Hyers–Ulam stability of a generalized Apollonius type quadratic mapping

Chun-Gil Park \(^a,\ast\), Themistocles M. Rassias \(^b\)

\(^a\) Department of Mathematics, Chungnam National University, Daejeon 305–764, Republic of Korea
\(^b\) Department of Mathematics, National Technical University of Athens, Zografou Campus, 15780 Athens, Greece

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Abstract

Let \( X, Y \) be linear spaces. It is shown that if a mapping \( Q : X \rightarrow Y \) satisfies the following functional equation:

\[
Q\left( \sum_{i=1}^{n} z_i - \left( \sum_{i=1}^{n} x_i \right) \right) + Q\left( \sum_{i=1}^{n} z_i - \sum_{i=1}^{n} y_i \right) = \frac{1}{2} Q\left( \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} y_i \right) + 2 Q\left( \sum_{i=1}^{n} z_i - \frac{1}{2} \left( \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i \right) \right)
\]

(0.1)

then the mapping \( Q : X \rightarrow Y \) is quadratic. We moreover prove the Hyers–Ulam stability of the functional equation (0.1) in Banach spaces.

Keywords: Hyers–Ulam stability; Quadratic mapping of Apollonius type

1. Introduction

The stability problem of functional equations originated from a question of S.M. Ulam [21] concerning the stability of group homomorphisms: Let \((G_1, \ast)\) be a group and let \((G_2, \circ, d)\) be...
a metric group with the metric \(d(\cdot, \cdot)\). Given \(\epsilon > 0\), does there exist a \(\delta(\epsilon) > 0\) such that if a mapping \(h : G_1 \to G_2\) satisfies the inequality
\[
d(h(x \ast y), h(x) \odot h(y)) < \delta
\]
for all \(x, y \in G_1\), then there is a homomorphism \(H : G_1 \to G_2\) with
\[
d(h(x), H(x)) < \epsilon
\]
for all \(x \in G_1\)? If the answer is affirmative, we would say the equation of homomorphism \(H(x \ast y) = H(x) \odot H(y)\) stable. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation?

D.H. Hyers [9] gave a first affirmative answer to the question of Ulam for Banach spaces. Let \(X\) and \(Y\) be Banach spaces. Assume that \(f : X \to Y\) satisfies
\[
\|f(x + y) - f(x) - f(y)\| \leq \epsilon
\]
for all \(x, y \in X\) and some \(\epsilon \geq 0\). Then there exists a unique additive mapping \(T : X \to Y\) such that
\[
\|f(x) - T(x)\| \leq \epsilon
\]
for all \(x \in X\). Th.M. Rassias [17] succeeded in extending the result of Hyers by weakening the condition for the Cauchy difference to be unbounded. A number of mathematicians were attracted to this result of Th.M. Rassias and stimulated to investigate the stability problems of functional equations. The stability phenomenon that was introduced and proved by Th.M. Rassias in his 1978 paper is called the Hyers–Ulam stability. G.L. Forti [5] and P. Gavruta [8] have generalized the result of Th.M. Rassias, which permitted the Cauchy difference to become arbitrary unbounded. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem. A large list of references can be found, for example, in the papers [2,6,7,10–19].

Now, a square norm on an inner product space satisfies the important parallelogram equality
\[
\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)
\]
for all vectors \(x, y\). The following functional equation, which was motivated by this equation,
\[
Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y),
\]
(1.1)
is called a quadratic functional equation, and every solution of Eq. (1.1) is said to be a quadratic mapping.

F. Skof [20] proved the Hyers–Ulam stability of the quadratic functional equation (1.1) for mappings \(f : E_1 \to E_2\), where \(E_1\) is a normed space and \(E_2\) is a Banach space. In [4], S. Czerwik proved the Hyers–Ulam stability of the quadratic functional equation. C. Borelli and G.L. Forti [3] generalized the stability result as follows: let \(G\) be an abelian group, \(E\) a Banach space. Assume that a mapping \(f : G \to E\) satisfies the functional inequality
\[
\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \varphi(x, y)
\]
for all \(x, y \in G\), and \(\varphi : G \times G \to [0, \infty)\) is a function such that
\[ \Phi(x, y) := \sum_{i=0}^{\infty} \frac{1}{4^{i+1}} \varphi(2^i x, 2^i y) < \infty \]

for all \( x, y \in G \). Then there exists a unique quadratic mapping \( Q : G \to E \) with the properties

\[ \| f(x) - Q(x) \| \leq \Phi(x, x) \]

for all \( x \in G \). Jun and Lee [14] proved the Hyers–Ulam stability of the Pexiderized quadratic equation

\[ f(x + y) + g(x - y) = 2h(x) + 2k(y) \]

for mappings \( f, g, h \) and \( k \).

In an inner product space, the equality

\[ \| z - x \|^2 + \| z - y \|^2 = \frac{1}{2} \| x - y \|^2 + 2 \left\| z - \frac{x + y}{2} \right\|^2 \]

holds, and is called the Apollonius’ identity. The following functional equation, which was motivated by this equation,

\[ Q(z - x) + Q(z - y) = \frac{1}{2} Q(x - y) + 2Q \left( z - \frac{x + y}{2} \right) \tag{1.2} \]

is quadratic. For this reason, the functional equation (1.2) is called a quadratic functional equation of Apollonius type, and each solution of the functional equation (1.2) is said to be a quadratic mapping of Apollonius type. The quadratic functional equation and several other functional equations are useful to characterize inner product spaces [1].

In this paper, employing the above equality (1.2), we introduce the new functional equation, which is called the generalized Apollonius type quadratic functional equation and whose solution of the functional equation is said to be a generalized Apollonius type quadratic mapping,

\[
\begin{align*}
Q\left(\left(\sum_{i=1}^{n} z_i\right) - \left(\sum_{i=1}^{n} x_i\right)\right) + Q\left(\left(\sum_{i=1}^{n} z_i\right) - \left(\sum_{i=1}^{n} y_i\right)\right) \\
= \frac{1}{2} Q\left(\left(\sum_{i=1}^{n} x_i\right) - \left(\sum_{i=1}^{n} y_i\right)\right) + 2Q\left(\left(\sum_{i=1}^{n} z_i\right) - \left(\sum_{i=1}^{n} x_i\right) + \left(\sum_{i=1}^{n} y_i\right)\right) \tag{1.3}
\end{align*}
\]

As a special case, if \( n = 1 \) in (1.3), then the functional equation (1.3) reduces to

\[ Q(z - x) + Q(z - y) = \frac{1}{2} Q(x - y) + 2Q \left( z - \frac{x + y}{2} \right) \]

We are going to show that the generalized Apollonius type quadratic functional equation (1.3) is quadratic, and prove the Hyers–Ulam stability of generalized Apollonius type quadratic mappings in Banach spaces.

2. Hyers–Ulam stability of a generalized Apollonius type quadratic mapping

Throughout this section, let \( X \) be a normed space with norm \( \| \cdot \| \) and \( Y \) a Banach space with norm \( \| \cdot \| \).
Lemma 2.1. If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and
\[
 f \left( \left( \sum_{i=1}^{n} z_i \right) - \left( \sum_{i=1}^{n} x_i \right) \right) + f \left( \left( \sum_{i=1}^{n} z_i \right) - \left( \sum_{i=1}^{n} y_i \right) \right) \\
= \frac{1}{2} f \left( \left( \sum_{i=1}^{n} x_i \right) - \left( \sum_{i=1}^{n} y_i \right) \right) + 2 f \left( \sum_{i=1}^{n} z_i - \frac{\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i}{2} \right)
\]
for all $x_1, \ldots, x_n \in X$, then the mapping $f : X \to Y$ is quadratic.

Proof. Suppose that a mapping $f : X \to Y$ satisfies the equality (2.1). Letting $z_2 = \cdots = z_n = x_2 = \cdots = x_n = y_2 = \cdots = y_n = 0$ and $y_1 = -x_1$, we get
\[
f(z_1 - x_1) + f(z_1 + x_1) = \frac{1}{2} f(2x_1) + 2 f(z_1)
\]
for all $x_1, z_1 \in X$. Letting $z_1 = x_1$ in (2.2), we get
\[
f(2x_1) = \frac{1}{2} f(2x_1) + 2 f(x_1)
\]
for all $x_1 \in X$. So
\[
f(2x_1) = 4 f(x_1)
\]
for all $x_1 \in X$. It follows from (2.2) that
\[
f(z_1 + x_1) + f(z_1 - x_1) = 2 f(z_1) + 2 f(x_1)
\]
for all $x_1, z_1 \in X$. Thus the mapping $f : X \to Y$ is quadratic. \(\square\)

Given a mapping $f : X \to Y$, we define $Df : X^{3n} \to Y$ by
\[
Df(z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n) \\
= f \left( \left( \sum_{i=1}^{n} z_i \right) - \left( \sum_{i=1}^{n} x_i \right) \right) + f \left( \left( \sum_{i=1}^{n} z_i \right) - \left( \sum_{i=1}^{n} y_i \right) \right) \\
- \frac{1}{2} f \left( \left( \sum_{i=1}^{n} x_i \right) - \left( \sum_{i=1}^{n} y_i \right) \right) + 2 f \left( \sum_{i=1}^{n} z_i - \frac{\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i}{2} \right)
\]
for all $z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n \in X$.

Theorem 2.2. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^{3n} \to (0, \infty)$ such that
\[
\tilde{\varphi}(z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n) \\
= \sum_{j=0}^{\infty} 4^j \varphi \left( \frac{z_1}{2^j}, \ldots, \frac{z_n}{2^j}, \frac{x_1}{2^j}, \ldots, \frac{x_n}{2^j}, \frac{y_1}{2^j}, \ldots, \frac{y_n}{2^j} \right) < \infty, \quad (2.3)
\]
\[
\|Df(z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n)\| \leq \varphi(z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n) \quad (2.4)
\]
for all \( z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n \in X \). Then there exists a unique generalized Apollonius type quadratic mapping \( Q : X \rightarrow Y \) such that

\[
\| f(x) - Q(x) \| \leq 2 \tilde{\varphi} \left( \frac{x}{n}, \ldots, \frac{x}{n}, 0, \ldots, 0 \right)
\]

(2.5)

for all \( x \in X \).

**Proof.** Letting \( z_1 = \cdots = z_n = x_1 = \cdots = x_n = x \) and \( y_1 = \cdots = y_n = 0 \) in (2.4), we get

\[
\| f(nx) - \frac{1}{2} f(nx) - 2 f \left( \frac{n}{2} x \right) \| \leq \varphi(x, \ldots, x, 0, \ldots, 0)
\]

for all \( x \in X \). So

\[
\| f(x) - 4 f \left( \frac{1}{2} x \right) \| \leq 2 \varphi \left( \frac{x}{n}, \ldots, \frac{x}{n}, 0, \ldots, 0 \right)
\]

(2.6)

for all \( x \in X \). Hence

\[
\| 4^l f \left( \frac{x}{2^l} \right) - 4^m f \left( \frac{x}{2^m} \right) \| \leq \sum_{j=l}^{m-1} 2 \cdot 4^j \varphi \left( \frac{x}{2^j n}, \ldots, \frac{x}{2^j n}, 0, \ldots, 0 \right)
\]

(2.7)

for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (2.3) and (2.7) that the sequence \( \{4^d f(x/2^d)\} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{4^d f(x/2^d)\} \) converges. So one can define the mapping \( Q : X \rightarrow Y \) by

\[
Q(x) := \lim_{d \to \infty} 4^d f \left( \frac{x}{2^d} \right)
\]

for all \( x \in X \).

By (2.3) and (2.4),

\[
\| DQ(z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n) \|
\leq \lim_{d \to \infty} 4^d \left\| Df \left( \frac{z_1}{2^d}, \ldots, \frac{z_n}{2^d}, \frac{x_1}{2^d}, \ldots, \frac{x_n}{2^d}, \frac{y_1}{2^d}, \ldots, \frac{y_n}{2^d} \right) \right\|
\leq \lim_{d \to \infty} 4^d \varphi \left( \frac{z_1}{2^d}, \ldots, \frac{z_n}{2^d}, \frac{x_1}{2^d}, \ldots, \frac{x_n}{2^d}, \frac{y_1}{2^d}, \ldots, \frac{y_n}{2^d} \right) = 0
\]

for all \( z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n \in X \). So

\[
DQ(z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n) = 0.
\]

By Lemma 2.1, the mapping \( Q : X \rightarrow Y \) is quadratic. Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (2.7), we get (2.5). So there exists a generalized Apollonius type quadratic mapping \( Q : X \rightarrow Y \) satisfying (2.5).

Now let \( Q' : X \rightarrow Y \) be another generalized Apollonius type quadratic mapping satisfying (2.5). Then we have
\[ \left\| Q(x) - Q'(x) \right\| = 4^d \left\| Q\left(\frac{x}{2^d}\right) - Q'\left(\frac{x}{2^d}\right) \right\| \]
\[ \leq 4^d \left( \left\| Q\left(\frac{x}{2^d}\right) - f\left(\frac{x}{2^d}\right) \right\| + \left\| Q'\left(\frac{x}{2^d}\right) - f\left(\frac{x}{2^d}\right) \right\| \right) \]
\[ \leq 2 \cdot 4^d \tilde{\varphi}\left(\frac{x}{2^{dn}}, \ldots, \frac{x}{2^{dn}}, 0, \ldots, 0\right), \]

which tends to zero as \( d \to \infty \) for all \( x \in X \). So we can conclude that \( Q(x) = Q'(x) \) for all \( x \in X \). This proves the uniqueness of \( Q \). \( \square \)

**Corollary 2.3.** Let \( p > 2 \) and \( \theta \) be positive real numbers, and let \( f : X \to Y \) be a mapping satisfying \( f(0) = 0 \) and
\[ \left\| Df(z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n) \right\| \leq \theta \left( \sum_{j=1}^{n} \| z_j \|^p + \sum_{j=1}^{n} \| x_j \|^p + \sum_{j=1}^{n} \| y_j \|^p \right) \]
for all \( z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n \in X \). Then there exists a unique generalized Apollonius type quadratic mapping \( Q : X \to Y \) such that
\[ \left\| f(x) - Q(x) \right\| \leq \frac{2^{p+2} \theta}{(2^p - 4)n^{p-1}} \| x \|^p \]
for all \( x \in X \).

**Proof.** Define
\[ \varphi(z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n) = \theta \left( \sum_{j=1}^{n} \| z_j \|^p + \sum_{j=1}^{n} \| x_j \|^p + \sum_{j=1}^{n} \| y_j \|^p \right), \]
and apply Theorem 2.2. \( \square \)

**Theorem 2.4.** Let \( f : X \to Y \) be a mapping satisfying \( f(0) = 0 \) for which there exists a function \( \varphi : X^{3n} \to [0, \infty) \) satisfying (2.4) such that
\[ \tilde{\varphi}(z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n) \]
\[ = \sum_{j=1}^{\infty} \frac{1}{4^j} \varphi(2^j z_1, \ldots, 2^j z_n, 2^j x_1, \ldots, 2^j x_n, 2^j y_1, \ldots, 2^j y_n) < \infty \] (2.8)
for all \( z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n \in X \). Then there exists a unique generalized Apollonius type quadratic mapping \( Q : X \to Y \) such that
\[ \left\| f(x) - Q(x) \right\| \leq 2 \tilde{\varphi}\left(\frac{x}{n}, \ldots, \frac{x}{n}, 0, \ldots, 0\right) \] (2.9)
for all \( x \in X \).
Proof. It follows from (2.6) that
\[
\left\| f(x) - \frac{1}{4} f(2x) \right\| \leq \frac{1}{2} \varphi \left( \frac{2x}{n}, \ldots, \frac{2x}{n}, 0, \ldots, 0 \right)
\]
for all \( x \in X \). Hence
\[
\left\| \frac{1}{4^l} f(2^l x) - \frac{1}{4^m} f(2^m x) \right\| \leq \sum_{j=l+1}^{m} \frac{2}{4^j} \varphi \left( \frac{2^j x}{n}, \ldots, \frac{2^j x}{n}, 0, \ldots, 0 \right)
\]
(2.10)
for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (2.8) and (2.10) that the sequence \( \left\{ \frac{1}{4^l} f(2^l x) \right\} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \left\{ \frac{1}{4^l} f(2^l x) \right\} \) converges. So one can define the mapping \( Q : X \to Y \) by
\[
Q(x) := \lim_{d \to \infty} \frac{1}{4^d} f(2^d x)
\]
for all \( x \in X \).

By (2.8) and (2.4),
\[
\left\| DQ(z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n) \right\| = \lim_{d \to \infty} \frac{1}{4^d} \left\| D f \left( 2^d z_1, \ldots, 2^d z_n, 2^d x_1, \ldots, 2^d x_n, 2^d y_1, \ldots, 2^d y_n \right) \right\|
\leq \lim_{d \to \infty} \frac{1}{4^d} \varphi \left( 2^d z_1, \ldots, 2^d z_n, 2^d x_1, \ldots, 2^d x_n, 2^d y_1, \ldots, 2^d y_n \right) = 0
\]
for all \( z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n \in X \). So
\[
D Q(z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n) = 0.
\]

By Lemma 2.1, the mapping \( Q : X \to Y \) is quadratic. Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (2.10), we get (2.9). So there exists a generalized Apollonius type quadratic mapping \( Q : X \to Y \) satisfying (2.9).

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. Let \( p < 2 \) and \( \theta \) be positive real numbers, and let \( f : X \to Y \) be a mapping satisfying \( f(0) = 0 \) and
\[
\left\| D f(z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n) \right\| \leq \theta \left( \sum_{j=1}^{n} \|z_j\|^p + \sum_{j=1}^{n} \|x_j\|^p + \sum_{j=1}^{n} \|y_j\|^p \right)
\]
for all \( z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n \in X \). Then there exists a unique generalized Apollonius type quadratic mapping \( Q : X \to Y \) such that
\[
\left\| f(x) - Q(x) \right\| \leq \frac{2^{p+2} \theta}{(4 - 2^p) n^{p-1}} \|x\|^p
\]
for all \( x \in X \).
Proof. Define

$$\varphi(z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n) = \theta \left( \sum_{j=1}^{n} \|z_j\|^p + \sum_{j=1}^{n} \|x_j\|^p + \sum_{j=1}^{n} \|y_j\|^p \right),$$

and apply Theorem 2.4. □

**Theorem 2.6.** Let \( f : X \to Y \) be a mapping satisfying \( f(0) = 0 \) for which there exists a function \( \varphi : X^{3n} \to [0, \infty) \) satisfying (2.4) such that

$$\tilde{\varphi}(z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum_{j=0}^{\infty} \frac{1}{4^j} \varphi(2^j z_1, \ldots, 2^j z_n, 2^j x_1, \ldots, 2^j x_n, 2^j y_1, \ldots, 2^j y_n) < \infty \quad (2.11)$$

for all \( z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n \in X \). Then there exists a unique generalized Apollonius type quadratic mapping \( Q : X \to Y \) such that

$$\| f(x) - Q(x) \| \leq \frac{1}{2} \varphi \left( \frac{x}{n}, \ldots, \frac{x}{n}, -\frac{x}{n}, \ldots, -\frac{x}{n} \right) \quad (2.12)$$

for all \( x \in X \).

Proof. Letting \( z_1 = \cdots = z_n = x_1 = \cdots = x_n = x \) and \( y_1 = \cdots = y_n = -x \) in (2.4), we get

$$\left\| f(2nx) - \frac{1}{2} f(2nx) - 2 f(nx) \right\| \leq \varphi(\underbrace{x, \ldots, x}_{2n \text{ times}}, \underbrace{-x, \ldots, -x}_{n \text{ times}})$$

for all \( x \in X \). So

$$\left\| f(x) - \frac{1}{4} f(2x) \right\| \leq \frac{1}{2} \varphi \left( \frac{x}{n}, \ldots, \frac{x}{n}, -\frac{x}{n}, \ldots, -\frac{x}{n} \right) \quad (2.13)$$

for all \( x \in X \). Hence

$$\left\| \frac{1}{4^l} f(2^l x) - \frac{1}{4^m} f(2^m x) \right\| \leq \sum_{j=l}^{m-1} \frac{1}{2 \cdot 4^j} \varphi \left( \frac{2^j x}{n}, \ldots, \frac{2^j x}{n}, -\frac{2^j x}{n}, \ldots, -\frac{2^j x}{n} \right) \quad (2.14)$$

for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (2.11) and (2.14) that the sequence \( \left\{ \frac{1}{4^d} f(2^d x) \right\} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \left\{ \frac{1}{4^d} f(2^d x) \right\} \) converges. So one can define the mapping \( Q : X \to Y \) by

$$Q(x) := \lim_{d \to \infty} \frac{1}{4^d} f(2^d x)$$

for all \( x \in X \).

By (2.11) and (2.4),
\[\|DQ(z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n)\| = \lim_{d \to \infty} \frac{1}{4^d} \|DF(2^dz_1, \ldots, 2^dz_n, 2^dx_1, \ldots, 2^dx_n, 2^dy_1, \ldots, 2^dy_n)\| \leq \lim_{d \to \infty} \frac{1}{4^d} \varphi(2^dz_1, \ldots, 2^dz_n, 2^dx_1, \ldots, 2^dx_n, 2^dy_1, \ldots, 2^dy_n) = 0\]

for all \(z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n \in X\). So

\[DQ(z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n) = 0.\]

By Lemma 2.1, the mapping \(Q : X \to Y\) is quadratic. Moreover, letting \(l = 0\) and passing the limit \(m \to \infty\) in (2.14), we get (2.12). So there exists a generalized Apollonius type quadratic mapping \(Q : X \to Y\) satisfying (2.12).

The rest of the proof is similar to the proof of Theorem 2.2. □

**Corollary 2.7.** Let \(p < 2\) and \(\theta\) be positive real numbers, and let \(f : X \to Y\) be a mapping satisfying \(f(0) = 0\) and

\[\|Df(z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n)\| \leq \theta \left( \sum_{j=1}^{n} \|z_j\|^p + \sum_{j=1}^{n} \|x_j\|^p + \sum_{j=1}^{n} \|y_j\|^p \right)\]

for all \(z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n \in X\). Then there exists a unique generalized Apollonius type quadratic mapping \(Q : X \to Y\) such that

\[\|f(x) - Q(x)\| \leq \frac{6\theta}{(4 - 2^p)n^{p-1}} \|x\|^p\]

for all \(x \in X\).

**Proof.** Define

\[\varphi(z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n) = \theta \left( \sum_{j=1}^{n} \|z_j\|^p + \sum_{j=1}^{n} \|x_j\|^p + \sum_{j=1}^{n} \|y_j\|^p \right),\]

and apply Theorem 2.6. □

**Theorem 2.8.** Let \(f : X \to Y\) be a mapping satisfying \(f(0) = 0\) for which there exists a function \(\varphi : X^{3n} \to [0, \infty)\) satisfying (2.4) such that

\[\tilde{\varphi}(z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum_{j=1}^{\infty} 4^j \varphi\left(\frac{z_1}{2^j}, \ldots, \frac{z_n}{2^j}, \frac{x_1}{2^j}, \ldots, \frac{x_n}{2^j}, \frac{y_1}{2^j}, \ldots, \frac{y_n}{2^j}\right) < \infty\] (2.15)

for all \(z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n \in X\). Then there exists a unique generalized Apollonius type quadratic mapping \(Q : X \to Y\) such that

\[\|f(x) - Q(x)\| \leq \frac{1}{2} \tilde{\varphi}\left(\frac{x}{n}, \ldots, \frac{x}{n}, \frac{-x}{n}, \ldots, \frac{-x}{n}\right)\] (2.16)

for all \(x \in X\).
Proof. It follows from (2.13) that
\[
\left\| f(x) - 4f\left(\frac{x}{2}\right) \right\| \leq 2\phi\left(\frac{x}{2^{2n}}, \cdots, \frac{x}{2^{2n}}, \cdots, \frac{x}{2^{2n}}\right)_{2n \text{ times}}
\]
for all \( x \in X \). Hence
\[
\left\| 4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right) \right\| \leq \sum_{j=l+1}^{m} \frac{4^j}{2^n} \phi\left(\frac{x}{2^{j+1}}, \cdots, \frac{x}{2^{j+1}}, \cdots, \frac{x}{2^{j+1}}\right)_{2n \text{ times}}
\]
(2.17)
for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (2.15) and (2.17) that the sequence \( \{4^d f(x/2^d)\} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{4^d f(x/2^d)\} \) converges. So one can define the mapping \( Q : X \rightarrow Y \) by
\[
Q(x) := \lim_{d \rightarrow \infty} 4^d f\left(\frac{x}{2^d}\right)
\]
for all \( x \in X \).

By (2.15) and (2.4),
\[
\|DQ(z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n)\| = \lim_{d \rightarrow \infty} 4^d \|Df\left(\frac{z_1}{2^d}, \cdots, \frac{z_n}{2^d}, \frac{x_1}{2^d}, \cdots, \frac{x_n}{2^d}, \frac{y_1}{2^d}, \cdots, \frac{y_n}{2^d}\right)\| \\
\leq \lim_{d \rightarrow \infty} 4^d \phi\left(\frac{z_1}{2^d}, \cdots, \frac{z_n}{2^d}, \frac{x_1}{2^d}, \cdots, \frac{x_n}{2^d}, \frac{y_1}{2^d}, \cdots, \frac{y_n}{2^d}\right) = 0
\]
for all \( z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n \in X \). So
\[
DQ(z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n) = 0.
\]
By Lemma 2.1, the mapping \( Q : X \rightarrow Y \) is quadratic. Moreover, letting \( l = 0 \) and passing the limit \( m \rightarrow \infty \) in (2.17), we get (2.16). So there exists a generalized Apollonius type quadratic mapping \( Q : X \rightarrow Y \) satisfying (2.16).

The rest of the proof is the same as in the proof of Theorem 2.2. □

Corollary 2.9. Let \( p > 2 \) and \( \theta \) be positive real numbers, and let \( f : X \rightarrow Y \) be a mapping satisfying \( f(0) = 0 \) and
\[
\|Df(z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n)\| \leq \theta \left(\sum_{j=1}^{n} \|z_j\|^p + \sum_{j=1}^{n} \|x_j\|^p + \sum_{j=1}^{n} \|y_j\|^p\right)
\]
for all \( z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n \in X \). Then there exists a unique generalized Apollonius type quadratic mapping \( Q : X \rightarrow Y \) such that
\[
\|f(x) - Q(x)\| \leq \frac{6\theta}{(2^p - 4)n^p-1} \|x\|^p
\]
for all \( x \in X \).
Proof. Define
\[
\varphi(z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n) = \theta \left( \sum_{j=1}^{n} \|z_j\|^p + \sum_{j=1}^{n} \|x_j\|^p + \sum_{j=1}^{n} \|y_j\|^p \right),
\]
and apply Theorem 2.8. □

References