Abstract

Let $\Phi$ be the golden ratio $(\sqrt{5} + 1)/2$, $f_n$ the $n$th Fibonacci finite word and $f$ the Fibonacci infinite word. Let $r$ be a rational number greater than $(2 + \Phi)/2$ and $u$ a non-empty word. If $u^r$ is a factor of $f$, then there exists $n \geq 1$ such that $u$ is a conjugate of $f_n$ and, moreover, each occurrence of $u^r$ is contained in a maximal one of $(f_n)^s$ for some $s \in [2, 2 + \Phi)$. Several known results on the Fibonacci infinite word follow from this.

1. Introduction

In analogy with the definition of the Fibonacci numbers, one sets $f_0 = b$, $f_1 = a$, and, for $n \geq 2$, one defines the $n$th Fibonacci finite word as the product $f_{n-1}f_{n-2}$ of the words $f_{n-1}$ and $f_{n-2}$ (see [4, 7]). The two products $f_{n-1}f_{n-2}$ and $f_{n-2}f_{n-1}$ are almost the same, being different only on the last two letters. This is the amusing, very simple but very interesting 'near-commutative property' used in [6] to study concrete algorithms. It plays an important role also in this paper.

The Fibonacci infinite word $f$ is the Sturmian word associated with the golden ratio $\Phi = (\sqrt{5} + 1)/2$ and can also be defined as the unique infinite word having, for each $n \geq 1$, $f_n$ as a left factor.

If a power $u^r$ has an occurrence in $f$, we try to extend it to the left and to the right as far as possible preserving periodicity. We call maximal the occurrences of the powers that cannot be locally extended and we prove that we always reach one of them, call it $v^s$, where $v$ is a conjugate of $u$. The main result of present paper says that: if $r > (2 + \Phi)/2$ then $v = f_n$ for some $n \geq 1$ and $s \in [2, 2 + \Phi)$. Several known results on $f$ are consequences of this.

2. Definitions and preliminary results

This paper is organized so as to be self-contained; terminology and notations are those currently used in theoretical computer science [4, 7].
We consider only the two-letter alphabet \{a, b\} and we call (finite) words the elements of the free monoid \{a, b\}*. We denote by 1 the empty word and by |u| the length of a word u. We consider a word u of length \(k \geq 1\) as a map \(u: \{0, 1, \ldots, k - 1\} \rightarrow \{a, b\}\); we write \(u = u(0) \ldots u(i) \ldots u(k - 1)\) where \(u(0), u(i)\) and \(u(k - 1)\) are, respectively, the first, the ith and the last letter of u.

A word u is a factor of a word v if there exist two words \(u', u'' \in \{a, b\}^*\) such that \(v = u'uu''\). When \(u' = 1\) (resp. \(u'' = 1\)) we say that u is a left factor (resp. right factor) of v. A proper factor, (resp. proper left factor, proper right factor) u of v is a factor (resp. left factor, right factor) u of v such that |u| < |v|.

A (right) infinite word on \{a, b\} is a map q from the set of non-negative integers into \{a, b\}. We write \(q = q(0)q(1)\ldots q(i)\ldots\) A word u is a factor of q if there exist a word u' and an infinite word q' such that \(q = u'q'\). If \(u' = 1\) we say that u is a left factor of q.

A non-empty word u may be a factor of another (finite or infinite) word w in several ways. So it is useful to speak about occurrences. For this reason, let i, j be integers such that \(0 \leq i \leq j\) (and that \(j < |w|\) if w is a finite word) and let us denote by \(w(i) \ldots w(j)\). We say that the pair of integers \((i, j)\) is an occurrence of the factor u in the word w if \(u = w(i, j)\). We say that an occurrence \((i_0, j_0)\) of u in w is contained in an occurrence \((i_1, j_1)\) of v in w if \(i_0 \leq i_1 \leq j_0 \leq j_1\). We only speak about occurrences of non-empty words.

Now, let \(\varphi: \{a, b\} \rightarrow \{a, b\}^*\) be the morphism whose restriction to \{a, b\} is given by \(\varphi(a) = ab, \varphi(b) = a\). Remark that \(\varphi\) is injective. Let us define the nth Fibonacci finite words \(f_n\) in the following way: \(f_0 = b\) and, for each \(n \geq 0\),

\[f_{n+1} = \varphi(f_n).\]

In particular, we have: \(f_1 = a, f_2 = ab, f_3 = aba, f_4 = abaab, f_5 = abaababa, f_6 = abaababaaba, f_7 = abaabababaababaabaaba \ldots\) It is clear that, for each \(n \geq 2\), \(f_n\) is the product (juxtaposition) \(f_{n-1}f_{n-2}\) of \(f_{n-1}\) and \(f_{n-2}\). Also, for each \(n \geq 0\), \(|f_n|\) is the nth element \(F_n\) of the sequence of Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377 \ldots\)

Remark now that, for each \(n \geq 1\), \(f_n\) is a left factor of \(f_{n+1}\). So there exists a unique infinite word, namely the Fibonacci infinite word f, such that, for each \(n \geq 1\), \(f_n\) is a left factor of f (see, [4, 7]) and we have

\[f = abaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababa... .

We denote by \(F(f)\) the set of the non empty factors of f and by \(LF(f)\) its subset containing the non empty left factors of f.

For each \(n \geq 2\), we denote by \(g_n\) the product \(f_{n-2}f_{n-1}\) and by \(h_n\) the common longest left factor of \(f_n\) and \(g_n\). In particular, we have: \(g_2 = ba, g_3 = aab, g_4 = ababa, g_5 = abaababa, \ldots\) and \(h_2 = 1, h_3 = a, h_4 = aba, h_5 = abaaba \ldots\).

Remark that a non-empty factor of an element of \(F(f)\) is again in \(F(f)\); for each \(n \geq 2, g_n \in F(f), \varphi(g_n) = g_{n+1}\) and \(h_{n+1} \in LF(f)\); if \(f(i) = b\) then \(i > 0\) and \(f(i - 1) = f(i + 1) = a\) and if \(f(i, i + 1) = aa\) then \(i > 0\) and \(f(i - 1) = f(i + 2) = b\) (i.e., \(bb, aaa \in F(f))\).
A factor $v$ of $f$ is special if $va, vb \in F(f)$. Let $k \geq 1$; we denote by $\tilde{u}$ the mirror image $u(k - 1)u(k - 2) \ldots u(1)u(0)$ of the word $u = u(0)u(1) \ldots u(k - 2)u(k - 1)$. We say that a non-empty word $v$ is a palindrome if $v = \tilde{v}$.

Lemma 1 belongs to the folklore (see e.g. [1, 2, 4, 5, 6]) and is very easy; so we can give just an hint of its proof. The point (i) is the 'near-commutative property' quoted in the introduction.

**Lemma 1.** For each $n \geq 2$,

(i) $f_{n+2} = f_{n+1}f_n = f_nf_{n+1} = h_{n+2}xy$ and $g_{n+2} = f_nf_{n+1} = f_{n+1}g_n = h_{n+2}yx$, where $x, y \in \{a, b\}$, $x \neq y$ and if $n$ is even then $xy = ab$ and if $n$ is odd then $xy = ba$;

(ii) $|h_n| = F_n - 2$;

(iii) $h_n$ is a special factor;

(iv) $f_{n+5} = h_{n+1}xyh_nyhx_{n+1}xy$, where $x, y \in \{a, b\}$, $x \neq y$;

(v) $h_n$ is a palindrome;

(vi) $h_{n+2} = f_nh_{n+1} = h_{n+1}\tilde{f}_n = h_n\tilde{f}_{n+1} = f_{n+1}h_n$;

(vii) for each integer $m \geq 0$, $h_n$ is a left and a right factor of $h_{n+m}$.

Moreover, (viii) if $v \in F(f)$ then $\tilde{v} \in F(f)$.

**Proof.** One can prove (i) easily by induction; (ii)–(iv) are consequence of (i); one can prove (v) by induction using (iv); (vi) is a consequence of (v); (vii) is a corollary of (vi) and finally one can prove (viii) using (v).

**Lemma 2.** For each $n \geq 2$,

(i) $f_n(0) = a$ ($= f_1$); $g_{n+1}(0) = a$;

(ii) if $f(i, j) = f_{n+2}$ then $f(j + 1) = a$; if $f(i, j) = g_{n+1}$ then $f(j + 1) = a$;

(iii) if $f(i, j) = h_{n+1}$ then $f(i, j + 2) = f_{n+1}$ or $f(i, j + 2) = g_{n+1}$.

**Proof.** (i) is trivial; to prove (ii) and (iii) use the fact that $aba$ is a right factor of each $h_{n+2}$ (point (vii) of Lemma 1) and the fact that $bb, aaaa \notin F(f)$.

**Lemma 3.** Let $u \in F(f)$. Then $\varphi^{-1}(u)$ exists and belongs to $F(f)$ if and only if one of these two conditions holds: (i) $u(0) = a$ and $u(|u| - 1) = b$; (ii) $u(0) = a$, $u(|u| - 1) = a$ and $ua \in F(f)$.

**Proof.** By induction on $|u|$.

**Remark.** We have $\varphi^{-1}(aa) = bb \notin F(f)$. We often use Lemma 3 together with points (i) and (ii) of Lemma 2 in order to prove the existence of $\varphi^{-1}(u)$ in $F(f)$ for suitable $u \in F(f)$.

The following Lemma 4 says that no occurrence of $g_n$ is too close to the left of $f$.

**Lemma 4.** For each $n \geq 2$, if $f(i,j) = g_n$ then $i \geq F_{n-1}$ and $f(i - F_{n-1}, i - 1) = f_{n-1}$.
Proof. By induction. Let \( n = 2 \). If \( f(i, i + 1) = ba = g_2 \) then \( i \neq 0 \) and \( f(i - 1) = a = f_1 \). Now, let \( n \geq 3 \) and \( f(i, j) = g_n \). As \( f(0, F_n - 1) = f_n \), \( i \neq 0 \) and so \( w = f(0, i - 1) \) is non empty. By Lemmas 2 and 3, one has \( \varphi^{-1}(w'g_n) = w'g_{n-1} \in F(f) \) for some non-empty \( w' \in LF(f) \). By induction hypothesis \( f_{n-2} \) is a right factor of \( w' \); so \( i \geq F_{n-1} \) and \( f(i - F_{n-1}, i - 1) = f_{n-1} \). \( \square \)

The following Lemma 5 belongs to the folklore. Point (i) is proved, in [1] for example, using an auxiliary morphism which is not necessary here. Point (ii) is a particular case of a more general result on sturmian words (see [4, 5]). For each \( k \geq 1 \), let us denote by \( s^{[k]} \) the mirror image of the left factor of \( f \) having length \( k \).

**Lemma 5.** For each \( k \geq 1 \), (i) the unique special factor of length \( k \) is \( s^{[k]} \); (ii) in \( F(f) \) there are exactly \( k + 1 \) elements of length \( k \).

**Proof.** (i) Remark that \( s^{[k]} \) is a right factor of \( h_m \) for each \( m \) such that \( F_m > k + 1 \) and so, by point (iii) of Lemma 1, \( s^{[k]} \) is special. Suppose now that for a given \( k \) there is a special factor \( v \) of length \( k \) which is different from \( s^{[k]} \). The last letter of \( v \) is necessarily \( a \), hence the greatest integer, such that \( h_n \) is a common right factor of \( s^{[k]} \) and \( v \), is greater than or equal to 3. Let also \( k' \) be the greatest integer such that \( s^{[k]}(k') \neq v(k') \). We have \( s^{[k]} = u'u'h_n \) and \( v = u''u'y'h_n \), for some \( u, u', u'' \in \{a, b\}^* \) and for some \( x, y \in \{a, b\}, x \neq y \). We have also \( |u'| < F_{n-1} \) otherwise, by point (vi) and (vii) of Lemma 1, we have a contradiction with the maximality of \( n \). Being a right factor of a special factor, \( u'y'h_n \) is special and so \( u'y'h_na, u'y'h_nb \in F(f) \). In both cases, \( n \) even or odd, \( u'y'g_n \in F(f) \) and, by Lemma 4, \( u'y' \) is a right factor of \( f_{n-1} \). Hence, \( xu' \) is not a right factor of \( f_{n-1} \). Contradiction. (ii) It is an easy consequence of (i). \( \square \)

Let us recall that an infinite word \( p \) is periodic (resp. ultimately periodic) if there exists \( k \geq 1 \) such that \( p(j + k) = p(j) \) for each \( j \geq 0 \) (resp. for each \( j \geq i \) for some \( i \geq 0 \)).

**Lemma 6.** The Fibonacci infinite word is not ultimately periodic.

**Proof.** This is easy by point (ii) of Lemma 5. \( \square \)

Let \( u, v, w, z, z' \in F(f) \). We say that \( (u, v, w) \) is a non empty overlap of \( z \) and \( z' \) if \( uw = z, vw = z' \) and \( uwv \in F(f) \). The possible non empty overlaps concerning \( f_n \) and \( g_n \) are considered in the following Lemma.

**Lemma 7.** Let \( n \geq 3 \). Then

(i) \( (f_{n-1}, f_{n-2}, g_{n-1}) \) is the unique non-empty overlap of \( f_n \) and \( f_n \);
(ii) \( (f_{n-1}, f_{n-2}, f_{n-1}) \) is the unique non-empty overlap of \( f_n \) and \( g_n \);
(iii) \( (f_{n-2}, f_{n-1}, f_{n-2}) \) is the unique non-empty overlap of \( g_n \) and \( f_n \);
(iv) there is no non-empty overlap of \( g_n \) and \( g_n \).
Proof. By induction. (i) Let \( n = 3 \). As \( a \) is the unique word which is a proper non-empty right and also left factor of \( aba, (ab, a, ba) = (f_2, f_1, g_2) \) is the unique non-empty overlap of \( f_3 \) and \( f_3 \); hence the statement is true for \( n = 3 \). Now, let \( n \geq 4 \). Clearly, \( (f_{n-1}, f_{n-2}, g_{n-1}) \) is a non-empty overlap of \( f_n \) and \( f_n \). By Lemmas 2 and 3 there exist \( u' = \varphi^{-1}(u), v' = \varphi^{-1}(v), w' = \varphi^{-1}(w) \) in \( F(f) \) such that \( (u', v', w') \) is a non-empty overlap of \( f_{n-1} \) and \( f_{n-1} \). By induction hypothesis, \( u' = f_{n-2}, v' = f_{n-3}, w' = g_{n-2} \). Hence \( u = \varphi(f_{n-2}) = f_{n-1}, v = \varphi(f_{n-3}) = f_{n-2}, w = \varphi(g_{n-2}) = g_{n-1} \); (ii) the argument is analogous, but starting with the fact that \( a \) is the unique word which is a proper non-empty right factor of \( aba \) and also a proper non-empty left factor of \( aab \); (iii) \( ab \) is the unique word which is a proper non-empty right factor of \( aab \) and also a proper non-empty left factor of \( aba \); (iv) no word is a proper non-empty right and also left factor of \( aab \).

Lemma 8. Let \( n \geq 5 \). There are exactly two non-empty overlaps of \( h_n \) and \( h_n \), namely just

\[
(f_{n-1}, h_{n-2}, f_{n-1})
\]

and

\[
(f_{n-2}, h_{n-1}, f_{n-2})
\]

Proof. This is an easy consequence of point (iii) of Lemma 2, Lemma 7 and point (vi) of Lemma 1.

Remark. In some sense, \((aba)(aba)\) can be considered as an 'overlap' of \( h_4 \) and \( h_4 \) but we have chosen to consider only non-empty overlaps. So there is a unique non-empty overlap of \( h_4 \) and \( h_4 \) and this is \((ab, a, ba)\), in accordance with point (i) of Lemma 7.

Lemma 9. Let \( v \in LF(f) \). Then the following two conditions are equivalent; (i) \( v \) is a palindrome and (ii) \( v = h_n \) for some \( n \geq 3 \).

Proof. (ii) \(\rightarrow\) (i) is point (v) of Lemma 1. We prove (i) \(\rightarrow\) (ii) by induction. The palindromes of \( LF(f) \), having length less than or equal to \( F_5 - 2 \), are \( a = h_3, aba = h_4 \) and \( abaaba = h_5 \). Let \( n \geq 4 \) and suppose, by induction hypothesis, that \( h_3, \ldots, h_n \) and \( h_{n+1} \) are all the palindromes of \( LF(f) \) having length less than or equal to \( F_{n+1} - 2 \). Suppose also, by way of contradiction, that there exists a proper left factor \( u \) of \( h_{n+2} \) such that \( h_{n+1} \) is a proper left factor of \( u \) with \( u \) a palindrome. By points (i) and (vii) of Lemma 1, we can write \( h_{n+2} = f_x h_{n-1} c d h_y b a \), where \( h_x \) is the left factor of \( h_n \) of length \( F_n - 4 \) and \( c, d \in \{a, b\}, c \neq d \). We can see that \( |u| \notin \{F_{n+1} - 1, F_{n+1}, F_{n+2} - 3\} \). So there exist a non-empty left factor \( u' \) of \( h_n, x, y \in \{a, b\} \) and \( u' \in \{a, b\}^{*} \) such that \( f_x = u' x y u' d c \) and \( u = u' x y u' d c h_{n+1} c d u' \). Being \( u' d c h_{n+1} c d u' \) a right factor of \( u, u \) a palindrome and a left factor of \( h_{n+2} \), one has that \( u' d c h_{n+1} c d u' \) is a right factor of \( h_{n+2} \). As \( |u' d c h_{n+1} c d u'| = F_{n+1} - 2 \), one has that \( u' d c h_{n+1} c d u' = h_{n+1} \). By Lemma 8 and by \( |u' x y| < F_n \), we have \( u' x y = f_{n-1} \) and so \( x y = c d \). Then \( u = u' c d u' d c h_{n+1} c d u' \) is not a palindrome. Contradiction.
Lemma 10. For each \( n \geq 2 \),
(i) \( f_n g_n \notin F(f) \);
(ii) if \( f(i, j) = g_{n+1} h_{n+1} \) then \( f(i - F_n, j) = f_n h_{n+1} \); 
(iii) if \( z, f_{n+1} z \in F(f) \) then \( |z| \geq F_n - 2 \);
(iv) if \( z, g_{n+1} z \in F(f) \) then \( |z| \geq F_{n-1} \).

Proof. (i) follows from point (i) of Lemma 1 and Lemma 4. (ii) Let \( f(i, j) = g_{n+1} h_{n+1} \). By Lemma 4, we have \( f(i - F_n, i - 1) = f_n \). Hence, \( f(i - F_n, j) = f_n h_{n+1} \).

The proofs of (iii) and (iv) are similar to that of Lemma 4.

For each non empty finite word \( w \), there exists a naturally associated periodic infinite word \( p_w \), defined as follows: \( p_w(0,|w| - 1) = w \) and, for each \( i \geq 0 \), \( p_w(i + |w|) = p_w(i) \). We say that a word \( u \) is a power of the (finite!) word \( w \) if \( u \) is a left factor of \( p_w \). We say that \( w \) is the base and \( k = |u|/|w| \) is the exponent of this power and we write \( u = w^k \). In general \( k \) is rational, but if \( k \) is an integer we obtain the usual notion of power. We consider only powers with exponent greater than or equal to 1. We say that two words \( u \) and \( v \) are conjugate if there exist two words \( u' \) and \( u'' \) such that \( u = u' u'' \) and \( v = u'' u' \). The following lemma is trivial.

Lemma 11. Let \( r, s \geq 1 \) and \( u, v \) be non-empty words of equal length. If \( u' \) is a factor of \( v^s \) then \( u \) is a conjugate of \( v \).

3. Maximal powers in the Fibonacci infinite word

Let \( u \in F(f) \), \( r \) rational and \( u' \in F(f) \). We say that the power \( u' \) is maximal if for each word \( v \) such that \( |u| = |v| \) and for each rational \( s \), if \( u' \) is a factor of \( v^s \in F(f) \), then \( u = v \) and \( r = s \). We say that an occurrence \((i_0, j_0)\) of \( u' \) in \( f \) is maximal if for each \( v \in F(f) \) such that \( |u| = |v| \) and for each \( i_1, j_1 \) such that \( i_0 \leq i_0 \leq j_0 \leq j_1 \), if \((i_0, j_1)\) is an occurrence of some power of \( v \) in \( f \) then \( u = v, i_0 = i_1 \) and \( j_0 = j_1 \). A power can have maximal occurrences even if it is not maximal. For example, \( a, (ab)^{3/2}, (aba)^2 \), and \((abaab)^{11/5}\) are not maximal powers but the pairs \((5, 5), (8, 10), (13, 18)\) and \((21, 31)\) are, respectively, maximal occurrences of them in \( f \).

Proposition 1. Let \( u \in F(f) \) and \( r \geq 1 \). If \((i_0, j_0)\) is an occurrence of \( u' \) in \( f \) then there exist a conjugate \( v \) of \( u \) and \( s \geq r \) such that \((i_0, j_0)\) is contained in a maximal occurrence of \( v^s \) in \( f \).

Proof. Let \( I \) be the set of all \( i \geq 0 \) such that \((i, j_0)\) is an occurrence of some \((u')^r \) such that \( |u| = |u'| \). Since \( I \) contains at least \( i_0 \), it is non empty. Let \( i_1 \) be its minimum. Now, let \( J \) be the set of all \( j \) such that \((i_1, j)\) is an occurrence of some \((u')^r \) such that \( |u| = |u'| \). Since \( J \) contains at least \( j_0 \), it is non empty. By Lemma 6, there is a maximum in \( J \), say \( j_1 \). Clearly, \((i_1, j_1)\) is a maximal occurrence of some \( v^s \) such that \( |v| = |u| \) and \( s \geq r \). By Lemma 11, \( u \) is a conjugate of \( v \).
Hereafter we denote by $\phi$ the golden ratio $(\sqrt{5} + 1)/2$. One of the arguments used proving the following Propositions 2–6 consists in reading some words in both directions, left–right and right–left.

**Proposition 2.** Let $v \in F(f)$ and $s$ be a rational number such that $(2 + \phi)/2 < s < 2$. Then no occurrence of $v^s$ in $f$ is a maximal one.

**Proof.** By way of contradiction, suppose that $(i_1, j_1)$ is a maximal occurrence of $v^s$. There exist a positive integer $k$, a proper non-empty left factor $v'$ of $v$ and a proper non-empty right factor $v''$ of $v$ such that $|v'| = k$, $v = v'v''$ and $v^s = v'v''v'$. We pose $x = v''(0)$ and $x' = v''(|v''| - 1)$ and we have $v'' = xv'' = v''x'$ for some $v''$, $v'' \in \{a, b\}^*$. 

(i) $i_1 = 0$. Let $f(j_1 + 1) = y$. By maximality of $v^s$ we have $x \neq y$, and so $v'$ is special. By Lemma 5, $v' = s[k]$ and, by definition of $s[k]$, $v' = \phi$. As $v' \in LF(f)$ and $v'$ is a palindrome, we have, by Lemma 9, $v' = h$, for some $n \geq 3$.

(ii) $i_1 \neq 0$. Let $f(i_1 - 1) = y$. By maximality of $v^s$ we have $x \neq y$ and $x' \neq y'$ and from this $v'$ and $\phi$ are both special factors. Then $v' = \phi = s[k]$ and so $v' \in LF(f)$ and $v'$ is a palindrome. Hence, by Lemma 9, $v' = h_n$ for some $n \geq 3$.

Thus in both cases (ia) and (ib) we have $v' = h_n$ for some $n \geq 3$.

Moreover, $n > 3$, otherwise we would have $v'v''v' = (av'')a = (av'')^s$ and $s = (2 + |v''|)/(1 + |v''|) \leq 3/2 < (2 + \phi)/2$ which is a contradiction. So $v' = h_n$ for $n \geq 4$.

(i) $|v''| = 1$. By point (vii) of Lemma 1, $v'$ begins and ends with aba. As $aaa$, abababa$\not\in F(f)$ we reach a contradiction in the case $v'' = a$ as well as in the case $v'' = b$.

(ii) $|v''| = 2$. By maximality of $v^s$ we have $v'xyv'yx \in F(f)$ where $x, y \in \{a, b\}$, $x \neq y$ and $v'' = xy$. By point (iii) of Lemma 2 we have to consider two cases: $v'xy = f_n$ and $v'xy = g_n$.

(i2a) If $v'xy = f_n$ then we have $v'xyv'yx = f_ng_n \in F(f)$ which is impossible by point (i) of Lemma 10.

(i2b) If $v'xy = g_n$ then we have $v'xyv'yx = g_nh_n \in F(f)$ which is impossible by point (ii) of Lemma 10.

(iii) $|v''| \geq 3$. We have $v'xyzv'yx \in F(f)$ where $x, y$ are letters and $z$ is a word. Again we have to consider two cases: $v'xy = f_n$ and $v'xy = g_n$.

(i3a) If $v'xy = f_n$ then, by point (iii) of Lemma 10, $|z| \geq F_n - 3$.

(i3b) If $v'xy = g_n$ then, by point (iv) of Lemma 10, $|z| \geq F_n - 2$.

So in both (i3a) and (i3b) we have $|z| \geq \min \{F_n - 2, F_n - 3\} = F_n - 3$. But then we have

$$s = (|v'| + |xy| + |v'|)/(|v'| + |xyz|) = 1 + (F_n - 2)/(F_n + |z|)$$

$$< 1 + (F_n - 2)/(2F_n - 1) < (2 + \phi)/2,$$

i.e., a contradiction. □
Proposition 3. Let \( v \in F(f) \). If \( v^2 \) has a maximal occurrence in \( f \) then \( v = a \) or \( v = aba \).

Proof. As in the previous case we first prove that \( v \) is a palindrome and \( v \in LF(f) \) and so that \( v = h_n \) for some \( n \geq 3 \). By point (vii) of Lemma 1, \( abaaba \) is a right and a left factor of \( h_n \) for each \( n \geq 5 \). Since \( abaabaabaab \in F(f) \), the only two possibilities are \( v = h_3 = a = f_1 \) or \( v = h_4 = aba = f_3 \). □

Proposition 4. Let \( v \in F(f) \) and \( s \) be a rational such that \( 2 < s < 3 \). If \( v^s \) has a maximal occurrence in \( f \) then either

\[
v = f_2 = ab \quad \text{and} \quad s = 5/2
\]

or

\[
v = f_n \quad \text{and} \quad s = 2 + (F_{n-1} - 2)/F_n
\]

for some \( n \geq 4 \).

Proof. In this case there exist a proper non-empty left factor \( v' \) of \( v \) and a proper non-empty right factor \( v'' \) of \( v \) such that \( v = v'v'' \) and \( v^s = (v'v'')^k \) for some integer \( k \geq 3 \). In analogy with the previous cases we first prove that \( v'v''v' = h_m \) for some \( m \geq 3 \).

Clearly the case \( m = 3 \) is impossible.

If \( m = 4 \) then \( v'v''v' = aba \), \( v'v'' = ab \) and \( v' = a \). Hence, \( v = ab \) and \( s = 5/2 \).

Now, let \( m \geq 5 \). By Lemma 8, we have \( v'v'' = f_{m-2} \) or \( v'v'' = f_{m-1} \).

(i) If \( v'v'' = f_{m-2} \) then, again by Lemma 8, we have \( F_{m-1} - 2 = |v'| < |v'v''| = F_{m-2} \) and so \( m = 4 \). Contradiction.

(ii) If \( v'v'' = f_{m-1} \) then, again by Lemma 8, \( v' = h_{m-2} \). Hence, for some \( n = m - 1 \geq 4 \), \( v = f_n \) and \( s = 2 + (F_{n-1} - 2)/F_n \). □

Proposition 5. Let \( v \in F(f) \). If \( v^3 \) has a maximal occurrence in \( f \) then \( v = aba \).

Proof. In analogy with the proof of Proposition 3, the unique possibility is \( v = h_4 = aba = f_3 \). □

Proposition 6. Let \( v \in F(f) \) and \( s \) be a rational such that \( 3 < s \). If \( v^s \) has a maximal occurrence in \( f \) then

\[
v = f_n \quad \text{and} \quad s = 3 + (F_{n-1} - 2)/F_n
\]

for some \( n \geq 4 \).

Proof. First, as in the proofs of Propositions 3 and 5, we have that the rational \( s \) is not an integer. So we can suppose that there exist a non-empty left factor \( v' \) of \( v \) and a non-empty right factor \( v'' \) of \( v \) such that \( v = v'v'' \) and \( v^s = (v'v'')^k v' \) for some integer \( k \geq 3 \). In analogy with the previous cases we prove that \( (v'v'')^{k-1} v' = h_m \) for some \( m \geq 3 \).
Clearly the cases \( m = 3 \) and \( m = 4 \) are impossible.

Let \( m \geq 5 \). By Lemma 8, we have \( v'v'' = f_{m-1} \) or \( v'v'' = f_{m-2} \).

(i) The case \( v'v'' = f_{m-1} \) is impossible. In fact, again by Lemma 8, we would have
\[
F_{m-1} = |v'v''| < |(v'v'')^k - 2| = F_{m-2} - 2,
\]
which is clearly a contradiction.

(ii) If \( v'v'' = f_{m-2} \) then, again by Lemma 8, we have that
\[
(v'v'')^k - 2v' = h_{m-1}
\]
and so
\[ k = 3 \text{ and } v' = h_{m-3}. \]
As \( 0 < |v'| = F_{m-3} - 2 \) we must have \( m \geq 6 \). Hence, for some \( n = m - 2 \geq 4 \), \( v = f_n \) and \( s = 3 + (F_{n-1} - 2)/F_n \).

Remark. The word \( abaabaa = (f_3)^2a \) is not a maximal power, being always a factor of a (maximal) occurrence of \( abaabaaha = (f_3)^3 \) in \( f \). The word \( abaabaabaa = (f_3)^3a \) does not belong to \( F(f) \).

Remark. Consider the sequences \( 2 + (F_{n-1} - 2)/F_n \) and \( 3 + (F_{n-1} - 2)/F_n \); their elements are exponents of powers having a maximal occurrence in \( f \) and the numbers \( 1 + \Phi \) and \( 2 + \Phi \) are their respective limits for \( n \) going to infinity. By Propositions 2–6, no other value greater than \( (2 + \Phi)/2 \) is the limit of such a sequence. On the other hand, as one can easily see, in the interval \([1, (2 + \Phi)/2]\) infinitely many values have such a property.

**Proposition 7.** Let \( s \) be a rational number greater than \( (2 + \Phi)/2 \) and \( v \in F(f) \). If \( v^s \) has a maximal occurrence in \( f \) then there exists \( n \geq 1 \) such that
\[
v = f_n.
\]

**Proof.** It follows by Propositions 2–6. 

**Proposition 8.** Let \( n \geq 1 \) and \( s \) be a rational number greater than \( (2 + \Phi)/2 \). If \((f_n)^s\) has a maximal occurrence in \( f \) then:
- if \( n = 1 \) then \( s = 2 \);
- if \( n = 2 \) then \( s = 5/2 \);
- if \( n = 3 \) then \( s = 2 \) or \( s = 3 \);
- if \( n \geq 4 \) then \( s = 2 + (F_{n-1} - 2)/F_n \) or \( s = 3 + (F_{n-1} - 2)/F_n \).

**Proof.** Again by Propositions 2–6.

**Remark.** For \( n \geq 4 \) the two values are effectively realized.

**Proposition 9.** Let \( r \) be a rational number greater than \( (2 + \Phi)/2 \) and \( u \in F(f) \). If \( u^r \in F(f) \), then there exists \( n \geq 1 \) such that \( u \) is a conjugate of \( f_n \) and, moreover, each occurrence of \( u^r \) is contained in a maximal one of \((f_n)^s\) for some \( s \in [2, 2 + \Phi] \).

**Proof.** By Propositions 1, 7 and 8 and by Lemma 11.
Proposition 10. For each $\varepsilon > 0$ there exist a rational $t \in [(2 + \Phi)/2 - \varepsilon, (2 + \Phi)/2)$ and a word $w$, such that $w^t$ has a maximal occurrence in $f$ and $|w| \neq F_n$ for each $n \geq 3$.

Proof. Consider, for $n \geq 3$, the factorization

$$f_{n+4} = f_{n+1} f_n f_s f_{n-1} f_{n-1} f_n.$$

Clearly, for $n \geq 3$, $(f_n f_s) f_n f_{n-1}$ has a maximal occurrence. As, for $n \geq 3$, $2F_n$ is not a Fibonacci number and $(2 + \Phi)/2$ is the limit of $(2F_n + F_{n+1} - 2)/2F_n$ for $n$ going to infinity, the statement is proved.

The following Propositions 11–13 are known results on the Fibonacci infinite word and are easy consequences of Proposition 9.

Proposition 11 (Séébold [9]). Let $u \in F(f)$. If $u^2 \in F(f)$ then $u$ is a conjugate of $f_n$ for some $n \geq 1$.

Proof. This follows immediately from Proposition 9 and from $2 > (2 + \Phi)/2$.

Proposition 12 (Karhumäki [3]). Let $u \in F(f)$ Then $u^4 \notin F(f)$.

Proof. As already remarked by Séébold, Proposition 12 is a consequence of Proposition 11.

Proposition 13 (Mignosi and Pirillo [8]). Let $u \in F(f)$ and $r$ rational such that $u^r \in F(f)$. Then $r < 2 + \Phi$.

Proof. By way of contradiction, suppose that for some $u \in F(f)$ and for some rational $r > 2 + \Phi$, $u^r \in F(f)$. By Proposition 9, each occurrence of $u^r$ in $f$ is contained in a maximal one of $(f_s)^r$ for some $s \in [2, 2 + \Phi)$ and some $n \geq 1$ such that $|u| = F_n$. So $r \leq s < 2 + \Phi$. Contradiction.

Remark. A previous version of this paper is in the Actes du Séminaire Lotharingien de Combinatoire (30e Session, 1993)

References