
1. INTRODUCTION

Let $G(q)$ be a finite simple group of Lie type over a field of order $q$ where $q = p^e$. We denote by $l(G(q))$ the degree of the smallest nontrivial projective irreducible representation of $G(q)$ over a field of characteristic $r \neq p$. The results of [LS, SZ] provide lower bounds for the numbers $l(G(q))$ and these bounds are the best possible in many cases. However, the results of [GT] for linear groups and those of [TZ] for representations in characteristic zero suggest that the bounds of [SZ] can be improved.

1 This work was part of my Ph.D. thesis at the University of Southern California. I am very grateful to my advisor, Robert Guralnick, for the help and support throughout this period. I also thank the referee, who carefully read the manuscript in record time and provided insightful comments.
We will improve the bounds for the case of the orthogonal groups, and we will say something about the gap between the smallest possible representations and the next family of such representations in the case of orthogonal and unitary groups. Note that for our purpose one can assume that the representations are defined over an algebraically closed field because increasing the field will decrease the dimensions of the representations. We will use the results of [SZ] together with the better bound for linear groups provided in [GPPS] to prove:

**Theorem 1.** Let $V$ be a nontrivial irreducible projective representation of $\Omega^\pm_{2n}(q)$ where $e = \pm$, or of $\Omega_{2n+1}(q)$ over a field of characteristic $r$ not dividing $q$. Then:

1. If $G = \Omega^\pm_{2n}(q)$, $n \geq 4$, $q > 30$, then
   \[
   \frac{(q^n - 1)(q^{n-1} + q)}{q^2 - 1} - 2 \leq \dim V \leq \frac{(q^n - 1)(q^{n-1} + q)}{q^2 - 1} + 1
   \]
   or
   \[
   \dim V \geq \frac{(q^n - 1)(q^{n-1} + q)}{q^2 - 1} + \frac{q^n - 1}{q - 1} - 4.
   \]
   Moreover, the smaller of these cases will not hold if $r$ does not divide $(q^n - 1)/(q - 1)$.

2. If $G = \Omega^\pm_{2n}(q)$, $n \geq 5$, $q \leq 3$, then
   \[
   \frac{(q^n - 1)(q^{n-1} - 1)}{q^2 - 1} \leq \dim V \leq \frac{(q^n - 1)(q^{n-1} - 1)}{q^2 - 1} + 1
   \]
   or
   \[
   \dim V \geq \frac{(q^n - 1)(q^{n-1} - 1)}{q^2 - 1} + \frac{q^n - 1}{q - 1} - 2.
   \]
   Moreover, if $r$ does not divide $(q^n - 1)/(q - 1)$, the latter bound is
   \[
   \frac{(q^n - 1)(q^{n-1} - 1)}{q^2 - 1} + \frac{q^n - 1}{q - 1} - 1.
   \]

3. If $G = \Omega_{2n+1}(q)$, $n \geq 4$, $(n, q) \neq (4, 2), (5, 2), (4, 4), (5, 3)$, then
   \[
   \frac{(q^n + 1)(q^{n-1} - q)}{q^2 - 1} - 1 \leq \dim V \leq \frac{(q^n + 1)(q^{n-1} - q)}{q^2 - 1} + 1
   \]
   or
   \[
   \dim V \geq \frac{(q^n + 1)(q^{n-1} - q)}{q^2 - 1} + \frac{q^{n-1} - 1}{q - 1} - 3.
   \]
Moreover, the smaller of these cases will hold if and only if \( r \) divides \( (q^n - 1)/(q - 1) \).

4. If \( G = P\Omega_{2n+1}(q), n \geq 3, q > 3, q \text{ odd}, \) then

\[
\frac{q^{2n} - 1}{q^2 - 1} - 2 \leq \dim V \leq \frac{q^{2n} - 1}{q^2 - 1}
\]

or

\[
\dim V \geq \frac{q^{2n} - 1}{q^2 - 1} + \frac{q^n - 1}{q - 1} - 4.
\]

Moreover, the smallest of these cases will not hold if \( r \) does not divide \( (q^n - 1)/(q - 1) \).

We note that our bounds are very close to the dimensions of the representations in characteristic zero obtained in [TZ]; hence after these representations pass to positive characteristic either their representations will stay irreducible or there will be at most two trivial modules among the composition factors. In particular in the case \( G = P\Omega_{2n}(q) \), we can determine exactly how this representation will factor.

For convenience we include a complete table with the bounds for \( l(G(q)) \) (Table I). In this table we mark with a star the cases where the bound is known to be sharp. Note also that the bounds in the cases \( E_i, i = 6, 7, 8 \), were obtained in [H].

2. ON LINEAR GROUPS

Before considering the orthogonal groups, we will investigate the small-dimensional indecomposable representations for the linear group of rank \( n > 2 \). In what follows let \( e(n, q) \) be \( (q^n - 1)/(q - 1) \). We will only consider modules \( N \) such that \( G = SL_n(q) \) will act on both \( N \) and \( N^* \) without fixing more than a one-dimensional space. If \( N \) is such a module and the length is more than 4, then at least two of the factors are nontrivial irreducible modules; hence the dimension is more than \( 2e(n, q) - 4 \). Therefore consider only those modules that have length at most 3 and only one nontrivial factor. In fact, after considering their radicals, we need to discuss the length 2 indecomposable with one nontrivial factor. We claim:

**Lemma 1.** Let \( G = SL_n(q), n > 2, (n, q) \neq (3, 2), (3, 4), (4, 2), (4, 3) \), and assume that \( N \) is a nontrivial indecomposable \( kG \) module of dimension smaller than \( 2e(n, q) - 4 \) and such that \( \dim C_N(G) \leq 1, \dim C_N^*(G) \leq 1 \). Then \( N \) either is irreducible or has exactly one nontrivial factor of dimension \( e(n, q) - 2 \) and at most two trivial factors. In particular, the dimension \( d \) of any such module satisfies \( e(n, q) - 2 \leq d \leq e(n, q) \).
<table>
<thead>
<tr>
<th>Group</th>
<th>Bound</th>
<th>Exceptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{PSL}_2(q)$*</td>
<td>$(q - 1)/\text{gcd}(2, q - 1)$</td>
<td>$l(\text{PSL}_2(4)) = 2$, $l(\text{PSL}_2(9)) = 3$</td>
</tr>
<tr>
<td>$\text{PSL}_n(q)$, $n \geq 3^*$</td>
<td>$(q^n - 1)/(q - 1) - 2$</td>
<td>$l(\text{PSL}_3(2)) = 2$, $l(\text{PSL}_4(2)) = 7$, $l(\text{PSL}_n(3)) = 26$</td>
</tr>
<tr>
<td>$\text{PSp}_n(q)$, $n \geq 2^*$</td>
<td>$(q^n - 1)/2$, $q$ odd $(q^n - 1)(q^n - q)/(2(q + 1))$, $q$ even</td>
<td>$l(\text{PSp}_4(2)) = 2$</td>
</tr>
<tr>
<td>$\text{PGL}_2(q)$, $n \geq 3$</td>
<td>$(q^n - 1)/(q^2 - 1) - 2$, $q \neq 3$ $(q^n - 1)(q^n - q)/(q^2 - 1)$, $q = 3$</td>
<td>$l(\text{PGL}_3(3)) = 27$</td>
</tr>
<tr>
<td>$\text{PGL}_2(q)$, $n \geq 4$</td>
<td>$(q^n - 1)(q^{n-1} + q)/(q^2 - 1) - 2$, $q &gt; 3$ $(q^n - 1)(q^n - 1)/(q^2 - 1)$, $q \leq 3$</td>
<td>$l(\text{PGL}_2(2)) = 8$</td>
</tr>
<tr>
<td>$\text{PS}L_3(q)$, $n \geq 4^*$</td>
<td>$(q^n + 1)(q^{n-1} - q)/(q^2 - 1) - 1$</td>
<td>$l(\text{PSL}_3(2)) \geq 32$, $l(\text{PSL}_3(4)) \geq 1026$, $l(\text{PSL}_3(2)) \geq 151$, $l(\text{PSL}_3(3)) \geq 2376$, $l(\text{PSL}_3(3)) = 3276$</td>
</tr>
<tr>
<td>$\text{PSU}_2(q)$, $n \geq 2$</td>
<td>$(q^{2n} - 1)/(q + 1)$</td>
<td>$l(\text{PSU}_3(2)) = 4$, $l(\text{PSU}_3(3)) = 6$, $l(\text{PSU}_n(2)) = 2$</td>
</tr>
<tr>
<td>$\text{PSU}_{2n+1}(q)$</td>
<td>$(q^{2n+1} - q)/(q + 1)$</td>
<td>$l(\text{PSU}_{2n+1}(q)) = 4$</td>
</tr>
<tr>
<td>$^2B_2(q)$</td>
<td>$(q - 1)/\sqrt{q/2}$</td>
<td>$l(\text{PSU}_{2n+1}(q)) = 10$</td>
</tr>
<tr>
<td>$^2G_2(q)$</td>
<td>$q(q - 1)$</td>
<td>$l(\text{PSU}_{2n+1}(q)) = 14$</td>
</tr>
<tr>
<td>$G_2(q)$</td>
<td>$q(q^2 - 1)$</td>
<td>$l(\text{PSU}_{2n+1}(q)) = 12$</td>
</tr>
<tr>
<td>$^3D_4(q)$</td>
<td>$q^3(q^2 - 1)$</td>
<td>$l(\text{PSU}_{2n+1}(q)) = 24$</td>
</tr>
<tr>
<td>$^2F_4(q)$</td>
<td>$q^4(q - 1)/\sqrt{q/2}$</td>
<td>$l(\text{PSU}_{2n+1}(q)) = 32$</td>
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<tr>
<td>$F_4(q)$</td>
<td>$q^6(q^2 - 1)$, $q$ odd $q^6(q^2 - 1)(q^2 - q)/(q^2 - q - 1)$, $q$ even</td>
<td>$l(\text{PSU}_{2n+1}(q)) = 48$</td>
</tr>
<tr>
<td>$^3E_6(q)$</td>
<td>$q^3(q^2 - 1)$</td>
<td>$l(\text{PSU}_{2n+1}(q)) = 72$</td>
</tr>
<tr>
<td>$E_6(q)$</td>
<td>$q^6(q^2 - 1)$</td>
<td>$l(\text{PSU}_{2n+1}(q)) = 144$</td>
</tr>
<tr>
<td>$E_8(q)$</td>
<td>$q^8(q^2 - 1)$</td>
<td>$l(\text{PSU}_{2n+1}(q)) = 288$</td>
</tr>
<tr>
<td>$E_8(q)$</td>
<td>$q^8(q^2 - 1)$</td>
<td>$l(\text{PSU}_{2n+1}(q)) = 576$</td>
</tr>
<tr>
<td>$E_8(q)$</td>
<td>$q^8(q^2 - 1)$</td>
<td>$l(\text{PSU}_{2n+1}(q)) = 1152$</td>
</tr>
</tbody>
</table>
Proof. By Theorem 9.1.5 of [GPPS], we know that there are only three possibilities for the dimensions of the nontrivial irreducible factors of $N$. These are $e(n, q) - 2$, $e(n, q) - 1$, and $e(n, q)$. Moreover, for the last two dimensions the corresponding modules are determined by Theorem 9.1.6 of [GPPS]. More precisely, if $P$ is the stabilizer of a 1-space in the natural action of $G$ and $M$ is an irreducible module then $\dim M = e(n, q) - 1$ implies $M \oplus k = kM$ and $\dim M = e(n, q)$ implies $M = \lambda M$, ($\lambda$ nontrivial).

By the preceding remarks, we need to investigate indecomposable modules of length 2 having small-dimensional factors, that is, having $M$ and the trivial module as factors. Consider in each case $\text{Ext}^1(k, M) = H^1(G, M)$ and $\text{Ext}^1(M, k) = H^1(G, M^*)$. By the Shapiro Lemma, $H^1(G, \lambda M) = H^1(P, \lambda) = 0$ (respectively, $H^1(G, kM) = H^1(P, k) = 0$) and if $\dim M = e(n, q) - 1$ (respectively, $e(n, q)$) then $M^* = M$ (respectively, $M^* = \tau M$ for some $\tau$). Assume $M \oplus k = kM$, then $H^1(G, kM) = H^1(G, k) \oplus H^1(G, M)$ so $\text{Ext}^1(k, M) = \text{Ext}^1(M, k) = H^1(G, M) = 0$.

The same result holds if $M = \lambda M$, hence any module of length 2 having $M$ and the trivial module as its factors is in fact the direct sum of these factors and so any indecomposable module of length $\geq 1$ and dimension smaller than $2e(n, q) - 4$ has an irreducible factor of dimension $e(n, q) - 2$, and by the remarks preceding the lemma it has length $\leq 3$. The result follows. ■

In [GT] Guralnick and Tiep proved that if $V$ is an irreducible representation of $SL_n(q)$, $n > 2$, of dimension smaller than $\delta(n, q)$ then it is one of the three types of representations mentioned above. Here

$$\delta(n, q) = \begin{cases} (q - 1)(q^2 - 1)/(3, q - 1) & \text{if } n = 3 \\ (q - 1)(q^3 - 1)/(2, q - 1) & \text{if } n = 4 \\ (q^{n-1} - 1)((q^{n-2} - q)/(q - 1) - k_{n-2}) & \text{if } n \geq 5, \end{cases}$$

where $k_r = 1$ if $r$ divides $(q^n - 1)/(q - 1)$ and is 0 otherwise. We will use this result in what follows.

### 3. UNITARY GROUPS

Before dealing with the proof of Theorem 1 we prove some estimates for the gap between the smallest representations for unitary groups. Recently, Malle and Hiss obtained much better results in this special case but the methods presented here are entirely elementary.

**PROPOSITION 1.** If $G = SU_{2n}(q)$, $n \geq 2$, and $V$ is a nontrivial irreducible $kG$ module (char $k$ does not divide $q$), then

$$\dim V = \frac{q^{2n} - 1}{q + 1}, \quad \frac{q^{2n} - 1}{q + 1} + 1$$
Proof. Let \( P \) be a stabilizer of a maximal totally singular space for the natural module of \( G, L \) the Levi complement, and \( Q \) the unipotent radical of \( P \). Then \( Q \) is abelian and can be viewed as the group of skew Hermitian \( n \times n \) matrices over a field with \( q^2 \) elements. Therefore \( L \) can be identified with \( GL_n(q^2) \) acting on \( Q \) as \( A \to gAg^{-1} \). Since \( r := \text{char } k \) does not divide the order of \( Q \), the irreducible \( G \) module \( V \) decomposes as \( V = [Q,V] \oplus C_V(Q) \) when regarded as a \( Q \) module (the same is true for \( V \) as a \( P \) module). Note that the smallest orbit of the action of \( L \) on \( Q \) has length \( (q^{2n} - 1)/(q + 1) \) (it corresponds to the rank-one skew Hermitian matrices) and therefore the same is true about the action of \( L \) on the characters of \( Q \). The length of the next orbit is at least twice as large. The conclusion is that \( \dim Q;V \geq \frac{q^{2n} - 1}{q + 1} \).

Assume that \( \dim Q;V = \frac{q^{2n} - 1}{q + 1} \) or is at least twice as large. We will prove that \( C \) is a trivial module for the derived group \( L' \). To see that we first note that \( [S] \) constructs a \( G \) module \( W \) of dimension \( \frac{q^{2n} - 1}{q + 1} \). The above argument shows that \( W = [Q,W] \).

Let \( x \in L' \) be a transvection and \( y \in Q \) a conjugate of \( x \). Note that \( M = [Q,V] = \alpha' \) as an \( L' \) module where \( S \) is the stabilizer of an eigenspace of \( Q \) and \( \alpha \) is a linear character of \( S \), hence \( \alpha \) is trivial on transvections of \( S \). In this case the value of the Brauer character \( t_\alpha(x) \) does not depend on \( x \) (it is the number of \( x \) invariant characters of \( Q \) occurring in \( M \)). A similar argument gives that \( t_\alpha(y) \) has the same value.

Next, \( y \) is a rational element in \( P \) so \( t_M(y) \) depends only on the dimension of \( C \) and thus not on the structure of \( M \). Also, \( W = [Q,W] \), so \( t_W(x) = t_W(y) \). Therefore \( t_M(x) = t_W(x) = t_W(y) = t_M(y) \); hence \( t_C(x) = t_C(y) \), and since \( y \) acts trivially on \( C \) so does \( x \). Since \( L' \) is generated by transvections it follows that \( L' \) acts trivially on \( C \).

\( M \) is a transitive permutation module for \( L' \), so \( \dim C_M(L') = 1 \). If \( \tilde{Q} \) is the opposite unipotent radical, \( \tilde{V} = [\tilde{Q},\tilde{V}] \oplus C_{\tilde{V}}(\tilde{Q}) \) and a similar argument gives \( \dim \tilde{C} = \dim C_{\tilde{V}}(\tilde{Q}) \). In particular, both \( C \) and \( C_{\tilde{V}}(\tilde{Q}) \) are hyperplanes in \( C_{\tilde{V}}(L') \) and cannot intersect, so they must have dimension at most one, hence we have the conclusion.

Note that in fact this give us similar estimates of the irreducible projective representations of the simple groups of type \( PSU_{2n}(q) \) with the exceptions of \( (n,q) = (2,2), (2,3) \), in which cases there are exceptional Schur multipliers.
4. PROOF OF THEOREM 1

Let $G(q)$ be one of $PO^+_2(q)$, where $\epsilon = \pm$, or $PO_{2n+1}(q)$, $q$ odd and $\hat{G}$ a perfect central extension of $G(q)$ acting irreducibly on $V$. If $S$ is a subgroup of $G(q)$ we will denote by $\bar{S}$ the subgroup of $\hat{G}$ such that $\bar{S}/Z(\hat{G}) = S$. We will denote by $P = JU$ the subgroup of $G(q)$ that is the stabilizer of a maximal totally singular subspace for the natural module, by $U$ the unipotent radical of $P$, and by $J$ the Levi complement ($J$ will be a linear group). Then since $U$ is a $p$ group, we have $\hat{U} = U_0 \times Z(\hat{G})$, where $U_0$ is $P$ isomorphic to $U$ by assumption on $(n, q)$. Let $V = [U_0, V] \oplus C_V(U_0)$ as a $P$ module. First we note that in fact $[U_0, V]$ is the sum of nontrivial representations of $U_0$ that occur in $V$. Therefore one can bound from below the dimension of this module by $l \cdot d$, where $l$ is the length of the smallest orbit of the action of $J$ on the characters of $U$ and $d$ is the degree of the corresponding characters. We will use the bound in [SZ] to see that in fact if $[U_0, V]$ is small then $M = C_V(U_0)$ must be quite large. This will actually give us that $M \neq C_M(J')$, which in particular means that $M$ as a $J'$ module will have a nontrivial irreducible factor and so we can use the bound in [GT] for the dimension of $M$.

**Proposition 2.** If $G(q) = PO^+_2(q)$ with $n \geq 4$ and $q > 3$, then Theorem 1 holds.

**Proof.** As above, pick $P$ to be the stabilizer of a maximal totally singular subspace of the natural orthogonal module, $U$ its unipotent radical, and $J$ the Levi complement. Then $J = GL_n(q)$, and if $q$ is odd, $U$ can be regarded as the space of $n \times n$ antisymmetric matrices on which $J$ acts in the natural way ($T^A = A T^T A$). If $q$ is even $U$ can be identified with the space of symmetric $n \times n$ matrices with zero diagonal under the same action of $J$; therefore the module structure is the same in both cases. The orbits are the matrices of a given rank $2i$; hence the size of such an orbit is $l(i) := q^{i(i-1)}(q^2 - 1) \cdots (q^{2i-2i+1} - 1)/(q^2 - 1)(q^4 - 1) \cdots (q^{2i} - 1)$. Therefore the smallest orbit has length $l = (q^n - 1)(q^{n-1} - 1)/(q^2 - 1)$, and after estimating term by term and noting that $i \geq 2$, we get

$$l(i)/l \geq \frac{q^{i(i-1)}(q^2 - 1)(q^3 - 1)(q^2 - 1)(q - 1)}{(q^4 - 1)(q^2 - 1)} \geq q \geq 2$$

and so $\dim[U_0, V] = l$ or is at least twice as big. If $q = 2, 3$ then the lower bound in Theorem 1 is obtained. (Note that in fact this is also the bound in [SZ] for $q = 3$.)

If $q > 3$ assume that $\dim[U_0, V] = l$. We know from [SZ] that $\dim V \geq (q^n - 1)(q^{n-1} + q)/(q^2 - 1) - n$, so $M = C_V(U_0)$ must have dimension larger than $(q^n - 1)/(q - 1) - n$. Note that $[U_0, V]$ is a transitive permutation module for $J'$; hence $C_{[U_0, V]}(J')$ is one-dimensional and so $C_M(J')$
is a hyperplane in $C_V(J)$. If we repeat the same argument for the opposite parabolic group we get that if $C_M(J')$ has dimension at least 1, then $C_V(G) \neq 0$, which contradicts the hypothesis. Therefore $\dim C_M(J') \leq 1$, and so since $n \geq 4$ we get that $C_V(U_0)$ has a nontrivial factor as a $J'$ module. The dimension of this factor must be at least $(q^n - 1)/(q - 1) - 2$ from [GT]; hence the lower bound.

For the rest of the theorem we consider once again $V = [U_0, V] \oplus M$ we consider the next possibility for the dimensions of the two modules. The next possible dimension of $[U_0, V]$ is very large and will already solve the problem. If we regard $M$ as a $J'$ module we see that the maximum dimension of $C_M(J')$ is 1. Therefore using Lemma 1 we get that an indecomposable factor of $M$ has dimension $e(n, q) - 2$, $e(n, q) - 1$, $e(n, q)$ or $\geq 2e(n, q) - 4$ and the result follows.

**Proposition 3.** Let $G(q) = P\Omega_{2n}(q)$ with $n \geq 4$; let $(n, q) \neq (4, 2), (5, 2), (4, 4), (5, 3)$; and let $V$ be a nontrivial projective irreducible representation in characteristic $r$ not dividing $q$. Then

$$\dim V \geq \frac{(q^n + 1)(q^n - q)}{q^2 - 1} - 1$$

**Proof.** Consider $P_n = JU$, the stabilizer of a maximal totally singular subspace of the natural module. Then $J = GL_{n-1}(q) \times D_{2(q+1)}$ and $U$ is nonabelian solvable, with $U' = \Phi(U)$ being the space of antisymmetric $(n - 1) \times (n - 1)$ matrices with the above-mentioned action of $J$. As before, $U = U_0 \times Z(G)$, $V$ regarded as a $U_0$ module has an irreducible component on which $Z(U_0)$ acts nontrivially or else the representation is not faithful.

Note also that the degree of any nonlinear character of $U_0$ is a multiple of $q^2$ (this is because $U/Z(U) = N \oplus N$ as a $J'$ module, where $N$ is the natural module, and $J$ acts as $(A, x)v = Avx$, where $v$ is a $2 \times (n - 1)$ matrix and we let $x \in O_2(q)$. The conclusion is that the dimension of $[U_0, V]$ is at least $q^2 \cdot l$; where $l$ is the length of the smallest orbit of the action of $P_n$ on $Z(U_0)$. As before, $l = (q^{n-1} - 1)(q^{n-2} - 1)/(q^2 - 1)$; hence $\dim [U_0, V] = q^2(q^{n-1} - 1)(q^{n-2} - 1)/(q^2 - 1)$ or it is at least twice as large (by the counting argument above). We know by [SZ] that $\dim V \geq (q^n + 1)(q^n - q)/(q^2 - 1) - n + 2$.

Let $M = C_V(U)$ and assume $\dim [U_0, V] = q^2(q^{n-1} - 1)(q^{n-2} - 1)/(q^2 - 1)$. It follows that $\dim M \geq q(q^{n-2} - 1)/(q - 1) - n + 2 \geq q^2 + 1$. Also, $C_{[U_0, V]}(J')$ has dimension $q^2$ and so by repeating the argument for the opposite parabolic subgroup $P$ and making the same assumptions on the dimension of $[U_0, V]$ we get that if $\dim C_M(J') > q^2$, then $C_V(P) \cap C_V(\bar{P}) \neq \emptyset$.

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2 As before, if $q$ is even we need to consider symmetric $(n - 1) \times (n - 1)$ matrices with zero diagonal and the module structure will be the same.
which leads to a contradiction. Therefore \( \dim C_M(\bar{V}) \leq q^2 \) and so \( M \) has a nontrivial factor as a \( \bar{V} \) module and the dimension of \( M \) is at least \((q^{n+1} - 1)/(q - 1) - 2\). If we add the lower bounds for \([U_0, V]\), respectively \( M \), we get that \( \dim V \geq (q^n + 1)(q^{n-1} - q)/(q^2 - 1) - 1 \); hence we have the lower bound. 

For the rest of the theorem note that \( C_M(\bar{V}) \) is at most \( q^2 \)-dimensional, and using Lemma 1 any indecomposable factor of \( M \) has dimension \( e(n - 1, q) - 2, e(n - 1, q) - 1, e(n - 1, q) \), or \( \geq 2e(n - 1, q) - 4 \). Therefore the proof of the theorem is complete once we prove the following:

**Proposition 4.** Let \( G = P\Omega_2(q) \) and let \( V, P_n \) be as before. Then either

\[
\dim V = \frac{(q^n + 1)(q^{n-1} - q)}{(q^2 - 1)} - 1 + k \quad (k = 0, 1, 2) \quad \text{and} \quad C_V(P_n) = 0,
\]

or

\[
\dim V \geq \frac{(q^n + 1)(q^{n-1} - q)}{(q^2 - 1)} + \frac{q^{n-1} - 1}{q - 1} - 3.
\]

**Proof.** We will proceed by induction. Let \( G = P\Omega_6(q) \), \( q \geq 3 \) and assume \( V \) is an irreducible representation of \( G \) such that \( \dim V \geq (q^4 + 1)(q^3 - q)/(q^2 - 1) \cdot 1 + k = q^3 + q - 1 + k \). Assume \( \dim C_V(P_n) = d \) and prove \( d = 0 \). By the proof of Proposition 3, \( d \leq q^2 \) and \( k - 2 \leq d \leq k \).

Let \( P = LQ \) be the stabilizer of a line in the natural module of \( G \). Note that the Levi complement \( L \) is \((GL_4(q) \times \Omega_6^-(q)) \) if \( q \) is even, \( \frac{1}{2}GL_4(q) \times \Omega_6^+(q) \) if \( q = 4k + 3 \), and \( \frac{1}{2}GL_4(q) \times \Omega_6^+(q) \cdot 2 \) if \( q = 4k + 1 \). As before, \( N = Q_0 \times Z(\bar{G}) \), where \( Q_0 \) and \( Q \) under the action of \( P \). Then \( V = [V, Q_0] \oplus C_V(Q_0) \) and since \( Q_0 \subset P_n \), \( C_V(P_n) \subset C_V(Q_0) = N \). Note that \( \dim [Q_0, V] = (q^3 + 1)(q^2 - 1) \) or it is at least twice as large and that will be more than the dimension of \( V \). Therefore \( \dim [Q_0, V] = (q^3 + 1)(q^2 - 1) \) and \( \dim N = q^3 - q^2 + q + k \).

Regard \( N \) as an \( L' \) module, and since \( L' = \Omega_6^-(q) = U_2(q) \), the minimal dimension of a nontrivial irreducible factor of \( N \) is (cf. [SZ]) \( q^3 - q^2 + q - 1 \), so \( N \) has exactly one nontrivial factor. Since \( \dim C_{Q_0, V}(L') = 1 \), we get that \( \dim C_N(L') \leq 1 \) (otherwise we can consider the opposite parabolic and get a submodule of \( V \)); hence \( N' \), the irreducible factor of \( N \), contains a \( d \)-dimensional vector space \( W \) on which \( P_n \) acts trivially (this is because \( C_N(L') \cap C_V(P_n) = \emptyset \)),

\[
q^3 - q^2 + q + k - 2 \leq \dim N' \leq q^3 - q^2 + q + k.
\]

Using Proposition 1, we get that \( k \leq 2 \) and so \( 0 \leq d \leq 2 \). Let us consider \( S \) to be the stabilizer of a maximal totally singular space, \( H \) its unipotent radical, and \( R \) its Levi complement. It follows that \( H \subset P_4 \) so \( W \subset C_N(H) \).
From the proof of Proposition 1 we can see that $C_{N'}(H)$ is a trivial $R'$ module and it is at most one-dimensional. It follows that $R'$ fixes $W$ and since $P'_1$ and $R'$ generate $G$, $W$ is a $G$ submodule of $V$ hence it is zero and thus $d = 0$ and so $C_V(P'_1) = 0$.

If $n > 4$, assume that $\dim V = (q^n + 1)(q^{n-1} - q)/(q^2 - 1) - 1 + s$ and, as before, $\dim C_V(P'_n) = d, d \leq q^2, s - 2 \leq d \leq s$. Let $P, L, Q$ be as above and note that $Q$ is abelian, $L' = \Omega^{2(n-1)}(q)$. If $V = [Q_0, V] \oplus C_V(Q_0)$, then $C_V(P'_n) \subset C_V(Q_0)$ and the smallest orbit of the action of $L'$ on $Q$ has length $(q^{(n-1)} + 1)(q^{(n-2)} - q)$, this being the only one that can occur. Consequently, $\dim C_V(Q_0) = (q^{(n-1)} + 1)(q^{(n-2)} - q)/(q^2 - 1) + s$ and so (cf. [SZ]) $C_V(Q)$ has only one nontrivial factor as an $L'$ module. After possibly passing to the dual of the module and factoring out by a trivial $L'$ module, we can obtain an irreducible $L'$ module that contains a $d$-dimensional vector space $W$ on which $P_n$ acts trivially. Since $P'_n \cap L'$ is the corresponding parabolic of $L'$, the induction hypothesis gives that $d = 0$, proving the proposition.

**Corollary 1.**

1. If $r$ divides $(q^{n-1} - 1)/(q - 1)$ then $P\Omega^{2n}(q)$ has an irreducible projective representation of dimension $(q^n + 1)(q^{n-1} - q)/(q^2 - 1) - 1$ over $F_r$. This representation is a factor of the reduction modulo $r$ of the smallest irreducible projective representation in characteristic 0.

2. If $r$ does not divide $(q^{n-1})/(q - 1)$ then the reduction mod $r$ of the smallest irreducible projective representation of $P\Omega^{2n}(q)$ in characteristic 0 is irreducible and there are no representations of dimension $(q^n + 1)(q^{n-1} - q)/(q^2 - 1) - 1$

**Proof.** Note that Part 2 follows immediately from Proposition 3 and the fact that by [GT] there are no nontrivial representations of the linear group of degree smaller than $(q^{n-1} - 1)/(q - 1) - 1$ if $r$ does not divide $(q^{n-1} - 1)/(q - 1)$.

For Part 1, let $M$ be the reduction modulo $r$ of the $(q^n + 1)(q^{n-1} - q)/(q^2 - 1)$-dimensional module in characteristic zero. If this is irreducible, then in the notation above, $\dim C_V(U_0) = (q^{n-1} - 1)/(q - 1) - 1$ and $C_M(P'_n) = 0$. Since $r/(q^{n-1} - 1)/(q - 1)$, there are no irreducible $SL_{n-1}(q)$ modules of dimension $(q^{n-1} - 1)/(q - 1) - 1$ by [GT] so we get a contradiction.

**Proposition 5.** If $G(q) = P\Omega^{2n+1}(q)$ with $n \geq 3$ and with $q > 3$ and odd, then the lower bound of Theorem 1 holds.

**Proof.** Again let $P = JU$ be the stabilizer of a maximal totally singular subspace of the natural module. It follows that $J = GL_n(q)$ and $U$ will have two composition factors that are elementary $p$ groups, $U'$ is the space of $n \times n$ antisymmetric matrices, and $U/U'$ is isomorphic to the natural $J$ module. As before $U = U_0 \times Z(G)$. As in the proof of Proposition 3
we can see that there is an irreducible factor $V_a$ in the decomposition of $V$ as a $U_0$ module on which $U_0$ acts nontrivially. This means that the $U_0$ action on $V_a$ factors through the action of an extraspecial group; hence \( \dim V_a \geq q \). The conclusion is that the dimension of $[U_0, V]$ is $l = q(q^n - 1)(q^{n-1} - 1)/(q^2 - 1)$ or it is at least twice as large. By [SZ], $\dim V \geq (q^{2n} - 1)/(q^2 - 1) - n$. Assuming $\dim [U_0, V] = l$, it will follow that the dimension of $M = \text{C}_F(U_0)$ is at least $(q^n - 1)/(q - 1) - n > q + 1$. Note that $C_{[U_0, V]}(J')$ has dimension $q$ and so, repeating the argument for the opposite parabolic subgroup $P$ and making the same assumptions on the dimension of $[\tilde{U}_0, V]$, we get that if $\dim M \leq q < \dim M$ and so $M$ has a nontrivial factor as a $J'$ module. By [GT] any such factor is at least of dimension $(q^n - 1)/(q - 1) - 2$. In particular this means that $\dim M \geq (q^n - 1)/(q - 1) - 2$ and so $\dim V \geq (q^{2n} - 1)/(q^2 - 1) - 2$.  

For the rest of the theorem note that Lemma 1 gives that any indecomposable factor of $M$ has dimension $1$, $e(n, q) - 2$, $e(n, q) - 1$, or $e(n, q)$, or $\geq 2e(n, q) - 4$; hence the result follows from

**Proposition 6.** With the notation of Proposition 5,

\[
\dim C_M(J') = \begin{cases} 
0 & \text{if } \text{char } k \neq 2, \\
\leq 1 & \text{if } \text{char } k = 2. 
\end{cases}
\]

**Proof.** Once again, in the notation of Proposition 5, if $V_a$ is a nonlinear irreducible factor of $V$ as a $U_0$ module then $\dim V_a = q$. The idea is to estimate the dimension of $C_{[U, V]}(J')$ because that will give a bound on the dimension of $C_M(J')$.

To do this we need to find the dimension of the fixed space of the normalizer of $V_a$ in $J'$. This group will have a factor isomorphic to $SL_2(q)$ acting nontrivially on $V_a$. In fact, using [S] we see that in fact the $SL_2(q)$ module $V_a$ has two nontrivial factors $\xi_1, \xi_2$ of dimensions $(q - 1)/2$ and $(q + 1)/2$ respectively. The smaller one will remain irreducible in any characteristic. Using the character table for $SL_2(q)$ (for example, [D]) one can see that

\[
\chi_{\xi_1}(-I_2) = -e(q - 1)/2,
\]

\[
\chi_{\xi_2}(-I_2) = e(q + 1)/2,
\]

where $e = (-1)^{(q-1)/2}$ and $\chi_{\xi}$ is the character associated to $\xi_i$.

Therefore, unless the characteristic of the base field is 2, $\chi_{\xi_1} + \chi_{\xi_2} \neq \chi_{\xi}$ and so the representations of dimension $(q + 1)/2$ cannot factor. Therefore the normalizer of $V_a$ does not fix any subspace if the characteristic of $k$ is odd and it could fix at most a one-dimensional space in characteristic 2.
Note that some choices for \((n, q)\) have been excluded from our treatment. It turns out that in these cases the corresponding bounds in [SZ] will be better. That is because our method uses bounds for \(l(L_n(q))\) and for those choices of \((n, q)\) the corresponding linear groups have exceptional multipliers; hence the bounds are much worse.

REFERENCES