \to \{1, 2, \ldots, \Delta\} \). Without loss of generality, suppose that \( \varphi(xz) = 1, \varphi(yz) = 2 \). Then the color missing at \( y \) must necessarily be the color 1, otherwise there would be a color missing simultaneously at \( x \) and \( y \), and hence the edge \( xy \) could be colored with that color, which is impossible. Hence without loss of generality we can assume that \( \varphi(yv) = \Delta \).

Case 1. \( \varphi(uv) \in \{1, 2\} \). Without loss of generality, we assume that \( \varphi(uv) = 1 \).

Claim 1. \( t \) must see \( \Delta \). Otherwise, we can recolor \( uv \) with \( 1 \), \( yv \) with 1, and color \( xy \) with \( \Delta \) to get a \( \Delta \)-coloring of \( G \).

Claim 2. \( t \) must see 2. Otherwise, we can recolor \( yz \) with 1, \( xz \) with 2 to get a new \( \Delta \)-coloring \( \varphi' \) of \( G' \). In \( \varphi' \), we consider \( L_{\Delta,2}(t) \). If it terminates at \( y \), then, exchanging the colors along \( L_{\Delta,2}(t) \) we have a coloring such that \( y \) is missing \( \Delta \) and \( x \) is missing \( \Delta \), thus allowing the edge \( xy \) to be colored \( \Delta \). Similarly if \( L_{\Delta,2}(t) \) terminates at \( x \), then, exchanging the colors along \( L_{\Delta,2}(t) \), we have a coloring such that \( y \) and \( x \) are missing color 2, thus allowing us to color the edge \( xy \) with color 2. Thus \( L_{\Delta,2}(t) \) ends at neither \( y \) nor \( x \). We interchange its colors; \( t \) cannot see \( \Delta \), a contradiction to Claim 1.

So there exists \( \alpha \in \{3, 4, \ldots, \Delta - 1\} \) such that \( t \) does not see \( \alpha \). We recolor \( yz \) with 1, \( xz \) with 2 to get a new \( \Delta \)-coloring \( \varphi'' \) of \( G' \). In \( \varphi'' \), \( L_{2,\alpha}(t) \) ends at neither \( y \) nor \( x \); otherwise, we exchange the colors along \( L_{\Delta,2}(t) \), and then \( xy \) can be colored with \( \alpha \) or 2, which is impossible. We interchange its colors; \( t \) cannot see 2, a contradiction to Claim 2.
Lemma 3. Without loss of generality, we assume that \( \varphi(vt) = \Delta - 1 \). We need to consider the following subcases:

Subcase 2.1. \( t \) sees \( 1, 2, \ldots, \Delta - 2, \Delta - 1 \).

First we interchange the colors of \( L_{1,\Delta}(t) \); this becomes Case 1.

Subcase 2.2. \( t \) sees \( 1, 3, 4, \ldots, \Delta - 2, \Delta - 1, \Delta \).

We can recolor \( yz \) with \( 1, xz \) with \( 2 \) to get a new \( \Delta \)-coloring \( \varphi' \) of \( G' \). In \( \varphi' \), \( L_{\Delta-1,2}(t) \) ends at neither \( y \) nor \( x \). We interchange its colors; this becomes Case 1.

Subcase 2.3. \( t \) sees \( 2, 3, 4, \ldots, \Delta - 2, \Delta - 1, \Delta \).

We interchange the colors of \( L_{2,1}(t) \); this becomes Subcase 2.2.

Subcase 2.4. \( t \) sees \( 1, 2, 4, 5, \ldots, \Delta - 2, \Delta - 1, \Delta \).

We interchange the colors of \( L_{1,3}(t) \); this becomes Subcase 2.3.

This proves Lemma 3. \( \square \)

3. Discharging method

Suppose that \( G = (V, E) \) is a 4-critical graph with \( |E| < \frac{7}{4}n \).

Denote \( 2d(x) - 7 \) by \( M(x) \), for each \( x \in V(G) \). Then

\[
\sum_{x \in V(G)} M(x) = \sum_{x \in V(G)} (2d(x) - 7) = 4|E| - 7|V| < 0
\]

(1)

We call the number \( M(x) \) the initial charge of \( x \) for \( x \in V \). We will assign a new charge denoted by \( M'(x) \) to each \( x \in V \) according to the discharging rule R below:

R1. \( x \) is a 2-vertex. Let \( N(x) = \{y, z\} \); then \( x \) gets \( \frac{3}{2} \) from each of \( y \) and \( z \).

A 3-vertex is called light if it is adjacent to only two 4-vertices; otherwise is called heavy.

R2. \( x \) is a 3-vertex.

If \( x \) is a light 3-vertex, \( x \) gets \( \frac{1}{2} \) from each of its adjacent 4-vertices.

If \( x \) is a heavy 3-vertex, \( x \) gets \( \frac{1}{3} \) from each of its adjacent 4-vertices.

R3. \( x \) is a 4-vertex.

1. If \( x \) is adjacent to one 2-vertex, \( x \) gives nothing to its three adjacent 4-vertices.

2. If \( x \) is not adjacent to any 2-vertex, by VAL, \( x \) is adjacent to at most two 3-vertices. If \( x \) is adjacent to some light 3-vertex (say \( v \)), then by Lemma 2, every vertex at distance 2 from \( v \) has degree \( \Delta \), so \( x \) is adjacent to just one 3-vertex. So it is not possible that \( x \) is adjacent to two light 3-vertices or one light 3-vertex and one heavy 3-vertex. So we need only consider the following four cases:

(2.1) \( x \) is adjacent to only 4-vertices. \( x \) gives \( \frac{1}{4} \) to each of its adjacent 4-vertices.

(2.2) \( x \) is adjacent to three 4-vertices and one light 3-vertex. \( x \) gives \( \frac{1}{6} \) to each of its adjacent 4-vertices.

(2.3) \( x \) is adjacent to three 4-vertices and one heavy 3-vertex. \( x \) gives \( \frac{1}{6} \) to each of its adjacent 4-vertices.

(2.4) \( x \) is adjacent to two 4-vertices and two heavy 3-vertices, \( x \) gives \( \frac{1}{6} \) to each of its adjacent 4-vertices.

Now, we prove that \( M'(x) \geq 0 \) for each \( x \in V \).

(a) \( d(x) = 2 \).

\( M'(x) = M(x) + 2 \times \frac{3}{2} = 0. \)

(b) \( d(x) = 3 \).

If \( x \) is a light 3-vertex, \( M'(x) = M(x) + 2 \times \frac{1}{2} = 0. \)

If \( x \) is a heavy 3-vertex, \( M'(x) = M(x) + 3 \times \frac{1}{3} = 0. \)

(c) \( d(x) = 4 \).

If \( x \) is adjacent to one 2-vertex (say \( y \)), we let \( N(y) = \{x, z\} \). If \( xz \notin E \), let \( N(x) \setminus \{y\} = \{u, v, w\} \). From Lemma 2, each of \( u, v, w \) is not adjacent to any 2-vertex. Each of \( u, v, w \) is adjacent to only 4-vertices or one 3-vertex (light or heavy) and three 4-vertices or two heavy 3-vertices and two 4-vertices. So the total charge that they give \( x \) is at least
\[ \frac{1}{6} \times 3 = \frac{1}{2}, \text{ so } M'(x) \geq M(x) - \frac{3}{2} + \frac{1}{2} \geq 0. \] If \( xz \in E \), let \( N(y) \setminus \{x, z\} = \{u, v\} \). From Lemma 3, each of \( u, v \) is not adjacent to any \( \leq 3 \)-vertex and the total charge that they give \( x \) is \( \frac{1}{2} \times 2 = \frac{1}{2} \), so \( M'(x) = M(x) - \frac{3}{2} + \frac{1}{2} = 0. \)

If \( x \) is adjacent to only 4-vertices, \( M'(x) = M(x) - 4 \times \frac{1}{4} = 0. \)

If \( x \) is adjacent to three 4-vertices and one light 3-vertex, \( M'(x) = M(x) - \frac{1}{4} - 3 \times \frac{1}{6} = 0. \)

If \( x \) is adjacent to three 4-vertices and one heavy 3-vertex, \( M'(x) = M(x) - \frac{1}{4} - 3 \times \frac{1}{6} > 0. \)

If \( x \) is adjacent to two 4-vertices and two heavy 3-vertices, \( M'(x) = M(x) - 2 \times \frac{1}{6} - 2 \times \frac{1}{3} = 0. \)

From the above rules, we can see that \( M'(x) \geq 0 \) for each \( x \in V \), a contradiction to (1).

This completes the proof.

Acknowledgements

The authors wish to thank the referees for their many valuable comments which greatly improved the presentation of this paper and led to the current form of this paper. This research is supported by China University of Mining and Technology Science Council, Number: OZK4566.

References