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Connectivity, Genus, and the Number of Components in Vertex-Deleted Subgraphs

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Let $c(G)$ denote the number of components in a graph G. It is shown that if G has genus γ and is k-connected with $k \geqslant 3$, then $c(G - X) \leqslant (2/(k-2))(|X| - 2 + 2\gamma)$, for all $X \subseteq V(G)$ with $|X| \geqslant k$. Some implications of this result for planar graphs ($\gamma = 0$) and toroidal graphs ($\gamma = 1$) are considered.

We consider only finite, undirected graphs. Multiple edges are permissible, but loops are not. Our terminology and notation are standard except as indicated.

If G is a graph and $X \subseteq V(G)$, we use $G - X$ to denote the graph obtained when the vertices of X are deleted from G. We use $c(G)$ to denote the number of components of G.

Our first result gives a sufficient condition for a graph to be k-connected when $k \geqslant 4$.

THEOREM 1. *Let* $|V(G)| > k \geqslant 4$, *and suppose that*

$$c(G - X) \leqslant \frac{2}{k-2}(|X| - 2), \quad \text{for all } X \subseteq V(G) \text{ with } |X| \geqslant k - 1. \tag{1}$$

Then G is k-connected.

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Proof. Let $X \subseteq V(G)$. If $|X| = k - 1$, then (1) guarantees that $c(G - X) = 1$. If $|X| < k - 1$ and $c(G - X) > 1$, then, since $|V(G)| > k$, there exists a superset X' of X with $|X'| = k - 1$ and $c(G - X') > 1$, which we just saw was impossible. So if $|X| \leqslant k - 1$, we have $c(G - X) = 1$, and thus G is k-connected.

The converse of this theorem will not be true in general. There exist k-connected graphs G which do not satisfy (1) for every $X \subseteq V(G)$ with $|X| \geqslant k$ (e.g., consider $K_{n,n}$ with $n \geqslant 4$). However, we now give a partial converse to Theorem 1 in terms of the genus of the graph G.

THEOREM 2. *Let G be a graph with genus γ. If G is k-connected with $k \geqslant 3$, then*

$$c(G - X) \leqslant \frac{2}{k-2}(|X| - 2 + 2\gamma), \quad \textit{for all } X \subseteq V(G) \textit{ with } |X| \geqslant k. \tag{2}$$

Proof. Let G be a k-connected graph with genus γ, and let $X \subseteq V(G)$ with $|X| \geqslant k$. Embed G in a sphere with γ handles. Add to G, in any manner, edges incident to at least one vertex of X until a situation is reached where the addition of any other edge of this type would lead to either crossing edges or a 2-gon (i.e., a face bounded by exactly two edges). In particular, multiple edges are permitted as long as they do not form a 2-gon. Call the resulting graph G'. Since $G' - X$ has exactly the same components as $G - X$, it suffices to prove (2) for G' instead of G.

Let $G'(X)$ denote the subgraph of G' induced by X. We first prove:

Each face of $G'(X)$ contains at most one component of $G' - X$. (*)

If $G'(X)$ does not satisfy (*), then $G'(X)$ has a face containing at least two components of $G' - X$. Choose any two of these components, say H and H'. Clearly there exist vertices $v \in H$ and $v' \in H'$ such that the edge (v, v') could be added to the embedding of G' without creating crossing edges.

Let f denote the face of G' inside of which the edge (v, v') can be thus added. Since G' is 3-connected, f will be bounded by a simple cycle containing v and v'. Let P_1, P_2 denote the two paths along this cycle joining v and v'. Each P_i contains a vertex in X, since P_i joins two components of $G' - X$. Let x_i be a vertex of $P_i \cap X$, for $i = 1, 2$. It is easy to see that the edge (x_1, x_2) could be added to the embedding of G' inside the face f without creating crossing edges or a 2-gon. This violates the definition of G', and completes the proof of (*).

We have established that each face of $G'(X)$ contains at most one component of $G' - X$. On the other hand, since G' is k-connected, any face of $G'(X)$ containing a component of $G' - X$ must be bounded by k or more edges. (We are assuming here that $c(G' - X) > 1$, since otherwise (2) is trivially satisfied.) Therefore we can complete the proof of (2) by showing that the number of faces in $G'(X)$ bounded by k or more edges is at most $(2/(k-2))(|X| - 2 + 2\gamma)$.

Let E and F denote the number of edges and faces in the graph $G'(X)$. By Euler's formula,

$$|X| - E + F = c(G'(X)) + 1 - 2\gamma. \tag{3}$$

Suppose the faces of $G'(X)$ are bounded by $f_1 \geqslant f_2 \geqslant \cdots \geqslant f_F$ edges. Since each edge in $G'(X)$ is a bounding edge of a face exactly twice in $G'(X)$, we have by (3)

$$\sum_{i=1}^{F} f_i = 2E = 2(|X| + F - c(G'(X)) - 1 + 2\gamma),$$

or

$$\sum_{i=1}^{F} (f_i - 2) = 2(|X| - c(G'(X)) - 1 + 2\gamma) \leqslant 2(|X| - 2 + 2\gamma). \tag{4}$$

Let $f_1, f_2, ..., f_r \geqslant k$. If $r > (2/k - 2)(|X| - 2 + 2\gamma)$, we have

$$\sum_{i=1}^{F} (f_i - 2) \geqslant \sum_{i=1}^{r} (f_i - 2) > (k-2) \cdot \frac{2}{k-2} (|X| - 2 + 2\gamma)$$
$$= 2(|X| - 2 + 2\gamma),$$

which contradicts (4). Hence, $r \leqslant (2/(k-2))(|X| - 2 + 2\gamma)$, and thus (2) is satisfied for G'. The proof of Theorem 2 is complete.

We can see that the bound in Theorem 2 is best possible in general by considering $K_{m,n}$, and using the result of Ringel [5] that:

$$\gamma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil, \qquad \text{if} \quad m, n \geqslant 2.$$

In the special case when G is a planar graph (i.e., when $\gamma = 0$), we have the following result.

COROLLARY. *If G is a k-connected planar graph with $k \geqslant 3$, then $c(G - X) \leqslant (2/(k-2))(|X| - 2)$ for all $X \subseteq V(G)$ with $|X| \geqslant k$.*

Combining this corollary with Theorem 1, we have the following characterization of the connectivity of planar graphs in terms of a component inequality.

THEOREM 3. *Let $k \geqslant 4$. A planar graph with $|V(G)| > k$ is k-connected if and only if* (1) *holds in G.*

Consider finally the special case when G is a toroidal graph (i.e., when $\gamma = 1$). Chvátal [2] has defined a graph to be *t-tough* if $c(G - X) \leqslant (1/t)|X|$ whenever $c(G - X) > 1$, and conjectures that every 2-tough graph is hamiltonian. It follows from Theorem 2 that a k-connected, toroidal graph with $k \geqslant 3$ is $((k - 2)/2)$-tough. In particular, a 6-connected toroidal graph is 2-tough. In line with Chvátal's conjecture, it has been shown (see [1, 3]) that every 6-connected, toroidal graph is indeed hamiltonian. On the other hand, we have no example of even a 4-connected, toroidal graph which is non-hamiltonian, and Grünbaum [4] conjectures that no such graph exists. The question of whether such a graph exists is especially interesting in light of a theorem of Tutte [6] that every 4-connected, planar graph is hamiltonian.

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