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ORIGINAL ARTICLE

On some generalizations of certain retarded nonlinear integral inequalities with iterated integrals and an application in retarded differential equation

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Abstract In this paper, we investigate some new nonlinear retarded integral inequalities of Gronwall–Bellman–Pachpatte type. These inequalities generalize some former famous inequalities and can be used as handy tools to study the qualitative as well as the quantitative properties of solutions of some nonlinear retarded differential and integral equations. An application is also presented to illustrate the usefulness of some of our results in estimation of solution of certain retarded nonlinear differential equations with the initial conditions.

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1. Introduction

Integral inequalities involving functions of one independent variable, which provide explicit bounds on unknown functions play a fundamental role in the development of the theory of linear and nonlinear ordinary differential equations, integral equations, and differential-integral equations, see for instance [1–3]. One of the best known and widely used inequalities in the study of nonlinear differential equations is Gronwall–Bellman inequality [4,5], which has become one of the very few classical and most influential results in the theory and applications of inequalities. Because of its fundamental importance, over the years, many generalizations and analogous results of Gronwall–Bellman inequality have been established, such as [6–28]. Gronwall–Bellman inequality can be stated as follows:

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Theorem 1.1. *Let $f(t)$ and $u(t)$ be a real-valued nonnegative continuous functions defined on $D_1 = [0, h]$, and let u_0 and h be positive constants for which the inequality*

$$u(t) \leqslant u_0 + \int_0^t f(s)u(s)ds, \quad \forall t \in D_1.$$

Then

$$u(t) \leqslant u_0 \exp\left(\int_0^t f(s)ds\right), \quad \forall t \in D_1.$$

However, in certain situations the bounds provided by the above-mentioned inequalities are not directly applicable, and it is desirable to find some new estimates which will be equally important in order to achieve a diversity of desired goals. In the present paper we establish explicit bounds on retarded Gronwall–Bellman and Pachpatte-like inequalities and extend certain results that were proved be El-Owaidy et al. in [13], which can be used to study the qualitative behavior of the solutions of certain classes of retarded differential equations. In our results, there are not only composite functions of unknown functions in iterated integrals on the right hand side of our inequalities, but also the composite functions of unknown function exist in every layer of the iterated integrals, also we illustrate an application of our results, which verifies that our results are handy tools to study the qualitative properties of nonlinear differential equations and integral equations.

Theorem 1.2 (*Lipovan [10]*). *Let $u, f \in \mathcal{C}([t_0, T_0], \mathbb{R}_+)$. Further, let $\alpha \in \mathcal{C}([t_0, T_0], [t_0, T_0])$ be a nondecreasing with $\alpha(t) \leqslant t$ on $[t_0, T_0]$, and let k be a nonnegative constant.*

Then the inequality

$$u(t) \leqslant k + \int_{\alpha(t_0)}^{\alpha(t)} f(s)u(s)ds, \quad t_0 < t < T_0,$$

implies that

$$u(t) \leqslant k \exp\left(\int_{\alpha(t_0)}^{\alpha(t)} f(s)ds\right), \quad t_0 < t < T_0.$$

Theorem 1.3 (*Agarwal [11]*). *Let $\phi \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ be an increasing function, $u, a, f \in \mathcal{C}([t_0, T_0], \mathbb{R}_+)$, $a(t)$ be an increasing function, and $\alpha(t) \in \mathcal{C}([t_0, T_0], [t_0, T_0])$ be a nondecreasing with $\alpha(t) \leqslant t$ on $[t_0, T_0]$ where $T_0 \in (0, \infty)$ is a constant . Then the inequality*

$$u(t) \leqslant a(t) + \int_{\alpha(0)}^{\alpha(t)} f(s)\phi(u(s))ds, \quad t_0 < t < T_0,$$

implies that

$$u(t) \leqslant W^{-1}\left(W(a(t)) \int_{\alpha(0)}^{\alpha(t)} f(s)ds\right), \quad t_0 < t < T_0,$$

where

$$W(t) = \int_1^t \frac{dt}{\phi(t)} ds, \quad t > 0,$$

W^{-1} is the inverse function of W, and T^* is the large number such that

$$W(a(T^*)) \int_{\alpha(0)}^{\alpha(T^*)} f(s)ds \leqslant \int_1^\infty \frac{dt}{\phi(t)} ds.$$

2. Main results

In this section, several new generalized retarded integral inequalities of Gronwall–Bellman type are introduced. Throughout this article, $\mathbb{R}$ denoted the set of real numbers, $I = [0, \infty)$ is the subset of $\mathbb{R}, '$ denotes the derivative. $\mathcal{C}(I, I)$ denotes the set of all continuous functions from I into I and $\mathcal{C}^1(I, I)$ denotes the set of all continuously differentiable functions from I into I.

Theorem 2.1. *Let $u(t), g(t), f(t) \in \mathcal{C}(I, I)$ be nonnegative functions. We suppose that $\varphi, \varphi', \alpha \in \mathcal{C}^1(I, I)$ are increasing functions, with $\varphi'(t) \leqslant k, \varphi > 0, \alpha(t) \leqslant t, \alpha(0) = 0$, for all $t \in I; k, u_0$ be positive constants, If the inequality*

$$u(t) \leqslant u_0 + \int_0^{\alpha(t)} f(s)\varphi(u(s))\left[\varphi(u(s)) + \int_0^s g(\lambda)\varphi(u(\lambda))d\lambda\right]ds, \quad \forall t \in I, \tag{2.1}$$

holds, for all $t \in I$. Then

$$u(t) \leqslant \Phi^{-1}\left(\Phi(u_0) + \int_0^{\alpha(t)} f(s)\beta(\alpha^{-1}(s))ds\right), \quad \forall t \in I, \tag{2.2}$$

where

$$\Phi(r) = \int_1^r \frac{dt}{\varphi(t)}, \quad r > 0, \tag{2.3}$$

and

$$\beta(t) = \exp\left(\int_0^{\alpha(t)} g(s)ds\right)\left((\varphi^{-1}(u_0)) - k\int_0^{\alpha(t)} f(s)\exp\left(\int_0^s g(\lambda)d\lambda\right)ds\right)^{-1}, \tag{2.4}$$

for all $t \in I$, such that $(\varphi^{-1}(u_0)) - k\int_0^{\alpha(t)} f(s)\exp\left(\int_0^s g(\lambda)d\lambda\right)ds > 0, \forall t \in I$.

Proof. Let $z(t)$ denotes the function on the right-hand side of (2.1), which is a nonnegative and nondecreasing function on I with $z(0) = u_0$. Then (2.1) is equivalent to

$$u(t) \leqslant z(t), u(\alpha(t)) \leqslant z(\alpha(t)) \leqslant z(t), \qquad \forall t \in I. \tag{2.5}$$

Differentiating $z(t)$, with respect to t, we get

$$\frac{dz}{dt} = \alpha'(t)f(\alpha(t))\varphi(u(\alpha(t)))[\varphi(u(\alpha(t))) + \int_0^{\alpha(t)} g(s)\varphi(u(s))ds], \forall t \in I.$$

Using (2.5), we get

$$\frac{dz}{dt} \leqslant \alpha'(t)f(\alpha(t))\varphi(z(\alpha(t)))y(t), \forall t \in I, \tag{2.6}$$

where $y(t) = \varphi(z(t)) + \int_0^{\alpha(t)} g(s)\varphi(z(s))ds, y(0) = \varphi(z(0)) = \varphi(u_0)$, $y(t)$ is a nonnegative and nondecreasing function on I. By the monotonicity φ, φ', z and $\alpha(t) \leqslant t$ we have $\varphi(z(t)) \leqslant y(t)$, $\varphi'(z(t)) \leqslant k$. Differentiating $y(t)$ with respect to t, and using (2.6), we have

$$\begin{aligned}\frac{dy}{dt} &\leqslant \varphi'(z(t))\alpha'(t)f(\alpha(t))y^2(t) + \alpha'(t)g(\alpha(t))\varphi(z(t)) \\ &\leqslant k\alpha'(t)f(\alpha(t))y^2(t) + \alpha'(t)g(\alpha(t))y(t), \quad \forall t \in I.\end{aligned} \tag{2.7}$$

But $y(t) > 0$, from (2.7) we get

$$y^{-2}(t)\frac{dy}{dt}-\alpha'(t)g(\alpha(t))y^{-1}(t)\leqslant k\alpha'(t)f(\alpha(t)),\quad \forall t\in I. \tag{2.8}$$

If we let

$$v(t)=y^{-1}(t),\quad \forall t\in I, \tag{2.9}$$

then we get $v(0)=\varphi^{-1}(u_0)$ and $y^{-2}(t)\frac{dy}{dt}=-\frac{dv}{dt}$, thus from (2.8) and (2.9), we have

$$\frac{dv}{dt}+\alpha'(t)g(\alpha(t))\geqslant -k\alpha'(t)f(\alpha(t)),\forall t\in I.$$

The above inequality implies an estimation for $v(t)$ as in the following

$$v(t)\geqslant \exp\left(-\int_0^{\alpha(t)}g(s)ds\right)\left((\varphi^{-1}(u_0))-\int_0^{\alpha(t)}f(s)\exp\left(\int_0^s g(\lambda)d\lambda\right)ds\right), \tag{2.10}$$

for all $t\in I$, from (2.4), (2.9) and (2.10), we get $y(t)\leqslant\beta(t)$, where $\beta(t)$ as defined in (2.4). Thus from (2.6) we have

$$\frac{dz}{dt}\leqslant\alpha'(t)f(\alpha(t))\varphi(z(t))\beta(t),\quad\forall t\in I. \tag{2.11}$$

By taking $t=s$ in (2.11) and integrating it from 0 to t, using (2.3), we obtain

$$\begin{aligned}z(t)&\leqslant\Phi^{-1}\left(\Phi(u_0)+\int_0^t\alpha'(s)f(\alpha(s))\beta(s)\right)ds,\\&\leqslant\Phi^{-1}\left(\Phi(u_0)+\int_0^{\alpha(t)}f(s)\beta(\alpha^{-1}(s))\right)ds,\quad\forall t\in I.\end{aligned} \tag{2.12}$$

Using (2.12) in (2.5), we get the required inequality in (2.2). This completes the proof. □

Theorem 2.2. *Let $u(t),g(t),h(t)\in\mathcal{C}(I,I)$ be nonnegative functions, and $f(t)$ is a positive, monotonic, nondecreasing function. We suppose that $\varphi,\varphi',\alpha\in\mathcal{C}^1(I,I)$ are increasing functions and $\frac{\varphi(u(t))}{f(t)}\leqslant\varphi(\frac{u(t)}{f(t)})$, with $\varphi'(t)\leqslant k,\alpha(t)\leqslant t,\alpha(0)=0$, for all $t\in I;k,u_0$ be positive constants. If the inequality*

$$u(t)\leqslant f(t)+\int_0^{\alpha(t)}g(s)\varphi(u(s))ds+\int_0^{\alpha(t)}g(s)\varphi(u(s))[\varphi(u(s))+\int_0^s h(\lambda)\varphi(u(\lambda))d\lambda]ds, \tag{2.13}$$

holds for all $t\in I$. Then

$$u(t)\leqslant f(t)\Phi^{-1}\left(\Phi(1)+\int_0^{\alpha(t)}g(s)[1+f(s)\Theta(\alpha^{-1}(s))]ds\right), \tag{2.14}$$

for all $t\in I$, where Φ as defined in (2.3) and

$$\Theta(t)=\frac{\exp(\int_0^{\alpha(t)}[kg(s)+h(s)]ds)}{\varphi^{-1}(1)-\int_0^{\alpha(t)}kg(s)f(s)\exp(\int_0^s[kg(\tau)+h(\tau)]d\tau)ds},\forall t\in I, \tag{2.15}$$

such that

$$\int_0^{\alpha(t)}g(s)f(s)\exp\left(\int_0^s[g(\tau)+h(\tau)]d\tau\right)ds<\varphi^{-1}(1),\forall t\in I.$$

Proof. Since $f(t)$ is a positive, monotonic, nondecreasing function, we observe from (2.13) that

$$\frac{u(t)}{f(t)}\leqslant 1+\int_0^{\alpha(t)}g(s)\frac{\varphi(u(s))}{f(s)}ds+\int_0^{\alpha(t)}g(s)f(s)\frac{\varphi(u(s))}{f(s)}\times\left[\frac{\varphi(u(s))}{f(s)}+\int_0^s h(\lambda)\frac{\varphi(u(\lambda))}{f(\lambda)}d\lambda\right]ds,$$

for all $t\in I$. By the relation $\frac{\varphi(u(t))}{f(t)}\leqslant\varphi(\frac{u(t)}{f(t)})$, then the above inequality can be written as

$$\frac{u(t)}{f(t)}\leqslant 1+\int_0^{\alpha(t)}g(s)\varphi\left(\frac{u(t)}{f(t)}\right)ds+\int_0^{\alpha(t)}g(s)f(s)\varphi\left(\frac{u(t)}{f(t)}\right)\times\left[\varphi\left(\frac{u(t)}{f(t)}\right)+\int_0^s h(\lambda)\varphi\left(\frac{u(\lambda)}{f(\lambda)}\right)d\lambda\right]ds,$$

for all $t\in I$. Let

$$r(t)=\frac{u(t)}{f(t)},\forall\ t\in I,\quad r(0)\leqslant 1, \tag{2.16}$$

hence

$$r(t)\leqslant 1+\int_0^{\alpha(t)}g(s)\varphi(r(s))ds+\int_0^{\alpha(t)}g(s)f(s)\varphi(r(s))[\varphi(r(s))+\int_0^s h(\lambda)\varphi(r(\lambda))d\lambda]ds, \tag{2.17}$$

for all $t\in I$.

Let $V(t)$ denotes the function on the right-hand side of (2.17), which is a nonnegative and nondecreasing function on I with $V(0)=1$. Then (2.17) is equivalent to

$$r(t)\leqslant V(t),r(\alpha(t))\leqslant V(\alpha(t))\leqslant V(t),\quad\forall t\in I. \tag{2.18}$$

Differentiating $V(t)$, with respect to t and using (2.18), we get

$$V'(t)\leqslant g(\alpha(t))\alpha'(t)\varphi(V(t))[1+f(\alpha(t))\gamma(t)],\quad\forall\ t\in I, \tag{2.19}$$

where $\gamma(t)=\varphi(V(t))+\int_0^{\alpha(t)}h(s)\varphi(V(s))ds$, hence $\gamma(0)=\varphi(1)$, and $\varphi(V(t))\leqslant\gamma(t),\gamma(t)$ is a nonnegative and nondecreasing function on I. By the monotonicity of φ,φ',V and $\alpha(t)\leqslant t$ we have $\varphi(V(t))\leqslant\gamma(t),\varphi'(V(t))\leqslant k$. Differentiating $\gamma(t)$ with respect to t, and using (2.19), we get

$$\gamma'(t)\leqslant[kg(\alpha(t))+h(\alpha(t))]\alpha'(t)\gamma(t)+kg(\alpha(t))\alpha'(t)f(\alpha(t))\gamma^2(t),\quad\forall\ t\in I,$$

but $\gamma(t)>0$, thus from the above inequality, we get

$$\gamma^{-2}(t)\gamma'(t)-[kg(\alpha(t))+h(\alpha(t))]\alpha'(t)\gamma^{-1}(t)\leqslant kg(\alpha(t))\alpha'(t)f(\alpha(t)),\quad\forall\ t\in I. \tag{2.20}$$

If we let

$$l(t)=\gamma^{-1}(t),\quad\forall\ t\in I, \tag{2.21}$$

then we get $l(0)=\varphi^{-1}(1)$ and $\gamma^{-2}\gamma'(t)=-l'(t)$, thus from (2.20), we have

$$l'(t)+[kg(\alpha(t))+h(\alpha(t))]\alpha'(t)l(t)\geqslant -kg(\alpha(t))\alpha'(t)f(\alpha(t)),\quad\forall\ t\in I.$$

The above inequality implies the estimation for $l(t)$ such that

$$l(t)\geqslant\frac{\varphi^{-1}(1)-k\int_0^{\alpha(t)}g(s)f(s)\exp(\int_0^s[kg(\tau)+h(\tau)]d\tau)ds}{\exp(\int_0^{\alpha(t)}[kg(s)+h(s)]ds)},\quad\forall t\in I.$$

Then from the above inequality in (2.21), we have

$$\gamma(t)\leqslant\Theta(t),\forall\ t\in I,$$

where $\Theta(t)$ as defined in (2.15), thus from (2.19) and the above inequality, we obtain

$$V'(t) \leqslant g(\alpha(t))\alpha'(t)\varphi(V(t))[1+f(\alpha(t))\Theta(t)], \quad \forall\ t \in I. \tag{2.22}$$

Since $\varphi(V(t)) > 0$, for all $t > 0$, then from (2.22) we have

$$\frac{V'(t)}{\varphi(V(t))} \leqslant g(\alpha(t))\alpha'(t)[1+f(\alpha(t))\Theta(t)],$$

for all $t \in I$. By taking $t = s$ in the above inequality and integrating it from 0 to t, and using the definition of Φ in (2.3), we get

$$\Phi(V(t)) \leqslant \Phi(1) + \int_0^{\alpha(t)} g(s)[1+f(s)\Theta(\alpha^{-1}(s))], \tag{2.23}$$

for all $t \in I$, where Φ is defined by (2.3), from (2.23), we have

$$V(t) \leqslant \Phi^{-1}\left(\Phi(1) + \int_0^{\alpha(t)} g(s)[1+f(s)\Theta(\alpha^{-1}(s))]ds\right), \tag{2.24}$$

for all $t \in I$, from (2.16), (2.18) and (2.24) we get the required inequality in (2.14). This completes the proof. $\square$

Remark 2.1. Theorem 2.2 gives the explicit estimation in Theorem 2.3 in [13], when $\varphi(u(t)) = u(t)$.

Theorem 2.3. *Let $u(t), g(t), f(t) \in \mathcal{C}(I, I)$ be nonnegative functions. We suppose that $\varphi_1, \varphi_2, \alpha \in \mathcal{C}^1(I, I)$ are increasing functions with $\alpha(t) \leqslant t, \varphi_i(t) > 0, i = 1, 2, \alpha(0) = 0$ and $\varphi_1'(t) = \varphi_2(t)$, for all $t \in I; u_0$ be positive constant. If the inequality*

$$\varphi_1(u(t)) \leqslant u_0 + \int_0^{\alpha(t)} f(s)\varphi_2(u(s))\left[u(s) + \int_0^s g(\lambda)\varphi_1(u(\lambda))d\lambda\right]^p ds, \quad \forall t \in I, \tag{2.25}$$

holds, for all $t \in I$. Then

$$u(t) \leqslant \varphi_1^{-1}(u_0) + \int_0^{\alpha(t)} f(s)\beta_1(\alpha^{-1})(s)ds, \forall t < T_1, \tag{2.26}$$

where

$$\beta_1(t) = \Omega^{-1}\left(\Omega\left(\left[\varphi_1^{p-1}(u_0) + (1-p)\int_0^{\alpha(t)} f(s)ds\right]^{\frac{1}{1-p}}\right) + \int_0^{\alpha(t)} g(s)ds\right), \tag{2.27}$$

$$\Omega(t) = \int_1^t \frac{ds}{\varphi_1(s)}, \forall t > 0, \tag{2.28}$$

$\Omega^{-1}, \varphi_1^{-1}$ are the inverse functions of Ω, φ_1 respectively and T_1 is the largest number such that

$$\Omega\left(\left[\varphi_1^{p-1}(u_0) + (1-p)\int_0^{\alpha(t)} f(s)ds\right]^{\frac{1}{1-p}}\right) + \int_0^{\alpha(t)} g(s)ds \leqslant \int_1^{\infty} \frac{ds}{\varphi_1(s)}, \tag{2.29}$$

for all $t < T_1$.

Proof. Let $\varphi_1(J(t))$ denotes the function on the right-hand side of (2.25), which is a nonnegative and nondecreasing function on I with $J(0) = \varphi_1^{-1}(u_0)$. Then (2.25) is equivalent to

$$u(t) \leqslant J(t), u(\alpha(t)) \leqslant J(\alpha(t)) \leqslant J(t), \quad \forall t \in I. \tag{2.30}$$

Differentiating $\varphi_1(J(t))$, with respect to t, and using (2.30), we get

$$\varphi_1'(J(t))\frac{dJ(t)}{dt} = \alpha'(t)f(\alpha(t))\varphi_2(u(\alpha(t)))[u(\alpha(t))) + \int_0^{\alpha(t)} g(\lambda)\varphi_1(u(\lambda))d\lambda]^p \leqslant \alpha'(t)f(\alpha(t))\varphi_2(J(t)) \times [J(t) + \int_0^{\alpha(t)} g(\lambda)\varphi_1(J(\lambda))d\lambda]^p, \quad \forall t \in I.$$

Using the relation $\varphi_1'(J(t)) = \varphi_2(J(t))$, then from the above inequality, we obtain

$$\frac{dJ(t)}{dt} \leqslant \alpha'(t)f(\alpha(t))w^p(t), \tag{2.31}$$

where $w(t) = J(t) + \int_0^{\alpha(t)} g(s)\varphi_1(J(s))ds$, $w(0) = J(0) = \varphi_1^{-1}(u_0)$ and $J(t) \leqslant w(t)$, w is a nonnegative and nondecreasing function on I. Differentiating $w(t)$ with respect to t, and using (2.31), we have

$$\frac{dw}{dt} \leqslant \alpha'(t)f(\alpha(t))w^p(t) + \alpha'(t)g(\alpha(t))\varphi_1(J(\alpha(t))) \leqslant \alpha'(t)f(\alpha(t))w^p(t) + \alpha'(t)g(\alpha(t))\varphi_1(w(\alpha(t))), \quad \forall t \in I. \tag{2.32}$$

By $w(t) > 0$, from (2.32), we get

$$\frac{dw}{w^p(t)} \leqslant \alpha'(t)f(\alpha(t))dt + \alpha'(t)g(\alpha(t))\frac{\varphi_1(w(\alpha(t)))}{w^p(\alpha(t))}dt, \quad \forall t \in I. \tag{2.33}$$

Integrating (2.33) from 0 to t, we have

$$w^{1-p}(t) \leqslant \varphi_1^{p-1}(u_0) + (1-p)\int_0^{\alpha(t)} f(s)ds + (1-p) \times \int_0^{\alpha(t)} g(s)\frac{\varphi_1(w(s))}{w^p(s)}ds, \tag{2.34}$$

for all $t \in I$, from (2.34), we have

$$w^{1-p}(t) \leqslant \varphi_1^{p-1}(u_0) + (1-p)\int_0^{\alpha(T)} f(s)ds + (1-p) \times \int_0^{\alpha(t)} g(s)\frac{\varphi_1(w(s)}{w^p(s)}ds, \tag{2.35}$$

for all $t \leqslant T$, where $0 \leqslant T < T_1$ is chosen arbitrarily, T_1 is defined by (2.29). Let $m^{1-p}(t)$ denote the function on the right-hand side of (2.35), which is a positive and nondecreasing function on I with $m(0) = \left[\varphi_1^{p-1}(u_0) + (1-p)\int_0^{\alpha(T)} f(s)ds\right]^{\frac{1}{1-p}}$ and

$$w(t) \leqslant m(t), \quad \forall t < T. \tag{2.36}$$

Differentiating $m^{1-p}(t)$ with respect to t and using (2.36), we get

$$\frac{dm}{\varphi_1(m(t))} \leqslant \alpha'(t)g(\alpha(t)), \quad \forall t < T, \tag{2.37}$$

by the definition of Ω in (2.28), then from (2.37), we obtain

$$\Omega(m(t)) \leqslant \Omega(m(0)) + \int_0^{\alpha(t)} g(s)ds$$
$$\leqslant \Omega\left(\left[\varphi_1^{p-1}(u_0) + (1-p)\int_0^{\alpha(T)} f(s)ds\right]^{\frac{1}{1-p}}\right) + \int_0^{\alpha(t)} g(s)ds,$$

for all $t < T$. Let $t = T$, then from the above inequality, we get

$$\Omega(m(t)) \leqslant \Omega\left(\left[\varphi_1^{p-1}(u_0) + (1-p)\int_0^{\alpha(T)} f(s)ds\right]^{\frac{1}{1-p}}\right) + \int_0^{\alpha(T)} g(s)ds. \tag{2.38}$$

Since $0 < T < T_1$ is chosen arbitrary, then from (2.38) in (2.36), we obtain

$$w(t) \leqslant \beta_1(t), \quad \forall t < T_1, \tag{2.39}$$

where $\beta_1(t)$ as defined in (2.27), thus from (2.31) and (2.39), we obtain

$$\frac{dJ(t)}{dt} \leqslant \alpha'(t)f(\alpha(t))\beta_1(t), \quad \forall t < T_1. \tag{2.40}$$

By taking $t = s$ in (2.40) and integrating it from 0 to t we have

$$J(t) \leqslant \varphi_1^{-1}(u_0) + \int_0^{\alpha(t)} f(s)\beta_1(\alpha^{-1})(s)ds, \quad \forall t < T_1. \tag{2.41}$$

Using (2.41) in (2.30), we get the required inequality in (2.26). This completes the proof. □

3. An application

In this section, we apply our result obtained in Theorem 2.1 to the following nonlinear retarded differential equation with the initial condition.

$$\begin{cases} \frac{du(t)}{dt} = M(t, u(\alpha(t)), H(t, u(\alpha(t)))), \forall t \in I, \\ \\ u(0) = u_0, \end{cases} \tag{3.1}$$

where u_0 is a constant, $M \in \mathcal{C}(I^3, \mathbb{R})$ $H \in \mathcal{C}(I \times I, \mathbb{R})$, satisfy the following conditions:

$$|M(t,u,H)| \leqslant f(\alpha(t))\varphi(u(\alpha(t)))$$
$$\left[|u(\alpha(t))| + \int_0^t |K(s, u(\alpha(s)))|ds\right], \tag{3.2}$$
$$|K(t, u(\alpha(t)))| \leqslant g(\alpha(t))\varphi(u(\alpha(t))), \tag{3.3}$$

where $f(t), g(t)$ as defined in Theorem 2.1.

Corollary 3.1. *Consider nonlinear system (3.1) and suppose that M, H satisfy the conditions (3.2) and (3.3). We suppose that $\varphi, \varphi', \alpha \in \mathcal{C}^1(I, I)$ are increasing functions with $\varphi_1'(t) \leqslant k, \alpha(t) \leqslant t, \alpha(0) = 0$, for all $t \in I; k, u_0$ are positive constants, then each solution $u(t)$ of (3.1) under discussion verifies the following estimation:*

$$u(t) \leqslant \Phi^{-1}\left(\Phi(u_0) + \int_0^{\alpha(t)} \frac{f(s)}{\alpha'(\alpha^{-1}(s))}\beta_2(\alpha^{-1}(s))ds\right), \quad \forall t \in I, \tag{3.4}$$

where Φ as defined in (2.3), and

$$\beta_2(t) = \exp\left(\int_0^{\alpha(t)} \frac{g(s)}{\alpha'(\alpha^{-1}(s))}ds\right) \times \left((\varphi^{-1}(u_0)) - k\int_0^{\alpha(t)} \frac{f(s)}{\alpha'(\alpha^{-1}(s))}\right.$$
$$\left. \times \exp\left(\int_0^s \frac{g(\lambda)}{\alpha'(\alpha^{-1}(\lambda))}d\lambda\right)ds\right)^{-1}, \quad \forall t \in I, \tag{3.5}$$

Proof. Integrating both sides of (3.1) from 0 to t, we have

$$u(t) = u_0 + \int_0^t M(s, u(\alpha(s)), H(s, u(\alpha(s))))ds, \quad \forall t \in I, \tag{3.6}$$

using the conditions (3.2) and (3.3), then from (3.6) we get

$$|u(t)| \leqslant u_0 + \int_0^t f(s)|\varphi(u(\alpha(s)))|\left[|\varphi(u(\alpha(s)))| + \int_0^s g(\alpha(\lambda))|\varphi(u(\alpha(\lambda)))|d\lambda\right]ds$$
$$\leqslant u_0 + \int_0^{\alpha(t)} \frac{f(s)|\varphi(u(s))|}{\alpha'(\alpha^{-1}(s))}\left[|\varphi(u(s))| + \int_0^s \frac{g(\lambda)|\varphi(u(\lambda))}{\alpha'(\alpha^{-1}(s))}|d\lambda\right]ds,$$

for all $t \in I$, applying Theorem 2.1 to the above inequality, we obtain the required estimation (3.4). This completes the proof. □

Remark 3.1. Gronwall-like inequality can be applied to the analysis of the behavior of the solutions of some retarded nonlinear differential equations. Our results also can be used to prove the global existence, uniqueness, stability, and other properties of the solutions of various nonlinear retarded differential and integral equations. The importance of these inequalities stem from the fact that it is applicable in certain situations in which other available inequalities do not apply directly.

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