Non-commutative shifts and crossed products

A. Kishimoto

Department of Mathematics, Hokkaido University, Sapporo 060-810, Japan

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Abstract

When $A$ is a unital simple AF $C^*$-algebra and has a unique tracial state, it is shown that the crossed product of the two-sided infinite tensor product $\otimes \mathbb{Z} A$ by the shift is a tracially AF $C^*$-algebra. A similar result is given to the crossed product of a certain non-unital two-sided infinite tensor product by the shift. Applying a far-reaching classification result of such $C^*$-algebras by H. Lin, we obtain an example of a one-parameter automorphism group on some AF $C^*$-algebra which is not approximately inner, a counter-example to the AF version of the so-called Powers–Sakai conjecture [23].

1. Introduction

By a non-commutative shift we mean the shift automorphism $\sigma$ of the two-sided infinite (minimal) tensor product $\otimes \mathbb{Z} A$ of copies of some unital (non-commutative) $C^*$-algebra $A$. Except for some special cases, the shift $\sigma$ induces a non-trivial action on K theory; e.g., if the $C^*$-algebra $A$ is AF, $\sigma_*$ fixes no non-zero elements of $K_0(\otimes \mathbb{Z} A)$ which are rationally independent of the class $[1]$ of the unit. Our general aim is to study the shift $\sigma$; more specifically in this note, the crossed product $(\otimes \mathbb{Z} A) \rtimes_\sigma \mathbb{Z}$, also as a typical example of crossed products by automorphisms inducing non-trivial actions on K theory.

Before going into details we list some results known for non-commutative shifts.

(1) If $A$ is completely non-commutative in the sense that $A$ contains a finite-dimensional $C^*$-subalgebra $D$ such that $1 \in D$ and $D$ has no one-dimensional direct
summands, then σ has the Rohlin property [17]; a fact which will be often used in this note. (2) If A is a UHF $C^\ast$-algebra or a full matrix algebra, then the crossed product $(\otimes Z_A) \times_\sigma Z$ is a unital simple AT algebra of real rank zero, since the shift is approximately inner (and so induces the identity map on $K_0(\otimes Z_A)$) [7,15,16]. (3) If A is a unital prime AF $C^\ast$-algebra (or a full matrix algebra), then the pure $\sigma$-invariant states are dense in the $\sigma$-invariant states of $\otimes Z_A$ [12,6]. (Only this property depends on $\sigma$ itself; all the other properties above and below depend just on its outer conjugacy class.) We will also state the following facts. (4) If A is a unital AF $C^\ast$-algebra, then $K_i(\otimes Z_A \times_\sigma Z)$ is a torsion-free abelian group for $i = 0, 1$. (5) If A is a finite-dimensional $C^\ast$-algebra with non-trivial center, then $\otimes Z_A \times_\sigma Z$ is not an AT $C^\ast$-algebra. But we should note that this is certainly AF embeddable (i.e., isomorphic to a $C^\ast$-subalgebra of an AF $C^\ast$-algebra); in general if A is AF embeddable, then $\otimes Z_A \times_\sigma Z$ is also AF embeddable (cf. [8]).

What we can add in this note is (6) if A is a unital simple AF $C^\ast$-algebra with a unique tracial state, then the crossed product $(\otimes Z_A) \times_\sigma Z$ is a unital simple trivially AF $C^\ast$-algebra. This is an attempt to extend the result (2) above but we are still short of showing that $(\otimes Z_A) \times_\sigma Z$ is an AT $C^\ast$-algebra.

The reason why we can prove the above assertion (6) is that if the unital simple AF $C^\ast$-algebra has a unique tracial state, then it is *tracially* UHF (see Lemma 2.4); anyway such a $C^\ast$-algebra is not very far from being UHF; but the fact of not being UHF also reflects on the conclusion in comparison with (2) above.

In the second part of this note we consider a non-unital version of the above result. This means that we specify two projections $e_-, e_+ \in A$ and we use $e_-$ (resp. $e_+$) as a unit as we move to the left (resp. right); i.e., setting an embedding $\otimes^n A \to \otimes^{n+1} A$ as $x \mapsto e_- \otimes x \otimes e_+$ for $n = 1, 2, \ldots$, we define the inductive limit $C^\ast$-algebra $\otimes \otimes Z(A, e_-, e_+)$, on which the shift σ is well-defined. In this setting we show the following: (7) If A is an AF $C^\ast$-algebra and $[e_-] \neq [e_+]$ in $K_0(A)$, then $K_1(\otimes Z(A, e_-, e_+) \times_\sigma Z) = \{0\}$. (8) If A is an AF $C^\ast$-algebra and $[e_-] \neq [e_+]$ and if $p|\tau([e_-] - [e_+])$ then both $p|\tau([e_-])$ and $p|\tau([e_+])$ for all prime numbers p, then $K_0(\otimes Z(A, e_-, e_+) \times_\sigma Z)$ is a torsion-free abelian group. (9) If A is a unital simple AF $C^\ast$-algebra with a unique tracial state $\tau$ and $\tau(e_-) = \tau(e_+)$, then $\otimes Z(A, e_-, e_+) \times_\sigma Z$ is a tracially AF $C^\ast$-algebra. We should hastily add that if $\tau(e_-) \neq \tau(e_+)$, then $\otimes Z(A, e_-, e_+) \times_\sigma Z$ is purely infinite [13,14,22,24]; so only the finite case $\tau(e_-) = \tau(e_+)$ has remained for clarification.

What I was hoping for was to prove that $\otimes Z(A, e_-, e_+) \times_\sigma Z$ is an AF $C^\ast$-algebra in the situation of (9) with $e_-, e_+$ satisfying the condition in (8). If successful, this would give an example of an action of T on some unital simple AF $C^\ast$-algebra whose fixed point algebra is again simple, a counter-example to the (AF version of) Powers–Sakai conjecture [23,25] (which says that any strongly continuous one-parameter automorphism group of an AF $C^\ast$-algebra is approximately inner; in particular that in the periodic case the fixed point algebra is not simple). This way of constructing possible counter-examples to this conjecture had perhaps been known (cf. [3–5]) but it became plausible only after Elliott’s paper [11] appeared—I owe this point to A. Kumjian.
In a recent preprint H. Lin [20] has shown a remarkable result: If $B$ is a unital separable nuclear simple tracially AF $C^*$-algebra and satisfies the Universal Coefficient Theorem, then $B$ is determined by its K theoretic data $(K_0(A), K_0(A)\oplus \mathbb{Z})$. (Earlier than this, he defined a notion of tracial topological rank; tracially AF $C^*$-algebras are those of tracial topological rank zero.) Note that unital simple AF $C^*$-algebras are in this class. Hence in the situation of (9) with (8), it follows that $B = (\otimes \mathbb{Z}(A, e_-)_+ \times \sigma \mathbb{Z})$ is an AF $C^*$-algebra. If $e$ is a non-zero projection in $\otimes \mathbb{Z}(A, e_+)_+$, then $D = (e \otimes 1)B(e \otimes 1)$ is a unital AF $C^*$-algebra which is left invariant under the dual action of $\sigma$. If we denote by $z$ the restriction of this action to $D$, we have the following situation: $D$ is a unital simple AF $C^*$-algebra, $z$ is an action of $T$ on $D$, and the fixed point algebra $D^z$ is simple (being isomorphic to $e(\otimes \mathbb{Z}(A, e_+)_+)e$). Thus relying on Lin’s result [20], we obtain a counter-example to the (AF version of) Powers–Sakai conjecture, i.e. $z$, as a one-parameter automorphism group of $D$, is not approximately inner. (Since $D$ cannot be a UHF $C^*$-algebra, this does not give a counter-example to the original Powers–Sakai conjecture.) Note also that if $E$ is a unital $C^*$-algebra and $\beta$ is a strongly continuous one-parameter automorphism group of $E$, then $\alpha \otimes \beta$ is never approximately inner on $B \otimes E$. (Because if it were, $\alpha \otimes \beta$ should have a ground state, which would give a ground state for $\alpha$.) In this way we could get more examples (see [25]). Formally we state:

**Theorem 1.1.** There exists a unital simple AF $C^*$-algebra $A$ and a strongly continuous one-parameter automorphism group $\alpha$ of $A$ such that $\alpha$ is not approximately inner.

Before concluding Introduction we will explain some definitions used above and to be used below. A (separable) $C^*$-algebra $A$ is called AF (approximately finite-dimensional) if there is an increasing sequence $(A_n)$ of $C^*$-subalgebras of $A$ with $A = \overline{\cup \cap A_n}$ such that $\dim A_n < \infty$ for all $n$; and is called AT [11] if we replace the condition $\dim A_n < \infty$ by $A_n \cong B_n \otimes C(T)$ with $\dim B_n < \infty$ in the above definition. We say that a unital (separable) $C^*$-algebra $A$ has real rank zero [9] if the invertible elements are dense in the self-adjoint part $A_{sa} = \{x \in A \mid x^* = x\}$ and that $A$ is approximately divisible [2] if $A$ has a central sequence $(D_n)$ of $C^*$-subalgebras such that $1 \in D_n$, $\dim D_n < \infty$, and $D_n$ has no one-dimensional direct summands for all $n$. If $A$ is a unital approximately divisible exact $C^*$-algebra with a unique tracial state, then $A$ has real rank zero [2]. If $A$ is not unital, $A$ has real rank zero if $A$ has an approximate unit $(p_n)$ consisting of projections such that $p_nAp_n$ has real rank zero. (This is equivalent to saying that $A + C1$ has real rank zero.)

The notion of tracially AF was introduced by H. Lin. A unital simple $C^*$-algebra $A$ is called tracially AF (approximately finite-dimensional) (or, more recently, of tracial topological rank zero) if for any finite subset $\mathcal{F}$ of $A$, any $\varepsilon > 0$, and any non-zero $q \in A_+$, there is a non-zero projection $p \in A$ and a finite-dimensional $C^*$-subalgebra $D$ of $pAp$ with $p \in D$ such that

1. $\forall a \in \mathcal{F} \quad ||p, a|| < \varepsilon,$
2. $\forall a \in \mathcal{F} \quad \text{dist}(pap, D) < \varepsilon,$
3. $\exists u \in \mathcal{U}(A) \quad u(1 - p)u^* \in qAQ,$
where $\mathcal{U}(A)$ is the unitary group of $A$ (see 1.2 of [18] and also [10,19,20]). (If furthermore $D$ can be chosen to be a full matrix algebra, then such a $C^*$-algebra may be called tracially UHF.) Since in our case the unital simple $C^*$-algebra $A$ is exact, approximately divisible, and has a unique tracial state, $A$ is tracially AF if for any finite subset $\mathcal{F}$ of $A$ and any $\varepsilon > 0$ there is a projection $p \in A$ and a finite-dimensional $C^*$-subalgebra $D$ of $pA p$ with $p \in D$ such that

1. $\forall a \in \mathcal{F} \exists \| [p, a] \| < \varepsilon$,
2. $\forall a \in \mathcal{F} \operatorname{dist}(p a p, D) < \varepsilon$,
3. $\tau(p) > 1 - \varepsilon$,

where $\tau$ is the unique tracial state of $A$ (see, e.g., [21]).

Lin [18] shows that if $A$ is a unital simple tracially AF $C^*$-algebra, then $A$ has real rank zero and stable rank one (though we will not need this fact).

If $A$ is a non-unital simple $C^*$-algebra, $A$ is tracially AF if $A$ has an approximate unit $\{p_i\}$ consisting of projections such that $p_i A p_i$ is tracially AF for each $i$.

### 2. Unital tensor products

Let $A$ be a unital AF $C^*$-algebra. We denote by $\bigotimes_{\mathbb{Z}} A$ the infinite tensor product $C^*$-algebra $\bigotimes_{i \in \mathbb{Z}} A(i)$ with $A(i) \equiv A$ and by $\sigma$ the automorphism of $\bigotimes_{i \in \mathbb{Z}} A(i)$ sending $x \in A(i)$ to $x \in A(i + 1)$; $\sigma$ will be called the shift (automorphism) of $\bigotimes_{\mathbb{Z}} A$.

Note that $K_0(\bigotimes_{\mathbb{Z}} A) = \bigcup_n \bigotimes_{-n}^n K_0(A)$, where $\bigotimes_{-n}^n K_0(A)$ is embedded into $\bigotimes_{n+1}^{n+1} K_0(A)$ by $g \mapsto [1] \otimes g \otimes [1]$.

**Proposition 2.1.** If $A$ is a unital AF $C^*$-algebra, then $K_1(\bigotimes_{\mathbb{Z}} A \times_{\sigma} \mathbb{Z})$ is a torsion-free abelian group for $i = 0, 1$. Moreover $K_1(\bigotimes_{\mathbb{Z}} A \times_{\sigma} \mathbb{Z})$ is isomorphic to the subfield of $\mathbb{Q}$ generated by $\{n \in \mathbb{N}; [1] = ng$ for some $g \in K_0(A)\}$.

**Proof.** Let $F$ denote the subfield of $\mathbb{Q}$ generated by $\{n \in \mathbb{N}; [1] = ng$ for some $g \in K_0(A)\}$. Then it follows that $K_0(\bigotimes_{\mathbb{Z}} A)$ is a module over $F$ and that $\sigma$ is a module homomorphism. Since by the Pimsner–Voiculescu exact sequence $K_1(\bigotimes_{\mathbb{Z}} A \times_{\sigma} \mathbb{Z})$ is the kernel of $\operatorname{id} - \sigma_*$ on $K_0(\bigotimes_{\mathbb{Z}} A)$, it suffices to show that if $g = \sigma_*(g)$, then $g \in F[1]$. Since $g = \sigma_k^*(g)$ for any $k \in \mathbb{N}$, this follows from the fact that if $G$ is a torsion-free abelian group and $h \otimes g = g \otimes h$ in $G \otimes G$, then $h$ and $g$ are rationally dependent.

By the Pimsner–Voiculescu exact sequence again $K_0(\bigotimes_{\mathbb{Z}} A \times_{\sigma} \mathbb{Z})$ is isomorphic to $K_0(\bigotimes_{\mathbb{Z}} A) / \operatorname{Range}(\operatorname{id} - \sigma_*)$, which we have to show is torsion-free. Suppose that $g - \sigma_*(g) = nh$ for some non-zero $g, h \in K_0(\bigotimes_{\mathbb{Z}} A)$ and $n = 2, 3, \ldots$. We have to show that $g$ is of the form $ng' + a[1]$ with $a \in F$. We may suppose that no prime factors of $n$ appear in $F$. Since $g - \sigma_k^*(g) = nh + \sigma_k^*(h)$ for any $k \in \mathbb{N}$, the problem can be stated as follows: If $G = K_0(A)^{\otimes m}$ and $g \otimes [1] - [1] \otimes g = nh$ in $G \otimes G$ for $g \in G$ and $h \in G \otimes G$ and no prime factors of $n$ divide $[1]$, then show that $g$ is of the form $ng' + a[1]$. To show this we may suppose that $G = \mathbb{Z}^k$ for some
Let $k > 1$ and that $g = (g_i)$ and $[1] = (\xi_i)$. Let $p$ be a prime number which divides $n$ and let $s$ denote the maximum integer such that $p^s$ divides $n$. Suppose that there is an $i$ such that $p^s$ does not divide $g_i$. Since $g_i[1] - \xi_i g = nh'$ for some $h' \in G$, $p^s$ does not divide $\xi_i$ (otherwise $p$ would divide $[1]$). The greatest common divisor of $\xi_i$ and $p^s$ is $p^t$ with some $0 \leq t < s$. Note that $p^t$ divides $g_i$. Since there exist $b, c \in \mathbb{Z}$ such that $b \xi_i + cp^t = p^t$, it follows that

$$p^t g = cp^t g + b \xi_i g = cp^t g + b(g_i[1] - nh') = p^t(cg - b(p^{-s} n)h') + b g_i[1]$$

and hence that $g = p^{s-t} g' + b(p^{-t} g_i)[1]$ with $g' = cg - b(p^{-s} n)h'$. Since $g' - \sigma_n(g') = (p^{-s+t} n)h$ and $p^{-s+t} n < n$, we can repeat this process for a finite number of times to get the conclusion. \square

**Proposition 2.2.** If $A$ is a finite-dimensional $C^*$-algebra with non-trivial center, then $(\otimes \mathbb{Z} A) \times_\sigma \mathbb{Z}$ is not an AT $C^*$-algebra.

**Proof.** If $\otimes \mathbb{Z} A \times_\sigma \mathbb{Z}$ were an AT algebra, then any quotient would be an AT algebra. If we denote by $A$ the maximal ideal space of $A$ (which is a finite set), then the center of $\otimes \mathbb{Z} A$ is identified with $C(\Pi \mathbb{Z} A)$. We take two distinct points $p_-, p_+ \in A$ and define a point $x \in \Pi \mathbb{Z} A$ by $x_n = p_-$ for $n < 0$ and $x_n = p_+$ for $n \geq 0$. Let $\Omega$ be the translation invariant closed subset of $\Pi \mathbb{Z} A$ generated by $x$. Then it follows that $\Omega$ consists of the translates of $x$ and two limit points $(p_-), (p_+)$, i.e., $\Omega$ is homeomorphic to $\{-\infty\} \cup \mathbb{Z} \cup \{+\infty\}$. Let $I$ denote the ideal of $\otimes \mathbb{Z} A$ generated by $C(\Omega')$, which is $\sigma$-invariant, and $E_+$ denote the characteristic function of $\{1, 2, \ldots\} \cup \{+\infty\}$. Then one can see that $U_\sigma E_+ U_\sigma^*$ is a proper subprojection of $E_+$ in the quotient $\otimes \mathbb{Z} A/I \times_\sigma \mathbb{Z}$ of $\otimes \mathbb{Z} A \times_\sigma \mathbb{Z}$, where $U_\sigma$ is the unitary implementing $\sigma$. Since this quotient contains a proper isometry $U_\sigma E_+ + 1 - E_+$, it cannot be an AT algebra. \square

**Theorem 2.3.** Let $A$ be a unital simple AF $C^*$-algebra with a unique tracial state and let $\sigma$ denote the shift automorphism of $\otimes \mathbb{Z} A$. Then the crossed product $(\otimes \mathbb{Z} A) \times_\sigma \mathbb{Z}$ is a unital simple tracially AF $C^*$-algebra with a unique tracial state.

If $A$ is an AF $C^*$-algebra, we denote by $T(A)$ the tracial state space of $A$. There is a natural order-preserving homomorphism $\phi$ of $K_0(A)$ into $\text{Aff}(T(A))$, the real affine continuous functions on $T(A)$, and if $g \in K_0(A)$ and $\phi(g) > 0$ (or $\phi(g)$ is strictly positive on $T(A)$), then $g > 0$ (or $g$ is positive and non-zero). We introduce an order on $\mathbb{R} \otimes K_0(A)$ by: $g > 0$ if $\phi(g) > 0$ for $g \in \mathbb{R} \otimes K_0(A)$. If $g \in K_0(A)$ and $g > 0$ in $\mathbb{R} \otimes K_0(A)$, then $g > 0$ in $K_0(A)$.

**Lemma 2.4.** Let $A$ be a unital simple AF $C^*$-algebra with a unique tracial state and let $(A_n)$ be an increasing sequence of finite-dimensional $C^*$-subalgebras of $A$ with $1 \in A_1$ and $A = \bigcup_n A_n$. For any $\varepsilon > 0$ there exist a $k \in \mathbb{N}$, a projection $p \in A_k \cap A_1$, and a full matrix $C^*$-subalgebra $D$ of $pAkp$ such that $D \supset A_1p$ and $|p| > (1 - \varepsilon)[1]$ in $\mathbb{R} \otimes K_0(A_k \cap A_1')$. 


Proof. From the inductive system $A_1 \to A_2 \to \cdots$, we obtain the inductive system of dimension groups $K_0(A_1) \to K_0(A_2) \to \cdots$. By identifying $K_n(A_n)$ with $\mathbb{Z}^{k_n}$, where $k_n$ is the dimension of the center of $A_n$, we obtain the $k_{n+1}$ by $k_n$ matrix $\chi_n$ with non-negative integer components such that $K_0(A_n) \to K_0(A_{n+1})$ is given by the multiplication of $\chi_n$. For $m < n$ we let $\chi_{mn}$ denote $\chi_{n-1}\chi_{n-2}\cdots\chi_m$, which gives the map $K_0(A_m) \to K_0(A_n)$. Let $\xi_n$ be the element of $\mathbb{Z}^{k_n}$ corresponding to $1 \in A_n$. By the assumption that $A$ has a unique tracial state, say $\tau$, it follows that for any $i_n \in \{1, 2, \ldots, k_n\}$,

$$
\left( \frac{\chi_n(i_n,j)\xi_n(j)}{\xi_n(i_n)} \right)_j
$$

converges to a fixed vector in $\mathbb{R}^{k_1}$ as $n \to \infty$. Denoting by $(p_{ni})$ the minimal central projections of $A_n$, indexed according to the indexing of $\mathbb{Z}^{k_n}$, we note that $\xi_n(i) = \text{rank}(p_{ni})$ in $A_n$ and that the above fixed vector is $(\tau(p_{1i}), \tau(p_{12}), \ldots, \tau(p_{1k_1})) \equiv (\mu_j)$. We notice that $\mu_j > 0$ and $\sum_j \mu_j = 1$.

Hence for any $\varepsilon > 0$ there exists a $k \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ with $n \geq k$, any $i = 1, 2, \ldots, k_n$, and any $j = 1, \ldots, k_1$,

$$
\left| \frac{\chi_n(i,j)}{\xi_n(i)} - \frac{\mu_j}{\xi_1(j)} \right| < \varepsilon.
$$

First we find a rational $c_j/d_j$ ($c_j, d_j \in \mathbb{N}$) such that $c_j > 1$ and

$$
\frac{c_j}{d_j} < \frac{\mu_j}{\xi_1(j)} < \frac{c_j + 1}{d_j}.
$$

Second we find an $n \in \mathbb{N}$ such that

$$
0 \leq \frac{\chi_n(i,j)}{\xi_n(i)} - \frac{c_j}{d_j} < \frac{1}{d_j}.
$$

Setting $d$ to be the least common multiple of $d_1, d_2, \ldots, d_{k_1}$, we express $\xi_n(i)$ as $a_i d + r_i$ with $a_i \in \mathbb{N}$ and $0 \leq r_i < d$ and deduce that

$$
0 \leq \chi_n(i,j) - a_i c_j d / d_j < a_i d / d_j + r_i (c_j + 1) / d_j \equiv b_{ij}.
$$

We note that

$$
\frac{b_{ij}}{\chi_n(i,j)} < \frac{1}{d_j} \left( 1 + \frac{c_j + 1}{a_i} \right) \frac{\xi_n(i)}{\chi_n(i,j)}.
$$

As $n \to \infty$, the right-hand side gets smaller than $1/c_j$, which we can assume is arbitrarily small. Set $b_j = c_j d / d_j \in \mathbb{N}$. Then, since $\text{rank}(p_{ni}p_{ij}) = \chi_n(i,j) \geq a_i b_j$ in $A_n \cap A'_1$, we find a projection $p \in A_n \cap A'_1$ such that $[p]$ corresponds to $(a_i b_j)$ in $K_0(A_n \cap A'_1) = \oplus K_0(p_{ni}p_{ij}A_n \cap A'_1)$. Since $[1] = (\chi_n(i,j))$ in $K_0(A_n \cap A'_1)$, we obtain
the estimate \(|p| > (1 - \varepsilon)[1]\). Finally we find a full matrix \(C^*\)-subalgebra \(D\) of \(pA_n p\) of the order \(\sum_j b_j \xi_j(j)\) such that \(D \supseteq A_1 p\). Note that \(pA_n p \cong \bigoplus_{i=1}^{k_n} D \otimes M_{a_i}\), where \(M_a\) is the \(a\) by \(a\) matrix algebra. \(\square\)

**Lemma 2.5.** Let \(A\) be a unital simple \(AF\) \(C^*\)-algebra with a unique tracial state and let \((A_n)\) be as before. For any \(n \in \mathbb{N}\) and \(\varepsilon > 0\) there exist a \(k \in \mathbb{N}\), a projection \(e \in \bigotimes Z(A_k \cap A'_1)\), a full matrix \(C^*\)-subalgebra \(D\) of \(\bigotimes Z A_k\), and a projection \(F\) and a unitary \(V\) in \((\bigotimes Z A_k) \times_{\varepsilon} \mathbb{Z}\) such that \(1_D \in \bigotimes Z (A_k \cap A'_1)\), \(F, e \in D\), \(1_D \geq F \geq e\), \(D \supseteq (\bigotimes^n A_1) 1_D\), \(||V - 1|| < \varepsilon\), \(\text{Ad} VU_\sigma(DF) = DF\), and \([e] > (1 - \varepsilon)[1]\) in \(\mathbb{R} \otimes K_0(\bigotimes Z (A_k \cap A'_1))\).

**Proof.** By the previous lemma we find a \(k \in \mathbb{N}\), a projection \(p \in A_k \cap A'_1\), and a full matrix \(C^*\)-subalgebra \(D\) of \(pA_k p\) such that \(D \supseteq A_1 p\). By changing indices we may suppose that \(k = 2\) and that \(A_n \cap A'_{n-1}\) has no one-dimensional direct summands for any \(m\). (As \(D \supseteq A\), this \(D\) is not intended to be the \(D\) in the statement.)

Let \(k \in \mathbb{N}\) be such that \(k \gg 1\). Let \(w \in \bigotimes Z (A_2 \cap \{p\}')\) be a unitary such that \(w(1 - \bigotimes_{n-k}^{n+1} p) = (1 - \bigotimes_{n-k}^{n+1} p), w(\bigotimes_{n-k}^{n+1} p) \in D \otimes (\bigotimes_{n-k+1}^{n} p) \otimes D\), and \(\text{Ad} w|\bigotimes_{n-k}^{n+1} p\) switches \(x \in D\) at \(-n - k\) and \(x \in D\) at \(n + 1\). This is possible because \(D\) is a full matrix algebra. Hence \(\text{Ad} w\) simulates the cyclical permutation, say \(\sigma', \) of the factors of \(\bigotimes_{n-k}^{n} p\) in the sense that \(\text{Ad} w\sigma(p \otimes x) = \sigma'(x) \otimes p\) for \(x \in \bigotimes_{n-k}^{n} p\). In particular we have that \((\text{Ad} w\sigma)^{2n+k+1}((\bigotimes_{-3n-2k}^{-n-k-1} p) \otimes x) = x \otimes (\bigotimes_{n+k+1}^{n+2k+1} p)\) for \(x \in \bigotimes_{-n-k}^{n} p\).

By using the Rohlin property for \(\sigma\) on \(\bigotimes Z (A_3 \cap A'_2)\) [17], we obtain, for any \(\varepsilon > 0\) and \(\ell \in \mathbb{N}\), a set of Rohlin towers \((e_y)\); \(e_y\)'s are projections for \(i = 0, 1, j = 0, 1, ..., \ell - i\) such that

\[
\sum_i \sum_j e_{ij} = 1, \\
||\sigma(e_{ij}) - e_{ij+1}|| < \varepsilon.
\]

We can then construct a unitary \(u \in \bigotimes Z A_3\) such that \(||w - u\sigma(u)|| \sim 1/\ell\), by using \((e_y)\) and short continuous paths from \(w_{\ell+1}, w_{\ell}\) to 1 applied by \(\sigma', 0 \leq j \leq \ell, \) where \(w_j\)'s are the unitaries defined by \(w_0 = 1, w_j = w_0\sigma(w_{j-1})\). By setting \(\ell = [(k - 1)/2]\) and choosing the paths in \(\bigotimes_{-n-k}^{n+k}(A_2 \cap \{p\}') \otimes (\bigotimes_{-n-k+1}^{n} \{1, p\}') \otimes (\bigotimes_{n+k+1}^{n} \{p\}')\), we thus obtain a unitary \(u \in \bigotimes Z A_3\) such that \(||w - u\sigma(u)|| \sim 1/k\) and \(u\) belongs to the \(C^*\)-subalgebra generated by \(\bigotimes Z A_3 \cap A'_2\) and \(\bigotimes_{-n-k}^{n}(A_2 \cap \{p\}') \otimes (\bigotimes_{-n-1}^{n} \{1, p\}') \otimes \bigotimes_{n+k}^{n+k-1} (A_2 \cap \{p\}')\). Note that \(u\) commutes with \(\bigotimes_{-n-1}^{n} A_1\) and \(p\) at any \(i \in \mathbb{Z}\).

Let

\[
D_1 = \text{Ad} u(\bigotimes_{-n-k}^{n} D \otimes \bigotimes_{n+k+1}^{n} p).
\]

Then \(D_1\) is a full matrix algebra containing \(\bigotimes_{-n-k}^{n} D \otimes (\bigotimes_{n-k}^{n} D \otimes (\bigotimes_{n+k}^{n} p)\) and its identity \(1_{D_1}\) equals \(\bigotimes_{-n-k}^{n+k} p\). (This \(D_1\) will be \(D\) in the statement.) We let
\[ v = u v \sigma (u^*) \], which is a unitary satisfying \[ ||v - 1|| \sim 1/k \] and \( v \) commutes with \( p \) at any point. Since \( \text{Ad } w \sigma = \text{Ad } u^* \text{Ad } v \sigma \text{Ad } u \), \( \text{Ad } v \sigma \) leaves \( D_1 \) invariant in the sense that

\[
\text{Ad } v \sigma(p_{-n-k-1}D_1) = D_1 p_{n+k+1},
\]

where \( p_n \) means \( p \in A \) at \( m \in \mathbb{Z} \). Since \( \text{Ad } v \sigma \) has period \( 2n + k + 1 \) in a sense, we obtain a unitary \( z \in D_1 \) such that \( \text{Ad } v \sigma(p_{-n-k-1}x) = \text{Ad } z(x)p_{n+k+1} \) for \( x \in D_1 \) and \( z^{2n+k+1} = 1 \) by regarding \( z \) as a unitary in \( \otimes \mathbb{Z} A_3 \) by adding \( 1 - \otimes_{n+k}^- p \). Note that

\[
\text{Ad } z^* v U_\sigma(p_{-n-k-1}x) = p_{n+k+1} x, \quad x \in D_1,
\]

where \( U_\sigma \) is the canonical unitary implementing \( \sigma \) in \( \otimes \mathbb{Z} A \times_\sigma \mathbb{Z} \).

Let \( \ell_1, \ell_2 \in \mathbb{N} \) be such that \( \ell_1 \gg \ell_2 \gg 1 \) and let \( \ell = \ell_1 + \ell_2 \). By the Rohlin property on \( \otimes \mathbb{Z} (A_4 \cap A_1) \) there exists an orthogonal family \( (f_i)_{i = -\ell}^{\ell} \) of projections in \( \otimes \mathbb{Z} (A_4 \cap A_1) \) and a unitary \( v_1 \in \otimes \mathbb{Z} (A_4 \cap A_1) \) such that \( v_1 \approx 1 \), \( \text{Ad } v_1 \sigma(f_i) = f_{i+1} \), and \([f_i] > 1/(2\ell + 2)[1]\) in \( R \otimes K_0(\otimes \mathbb{Z} (A_4 \cap A_1)) \) (see 2.11 of [17]). We define, for \( i = -\ell, -\ell + 1, \ldots, \ell \),

\[
F_i = f_i(\otimes_{-n-k-\ell+1}^{n+k+\ell+i} p).
\]

Then \( (F_i)_{i = -\ell}^{\ell} \) is an orthogonal family of projections in \( \otimes \mathbb{Z} A_4 \cap A_1 \) such that \( \text{Ad } v_1 \sigma(F_i) = F_{i+1} \). Since \([p] > (1 - \varepsilon)[1]\) in \( R \otimes K_0(A_2 \cap A_1) \), \([0(\otimes_{-n-k-\ell+1}^{n+k+\ell+i} p)]\) is greater than \((1 - \varepsilon)^{2n + 2k + 2\ell + 1}[1]\) in \( R \otimes K_0(\otimes \mathbb{Z} A_2 \cap A_1) \). Since \( \varepsilon \) can be made small independently of \( n, k, \ell, \ell_1, \ell_2 \), we can assume that \( \varepsilon \) is so small that we still have that \([F_i] > 1/(2\ell + 2)[1]\) in \( R \otimes K_0(\otimes \mathbb{Z} A_4 \cap A_1) \).

Note that \( z \) and \( v \) commute with \( \otimes_{-n-k-\ell+1}^{n+k+\ell+i} p \) and \( f_i \), and that \( \text{Ad } v_1 z^* v U_\sigma(F_i) = F_{i+1} \). We define an almost Ad \( \text{Ad } v_1 z^* v U_\sigma \)-invariant projection \( F \in (\otimes \mathbb{Z} A_4) \times_\sigma \mathbb{Z} \) as follows (cf. [15]):

\[
F = \sum_{i = -\ell_1}^{\ell_1} F_i + \sum_{i = 1}^{\ell_2 - 1} \frac{i}{\ell_2} (F_{-\ell_1 + i} + F_{\ell_1 - i}) + \sum_{i = 1}^{\ell_2 - 1} \sqrt{i(\ell_2 - i)} \left( (v_1 z^* v U_\sigma)^{2\ell_1 + \ell_2} F_{-\ell_1 + i} + F_{\ell_1 - i} (v_1 z^* v U_\sigma)^{2\ell_1 + \ell_2} \right).
\]

We can see that \( F \) is indeed a projection and that

\[
\text{Ad } v_1 z^* v U_\sigma(F) - F \sim 1/\sqrt{\ell_2},
\]

\[
F \geq \sum_{i = -\ell_1}^{\ell_1} F_i \equiv e,
\]

\[
F \leq \otimes_{-n-k-1}^{n+k+1} p \leq 1_{D_1}.
\]
Here we note that $e$ is a projection in $(\otimes Z A_4 \cap A_1') \cap D_1'$ and that $[e]$ can be assumed to be close to $[1]$ in $R \otimes K_0(\otimes Z A_4 \cap A_1')$ since $\ell_1 \gg \ell_2$. For $x \in D_1$ and $i = 1, 2, \ldots, \ell_2 - 1$ we compute:

\[
(v_1 z^* v U_\sigma)^{2\ell_1+i} F_{-\ell_1+i} = (v_1 z^* v U_\sigma)^{2\ell_1+i} (\otimes_{-n-k-2/\ell_1-2/\ell_2+i} p) x F_{-\ell_1-\ell_2+i} = (\otimes_{-n-k-\ell_2+i} p) x (v_1 z^* v U_\sigma)^{2\ell_1+i} F_{-\ell_1-\ell_2+i} = x (v_1 z^* v U_\sigma)^{2\ell_1+i} F_{-\ell_1+i}.
\]

Thus we obtain that $F \in D_1'$. By the same computation for $\text{Ad} v_1 z^* v U_\sigma(F)$, we also obtain that $\text{Ad} v_1 z^* v U_\sigma(F) \in D_1'$. Hence, since $v_1 z^* = z^* v_1$ and $z \in D_1$, it follows that $\text{Ad} v_1 v U_\sigma(F) = \text{Ad} v_1 z^* v U_\sigma(F) \in D_1'$. Then we get a unitary $V \in (\otimes Z A_4 \times_\sigma Z) \cap D_1'$ such that $\text{Ad} v_1 v U_\sigma(F) = F$ and $V - 1 \sim 1/\sqrt{\ell_2}$. Taking $V v_1 v$ for $V$ and $D_1$ for $D$ and noting that $v_1 - 1 \sim 0$ and $v - 1 \sim 1/k$, we conclude the proof.

**Lemma 2.6.** Let $A$ be a unital approximately divisible AF $C^*$-algebra. For any $\epsilon > 0$ there exists a 2 by 2 matrix $C^*$-subalgebra $C$ of $A$ such that $[1_C] > (1 - \epsilon)[1]$ in $R \otimes K_0(A)$.

**Proof.** Since $A$ is approximately divisible, the range of $K_0(A)$ is dense in $\text{Aff}(T(A))$ for the natural map $K_0(A) \to \text{Aff}(T(A))$. Hence there is a positive $g \in K_0(A)$ such that $2^{-1}(1 - \epsilon)[1] < g < 2^{-1}[1]$ in $R \otimes K_0(A)$. Then we find mutually orthogonal projections $e_1$ and $e_2$ in $A$ such that $[e_1] = [e_2] = g$, and choose a $v \in A$ such that $v^* v = e_1$ and $v v^* = e_2$. The $C^*$-subalgebra generated by $v$ gives the desired $C$.

**Lemma 2.7.** Let $A$ be a unital simple AF $C^*$-algebra with a unique tracial state and let $(A_n)$ be as usual. For any $n \in \mathbb{N}$ and $\epsilon > 0$ there exist a projection $e \in \otimes Z (A \cap A_1')$, a 2 by 2 matrix $C^*$-subalgebra $C$ of $\otimes Z A \cap A_1'$, and a projection $F$ and a unitary $V$ in $(\otimes Z A) \times_\sigma Z$ such that $e, F \in C'$, $F \in (\otimes_{-n} A_1)'$, $1_C \geq F \geq \epsilon$, $||V - 1|| < \epsilon$, $\text{Ad} V U_\sigma C F = \text{id}$, and $[e] > (1 - \epsilon)[1]$ in $R \otimes K_0(\otimes Z A \cap A_1')$.

**Proof.** This will be proven just as Lemma 2.5 is.

As in the proof of Lemma 2.5, we obtain a projection $p \in A_2 \cap A_1'$ and a full matrix $C^*$-subalgebra $D$ of $p A_2 p$ such that $D \supset A_1 p$ and then define $D_1 = \text{Ad} u(\otimes_{-n-k} D \otimes \otimes_{n+k} p) (\subset \otimes Z A_2)$ for some unitary $u (\otimes Z A_3 \cap \{p\}')$ and a large $k \in \mathbb{N}$, and unitaries $v (\otimes \otimes Z A_3 \cap \{p\}')$ and $z (\in D_1 + 1)$; in particular $||v - 1|| \sim 1/k$, $D_1 \supset (\otimes_{-n} A_1) p$, and $\text{Ad} z^* v U_\sigma (p_{-n-k-1} x) = p_{n+k+1} x$ for $x \in D_1$.

Since $A \cap A_3'$ is approximately divisible, the previous lemma gives a projection $g$ and a 2 by 2 matrix $C^*$-subalgebra $C$ in $A \cap A_3'$ such that $q = 1_C$ and $[q] > (1 - \epsilon)[1]$ in $R \otimes K_0(A \cap A_3')$ for a sufficiently small $\epsilon > 0$. Here we may replace $A$ by $A_4$. Then just as in the previous paragraph, we define $C_1 = \text{Ad} u' (\otimes_{-n-k} C \otimes \otimes_{n+k} q)$ for the same $k$ as above and for some unitary $u' (\in \otimes Z A_5 \cap A_1')$, and also
Lemma 2.5. and a unitary $V$ for any point of $z$. Let $\ell_1, \ell_2 \in \mathbb{N}$ be such that $\ell_1 \geq \ell_2 \geq 1$ and let $\ell = \ell_1 + \ell_2$. By the Rohlin property $\sigma$ on $\otimes \mathbb{Z}A_6 \cap A_3$, we obtain an orthogonal family $(f_i)_{i=-\ell}^{\ell}$ of projections and a unitary $v_1 \in \otimes \mathbb{Z}A_6 \cap A_3'$ such that $v_1 \sim 1$, $\text{Ad} v_1 \sigma(f_i) = f_{i+1}$, and $|f_i| > (2\ell + 2)^{-1}[1]$ in $R \otimes K_0(\otimes \mathbb{Z}A_6 \cap A_3')$. We define, for $i = -\ell, -\ell + 1, \ldots, \ell$, 

$$F_i = f_i(\otimes_{-n-k-\ell}^{-n+k+\ell+i} pq).$$

Here $(F_i)$ is an orthogonal family of projections such that $\text{Ad} v_1 z^* v^* v^* v U_\sigma(F_i) = F_{i+1}$. We assume that $|F_i| > (2\ell + 2)^{-1}[1]$ in $R \otimes K_0(\otimes \mathbb{Z}A_6 \cap A_3')$ by assuming that $[pq]$ is sufficiently close to $[1]$ in $R \otimes K_0(A_6 \cap A_3')$. We define an almost $\text{Ad} v_1 z^* v^* v^* v U_\sigma$-invariant projection $F \in \otimes \mathbb{Z}A_6 \times_\sigma \mathbb{Z}$ in terms of $(F_i)$ and $(v_1 z^* v^* v^* v U_\sigma)^{2\ell+\ell_2}$ just as in the proof of Lemma 2.5. Since $F$, $\text{Ad} v_1 z^* v^* v^* v U_\sigma(F) \in C^*(C_1, D_1)'$, we find a unitary $V_1 \in \otimes \mathbb{Z}A_6 \times_\sigma \mathbb{Z}$ such that $V_1 \in C^*(C_1, D_1)'$, $V_1 \sim 1$, and $\text{Ad} V_1 v_1 z^* v^* v^* v U_\sigma(F) = F$. Note that $\text{Ad} V_1 v_1 z^* v^* v^* v U_\sigma | C^*(C_1, D_1)F = \text{id}$. Since $\text{Ad} z' | C_1$ has period $2n + k + 1$, the spectral gap of $z'$ is no greater than $2n/(2n + k + 1)$. Hence there is a 2 by 2 matrix $C^*$-subalgebra $C_2$ of $C_1$ such that $|\|(\text{Ad} z' - \text{id})|C_2\|$ is at most of the order of $1/(2n + k + 1)$. Since $\text{id}|C_2 = \text{Ad} V_1 v_1 z^* v^* v^* v U_\sigma|C_2 = \text{Ad} V_1 z^* v^* v U_\sigma|C_2 \simeq \text{Ad} U_\sigma|C_2$, $C_2$ has the desired properties for $C$. The other properties are also satisfied just as in the proof of Lemma 2.5.

Lemma 2.8. Let $A$ be a unital simple AF $C^*$-algebra with a unique tracial state and let $(A_n)$ be as usual. For any $n \in \mathbb{N}$ and $\varepsilon > 0$ there exists a $C^*$-subalgebra $C$ of $(\otimes \mathbb{Z}A) \times_\sigma \mathbb{Z}$ and a unitary $V \in (\otimes \mathbb{Z}A) \times_\sigma \mathbb{Z}$ such that $C \ni 1$, $C \subset (\otimes_{-n}^{n} A_1)'$, $C \cong M_2 \oplus M_3$, $\|V - 1\| < \varepsilon$, and $\text{Ad} VU_\sigma C = \text{id}$.

Proof. We use the previous lemma for a sufficiently small $\varepsilon > 0$, where in the conclusion we may replace $A$ by $A_k$ for a sufficiently large $k$. By changing indices we may assume that $k = 2$.

Let $\ell \in \mathbb{N}$ be such that $\ell \geq 1$ and $\ell < (3\varepsilon)^{-1}$. By using the Rohlin property for $\sigma$ on $\otimes \mathbb{Z}A \cap A_2'$, we find an orthogonal family $(f_i)_{i=0}^{\ell-1}$ of projections and a unitary $v$ in $\otimes \mathbb{Z}A \cap A_2'$ such that $v \sim 1$, $\text{Ad} v U_\sigma(f_i) = f_{i+1}$, and $|f_i| > (1 + \ell)^{-1}[1]$ in $R \otimes K_0(\otimes \mathbb{Z}A \cap A_2')$. Let $(e_{ij})$ be a set of matrix units in $C$. We may assume that $[1 - e] \leq [e_{11}]f_1$ in $K_0(\otimes \mathbb{Z}A \cap A_1')$ and let $b \in \otimes \mathbb{Z}A \cap A_1'$ be a partial isometry such that $b^* b = 1 - e$ and $b b^* \leq e_{11}$. We define

$$y = \frac{1}{\sqrt{\ell}} \sum_{i=0}^{\ell-1} (v U_\sigma)^i b (v U_\sigma)^{-i} (1 - F).$$
Then $y$ belongs to $(\otimes \mathbb{Z}A \cap A_1')(1 - F) \subset (\otimes_{-n}^0 A_1)'$, is close to a partial isometry

$$y_1 = \frac{1}{\sqrt{\varepsilon}} \sum_{i=0}^{\varepsilon-1} (vU_\sigma)^i b(vU_\sigma)^{-i}(1 - F)$$

with $y_1^*y_1 = 1 - F$, and is close to

$$y_2 = \frac{1}{\sqrt{\varepsilon}} \sum_{i=0}^{\varepsilon-1} (VU_\sigma)^i b(vU_\sigma)^{-i}(1 - F)$$

with $y_2 = Fe_{11}y_2$. Note also that

$$||\mathrm{Ad} vU_\sigma(y) - y|| \leq 2/\sqrt{\varepsilon}.$$ 

Let $v_1$ be the partial isometry obtained by the polar decomposition of $Fe_{11}y \in (\otimes_{-n}^0 A_1)'$. Then it follows that $v_1^*v_1 = 1 - F$, $v_1v_1^* \leq Fe_{11}$, and $[U_\sigma, v_1] \sim 0$. Then the $C^*$-subalgebra $C_1$ generated by $CF$ and $v_1$ satisfies that $C_1 \subset (\otimes_{-n}^0 A_1)'$, $C_1 \ni 1$, $C_1 \cong M_2 \oplus M_3$, and $\mathrm{Ad} U_\sigma|C_1 \cong \mathrm{id}$. This concludes the proof. \(\square\)

**Proof of Theorem 2.3.** Since $\sigma$ is asymptotically abelian (or has the Rohlin property), the crossed product $(\otimes \mathbb{Z}A) \times_\sigma \mathbb{Z}$ has a unique tracial state (see [1] for more results).

By Lemma 2.8 we know that the crossed product $(\otimes \mathbb{Z}A) \times_\sigma \mathbb{Z}$ is approximately divisible. Since the crossed product has a unique tracial state, it has real rank zero [2]. In the situation of Lemma 2.5 let $z \in D$ be a unitary such that $\mathrm{Ad} VU_\sigma|DF = \mathrm{Ad} z|DF$. Then $z^*VU_\sigma F$ is a unitary in $FD'F$; so the $C^*$-subalgebra generated by $DF$ and $z^*VU_\sigma F$ is isomorphic to $D \otimes C(T)$ or a quotient of it and contains $(\otimes_{-n}^0 A_1)F$ exactly and $FU_\sigma F \sim U_\sigma F$ almost (since $V \approx 1$). (One can show that the $K_1$ class of $z^*VU_\sigma F + 1 - F$ is non-zero if $\varepsilon$ is small and so that its spectrum is full. But, since this fact is not required, we will refrain from proving it.) Thus $(\otimes \mathbb{Z}A) \times_\sigma \mathbb{Z}$ is tracially $AT$.

Since $W = z^*VU_\sigma F$ is a unitary in the $C^*$-algebra $F((\otimes \mathbb{Z}A) \times_\sigma \mathbb{Z})F \cap FD'F$ of real rank zero, one can approximate $W$ by a unitary $W_1 + W_2$ such that $G = W_1^*W_1 = W_1W_1^*$ is a projection close to $F$, i.e., $\tau(G) \approx \tau(F)$ with $\tau$ the tracial state, and $\mathrm{Sp}(W_1)$ is finite. Since $D \ni (\otimes_{-n}^0 A_1)1_D$ and $F \in (\otimes_{-n}^0 A_1)'$, we have that $G \in (\otimes_{-n}^0 A_1)'$. Since $[W, G] \approx 0$, we have that $GU_\sigma = zGz^*U_\sigma \approx zGz^*VU_\sigma = zGW \approx zWG \approx U_\sigma G$. Let $D_1$ be the $C^*$-subalgebra generated by $GDG$ and $W_1$. Then $D_1$ is finite-dimensional and its unit $G$ commutes with $\otimes_{-n}^0 A_1$ and almost commutes with $U_\sigma$. It follows that

$$G(\otimes_{-n}^0 A_1) \subset GDG \subset D_1$$

and

$$GU_\sigma G \approx GzW_1 \in D_1.$$ 

This concludes the proof that $(\otimes \mathbb{Z}A) \times_\sigma \mathbb{Z}$ is a tracially AF $C^*$-algebra.
3. Non-unital tensor products

If we are given a $C^*$-algebra $A(i)$ and a non-zero projection $e_i \in A(i)$ for each $i \in \mathbb{Z}$, we define a $C^*$-algebra $\otimes_{i \in \mathbb{Z}} (A(i), e_i)$ as the inductive limit of the following inductive system $(A_A, \varphi_{A_i, A_j})$: For each finite set $A$ of $\mathbb{Z}$ let $A_A$ be the (minimal) tensor product of $\{A(i), i \in A\}$ and for $A_1 \subset A_2$ define an embedding $\varphi_{A_1, A_2}$ of $A_A$ into $A_A$ by $x \mapsto x \otimes (\otimes_{i \in A_2 \setminus A_1} e_i)$. If $A(i)$ is unital and $e_i = 1$ for every $i$, then $\otimes_{i \in \mathbb{Z}} (A(i), e_i)$ is just the usual tensor product $\otimes_{i \in \mathbb{Z}} A(i)$.

Suppose that all $A(i)$’s are unital but $e_i \neq 1$ for infinitely many $i$. Then $\otimes_{i \in \mathbb{Z}} (A(i), e_i)$ is a non-unital $C^*$-algebra and $\otimes_{i \in \mathbb{Z}} A(i)$ can be naturally regarded as a multiplier algebra of $\otimes_{i \in \mathbb{Z}} (A(i), e_i)$. Suppose that we are given another projection $e_i' \in A(i)$ for each $i$ and if $e_i'$ is unitarily equivalent to $e_i$, then $\otimes_{i \in \mathbb{Z}} (A(i), e_i')$ is isomorphic to $\otimes_{i \in \mathbb{Z}} (A(i), e_i)$. If $e_i' = e_i$ except for a finite number of $i$, then $\otimes_{i \in \mathbb{Z}} (A(i), e_i')$ and $\otimes_{i \in \mathbb{Z}} (A(i), e_i)$ are identical.

Suppose that we are given a $C^*$-algebra $A$ and that $A(i) = A$. Let $e_-$ and $e_+$ be non-zero projections in $A$ and let $e_i = e_+$ for $i \geq 0$ and $e_i = e_-$ for $i < 0$. In this case since $e_{i+1} = e_i$ except for $i = -1$, we can define a shift automorphism $\sigma$ of $\otimes_{Z} (A, e_-, e_+)$ by $x \mapsto x \otimes (\otimes_{i \in \mathbb{Z}} A(i), e_i)$ for each $x \in A(i)$.

Suppose that we are given another pair $e_-, e_+$ of projections in $A$ such that $e_i'$ is unitarily equivalent to $e_\pm$ (in $A + C1$ if $A$ is non-unital). Then denoting by $\sigma'$ the shift automorphism of $\otimes_{Z} (A, e_-, e_+)$, it follows that $(\otimes_{Z} (A, e_-, e_+), \sigma)$ and $(\otimes_{Z} (A, e_-, e_+), \sigma')$ are outer conjugate. To see this let $U_\pm$ be unitaries in $A$ such that $\text{Ad} U_\pm (e_\pm) = e_\pm$. With $U_i = U_+$ for $i \geq 0$ and $U_i = U_-$ for $i < 0$, define a map $\varphi$ of $\otimes_{Z} (A, e_-, e_+)$ into $\otimes_{Z} (A, e_-, e_+)$ by $x \mapsto \text{Ad} U_i (x) \otimes (\otimes_{i \in \mathbb{Z}} A(i), e_i)$. This is indeed an isomorphism. Since $\varphi^{-1} \sigma' \varphi^{-1} = \otimes_{i} \text{Ad} U_i \varphi^{-1}$, and $\text{Ad} U_i \varphi^{-1} = 1$ except for $i = 0$, $\varphi^{-1} \sigma' \varphi^{-1}$ is inner, proving the assertion. Hence in particular $\otimes_{Z} (A, e_-, e_+) \times_\sigma Z$ is isomorphic to $\otimes_{Z} (A, e_-, e_+) \times_\sigma Z$.

Suppose that $A$ is a non-unital AF $C^*$-algebra and $e_-, e_+$ are projections in $A$. By changing $e_-, e_+$ by equivalent projections if necessary, we may assume that there is an approximate unit $(e_n)$ consisting of projections such that $e_1 \geq e_-, e_+$. Then

$$(\otimes_{Z} (e_n A e_n, e_-, e_+))_n$$

is an increasing sequence of $\sigma$-invariant hereditary $C^*$-subalgebras of $\otimes_{Z} (A, e_-, e_+)$ with dense union. Thus it follows that $(\otimes_{Z} (e_n A e_n, e_-, e_+) \times_\sigma Z)$ is also an increasing sequence of hereditary $C^*$-subalgebras of $\otimes_{Z} (A, e_-, e_+) \times_\sigma Z$ with dense union. Thus when we consider the crossed products, there will be no loss of generality by assuming that $A$ is unital.

**Proposition 3.1.** If $A$ is an AF $C^*$-algebra and $[e_-] \neq [e_+]$, then $K_1(\otimes_{Z} (A, e_-, e_+) \times_\sigma Z)$ is zero.

**Proof.** We have to show that the kernel of $\text{id} - \sigma_+$ on $K_0(\otimes_{Z} (A, e_-, e_+))$ is $\{0\}$. Suppose that $g = \sigma_+(g)$ for some $g \in K_0(\otimes_{Z} (A, e_-, e_+))$. There is an $n \in \mathbb{N}$ such that
Let $A$ be a unital simple $K$-algebra. Since $g = \sigma_n^{2n+1}(g)$ and the $n+1$st factor of $g$ (resp. $\sigma_n^{2n+1}(g)$) is $[e_-]$ (resp. $[e_-]$), if $[e_-]$ and $[e_-]$ are rationally independent, it follows that $g = 0$. If $m[e_+] = \ell[e_-]$ with $m \neq \ell$, then deleting the common factor $m[e_+] = \ell[e_-]$ at $n+1 \in \mathbb{Z}$ from $mg$ and $\ell\sigma_n^{2n+1}(g)$, we obtain $g$ and $\sigma_n^{2n}(g)$ respectively. Thus $mg = \ell\sigma_n^{2n+1}(g) = \ell g$. Hence it follows that $g = 0$. □

Proposition 3.2. If $A$ is an AF $C^*$-algebra, $[e_-] \neq [e_+]$, and if $p | [e_-] - [e_+]$ then $p | [e_-]$ and $p | [e_+]$ for all prime numbers $p$, then $K_0(\otimes Z(A, e_-, e_+))$ is torsion-free.

Proof. We have to show that $K_0(\otimes Z(A, e_-, e_+))/\Range(\id - \sigma_s)$ is torsion-free.

Let $F$ be the subfield of $Q$ generated by

$$\{n \in \mathbb{N}; n \text{ divides } [e_-] \text{ or } [e_+]\}.$$ 

Then $K_0(\otimes Z(A, e_-, e_+))$ is a module over $F$ and $\sigma_s$ is a module homomorphism.

Suppose that $K_0(\otimes Z(A, e_-, e_+))/\Range(\id - \sigma_s)$ has torsion and let $h$ be an element not in the range of $\id - \sigma_s$ but $nh = g - \sigma_s(g)$ for some $n > 1$ and $g \in K_0(\otimes Z(A, e_-, e_+))$. We suppose that $n$ is the smallest positive integer with this property. In particular no prime factors of $n$ appear in $F$. Since $g - \sigma_s(g)$ is divisible by $n$ for any $k \in \mathbb{N}$, we are led to the following situation: If $G = K_0(A)\otimes m$ for some $m \in \mathbb{N}$, there is a $g \in G$ such that $\xi_- \otimes g - g \otimes \xi_+$ is divisible by $n$, where $\xi_{\pm} = [e_{\pm}]\otimes m$. We may then replace $K_0(A)$ by $Z'$ for some $\ell \in \mathbb{N}$ and so $G$ by $Z'^m$. Our standing assumption says that no prime factors of $n$ divide either $\xi_-$ or $\xi_+$.

Let $p$ be a prime factor of $n$ and let $s$ be the maximum integer such that $p^s | n$. If there is a component $g_i$ of $g$ such that $p^s$ does not divide $g_i$, then by the same argument used in the proof of A1, we have that $g = p^{s-t}g' + d\xi_+$, where $s - t > 0$. If $p | d$ then $p | g$, which contradicts the choice of $n$. Thus we are led to the situation that $\xi_- \otimes \xi_+ - \xi_+ \otimes \xi_+$ is divisible by $p$, which means $\xi_- - \xi_+$ is divisible by $p$ (since $\xi_+$ is not). Since $[e_-] - [e_+]$ is not divisible by $p$ by the assumption, it follows that $[e_-]\otimes m - [e_+]\otimes m$ is not. Hence we have reached a contradiction. □

Theorem 3.3. Let $A$ be a unital simple AF $C^*$-algebra with a unique tracial state $\tau$ and let $e_-, e_+$ be non-zero projections in $A$ such that $\tau(e_-) = \tau(e_+) < 1$. Then $\otimes Z(A, e_-, e_+)\times_\sigma Z$ is a simple tracially AF $C^*$-algebra, which admits a densely defined lower semi-continuous trace, unique up to constant multiples.

There is a (unique up to constant multiples, densely defined lower semi-continuous) trace on $\otimes Z(A, e_-, e_+)$ which is left invariant under $\sigma$. Hence the part on trace is obvious in the above assertion.

Let $(A_n)$ be an increasing sequence of finite-dimensional $C^*$-subalgebras of $A$ with $A = \cup_n A_n$ and $A_1 \ni 1$. Let $e_-, e_+ \in A_1$ be projections such that $0 < \tau(e_-) = \tau(e_+) < 1$. For $m \leq n$ we denote the identity of $\otimes^n_m A$ in $\otimes Z(A, e_-, e_+)$ by $1_{(m,n)}$. We shall show that for any $n \in \mathbb{N}$ and $\varepsilon > 0$ there is a $C^*$ subalgebra $C$ of $\otimes Z(A, e_-, e_+)\times_\sigma Z$
such that

\[ 1_C = 1_{(-n,n)}; \]

\[ C \cong M_2 \oplus M_3, \]

\[ C \subset (\otimes_{-n}^{n}(A_1, e_-, e_+))', \]

\[ ||[x, U_1^{1_{(-n,n-1)}}]| < \varepsilon||x||, \quad x \in C. \]

This implies that for any finite subset \( \mathcal{F} \) of \( \bigotimes_{Z}(A, e_-, e_+) \times_{\sigma} \mathbb{Z} \) and any \( \varepsilon > 0 \) there is a \( C^* \) subalgebra \( C \) such that \( ||(1 - 1_C)y|| < \varepsilon \) for \( y \in \mathcal{F} \), \( ||[x, y]|| < \varepsilon||x|| \) for \( x \in C \) and \( y \in \mathcal{F} \), and \( C \cong M_2 \oplus M_3 \). Hence it follows that for any projection \( p \in \bigotimes_{Z}(A, e_-, e_+) \times_{\sigma} \mathbb{Z} \), \( p(\bigotimes_{Z}(A, e_-, e_+) \times_{\sigma} \mathbb{Z})p \) is approximately divisible and thus has real rank zero. Then one can conclude that \( \bigotimes_{Z}(A, e_-, e_+) \times_{\sigma} \mathbb{Z} \) has real rank zero (or \( \bigotimes_{Z}(A, e_-, e_+) \times_{\sigma} \mathbb{Z} + C_1 \) has real rank zero [9]).

In Lemma 3.5 we shall show that \( \bigotimes_{Z}(A, e_-, e_+) \times_{\sigma} \mathbb{Z} \) is tracially AT, i.e., for any \( n \in \mathbb{N} \) and \( \varepsilon > 0 \) there is a \( C^* \)-subalgebra \( D \) of \( \bigotimes_{Z}(A, e_-, e_+) \times_{\sigma} \mathbb{Z} \) such that \( D \) is isomorphic to (a quotient of) the tensor product of a full matrix algebra and \( C(T) \), and

\[ 1_D \in (\otimes_{-n}^{n} A_1)', \]

\[ 1_D \leq 1_{(-n,n)}, \]

\[ [1_{(-n,n)} - 1_D] \leq \varepsilon[1_{(-n,n)}], \]

\[ D \supset (\otimes_{-n}^{n} A_1)1_D, \]

\[ ||[1_D, U_1^{1_{(-n,n-1)}}]| < \varepsilon; \]

\[ \text{dist}(D, 1_D U_1^{1_{(-n,n-1)}} 1_D) < \varepsilon. \]

Since \( 1_{(-n,n)} \) forms an approximate identity for \( \bigotimes_{Z}(A, e_-, e_+) \times_{\sigma} \mathbb{Z} \), this implies that for any projection \( p \in \bigotimes_{Z}(A, e_-, e_+) \times_{\sigma} \mathbb{Z} \), the hereditary \( C^* \)-subalgebra cut down by \( p \) is tracially AT. Using the fact that \( \bigotimes_{Z}(A, e_-, e_+) \times_{\sigma} \mathbb{Z} \) has real rank zero, which will be shown later, we conclude that it is tracially AF, in the same way as in the unital case. Note also that if \( [e_-] \neq [e_+] \), then \( \bigotimes_{Z}(A_k, e_-, e_+) \times_{\sigma} \mathbb{Z} \) has purely infinite quotients for any \( k \).

First we shall give an analogue of 2.4.

**Lemma 3.4.** Let \( A \) be a unital simple AF \( C^* \)-algebra with a unique tracial state and let \( (A_n) \) and \( e_-, e_+ \in A_1 \) as above. Then for any \( \varepsilon > 0 \) there exist a \( k \in \mathbb{N} \), a projection
Let $A$ be a unital simple C*-algebra $p \in A \cap A_1'$, and a full matrix C*-subalgebra $D$ of $pA_k p$ such that $D \ni p$, $D \ni A_1 p$, $[pe_+] = [pe_-]$ in $K_0(A_k)$, and $[p] > (1 - \varepsilon)[1]$ in $R \otimes K_0(A_k \cap A_1')$.

**Proof.** If $[e_-] = [e_+]$ then we may assume that $[e_-] = [e_+]$ in $K_0(A_1)$ and then this is just Lemma 2.4. Assuming that $[e_-] \neq [e_+]$ we will extend Lemma 2.4, adopting the same notation as in its proof.

When we choose $c_j, d_j \in \mathbb{N}$ for $j = 1, 2, \ldots, k_1$ there (such that $c_j/d_j$ approximates $\mu_j/\xi_1(j)$), we have to impose an extra condition corresponding to $[pe_+] = [pe_-]$; i.e., we have to find $c_j, d_j \in \mathbb{N}$ such that $c_j \geq 1$ and

$$\frac{c_j}{d_j} < \frac{\mu_j}{\xi_1(j)} < \frac{c_j + 1}{d_j},$$

$$\sum_{j=1}^{k_1} c_j \dim(e_- p_{1j}) = \sum_{j=1}^{k_1} c_j \dim(e_+ p_{1j}),$$

where $\dim(e_\pm p_{1j})$ is the dimension of $e_\pm p_{1j}$ in $A_1 p_{1j}$. Since

$$\sum_j \frac{\mu_j}{\xi_1(j)} \dim(e_- p_{1j}) = \sum_j \frac{\mu_j}{\xi_1(j)} \dim(e_+ p_{1j}),$$

which follows from $\tau(e_-) = \tau(e_+)$, both

$$\{ j ; \dim(e_- p_{1j}) > \dim(e_+ p_{1j}) \}$$

and

$$\{ j ; \dim(e_- p_{1j}) < \dim(e_+ p_{1j}) \}$$

are non-empty. Hence we can find $c_j \in \mathbb{N}$ such that $c_j \geq 1$ and

$$\sum_{i=1}^{k_1} c_j (\dim(e_- p_{1j}) - \dim(e_+ p_{1j})) = 0.$$

By assuming that $\xi_1(j)/\mu_j > 1$, we can then find $d_j \in \mathbb{N}$ satisfying

$$\frac{\xi_1(j)}{\mu_j} c_j < d_j < \frac{\xi_1(j)}{\mu_j} (c_j + 1).$$

Having defined $c_j, d_j$ we can proceed as in the proof of 2.4. □

**Lemma 3.5.** Let $A$ be a unital simple AF C*-algebra with a unique tracial state $\tau$ and let $(A_n)$ and $e_-, e_+ \in A_1$ as above. For any $n \in \mathbb{N}$ and $\varepsilon > 0$ there exist a $k \in \mathbb{N}$, a projection $e \in \otimes \mathbb{Z}(A_k \cap A_1')$, a projection $g \in \otimes \mathbb{Z}(A_k, e_-, e_+)$, a full matrix C*-subalgebra $D$ of $\otimes \mathbb{Z} A_k$, a projection $F \in (\otimes \mathbb{Z} A_k) \times_\sigma \mathbb{Z}$, and a unitary $V$ in the multiplier algebra of $\otimes \mathbb{Z}(A_k, e_-, e_+) \times_\sigma \mathbb{Z}$ such that

$$1_D \in \otimes \mathbb{Z} A_k \cap A_1',$$

$$F, e, g \in D',$$

$$[F, g] = 0,$$
\[ [e, g] = 0, \]
\[ 1_D \supseteq F \supseteq e, \]
\[ g \supseteq 1_{(-n,n)}, \]
\[ D \supseteq (\otimes_{-n}^n A_1)1_D, \]
\[ Dg \supseteq (\otimes_{-n}^n (A_1, e_-, e_+))1_D, \]
\[ \|V - 1\| < \varepsilon, \]
\[ \text{Ad } VU_\sigma(DB) = DB, \]
\[ \text{Ad } VU_\sigma(Dg) = Dg, \]
\[ [e] > (1 - \varepsilon)[1] \text{ in } \mathbb{R} \otimes K_0(\otimes Z A_k \cap A_1'). \]

**Proof.** This will be proven just as Lemma 2.5 is. We use the same notation as in its proof.

We select a projection \( p \in A_2 \cap A_1' \) and a full matrix \( C^*\)-subalgebra \( D \supset p A_2 p \) with \( D \supset A_1 p \) as in Lemma 3.4.

Let \( k \in \mathbb{N} \) with \( k \gg 1 \). We work in the multiplier algebra of \( \otimes Z (A, e_-, e_+) \). Let \( w \) be a unitary in \( \otimes_{-n-k}^{n+1} A_2 \) such that \( w(1 - \otimes_{-n-k}^{n+1} p) = 1 - \otimes_{-n-k}^{n+1} p, \ w(\otimes_{-n-k}^{n+1} p) \in D \otimes (\otimes_{-n-k}^{n+1} p) \otimes D \), and \( \text{Ad } w | \otimes_{-n-k}^{n+1} p \) switches \( D \) at \(-n-k\) and \( D \) at \( n+1 \) in such a way that it switches \((pe_-)_{-n-k}\) and \((pe_+)_{n+1}\). This is possible because \( D \) is a full matrix algebra and \( \dim((pe_-)) = \dim((pe_+)) \) in \( D \). Hence in particular \( \text{Ad } w \) leaves \((pe_-)_{-n-k}(pe_+)_{n+1}\) invariant. By using a set of Rohlin towers for \( \sigma \) on \( \otimes Z A_3 \cap A_2' \) (which is in the multiplier algebra) we will choose a unitary \( u \) in the multiplier algebra just as in the proof of 2.5. For example \( u \) belongs to the \( C^*\)-subalgebra generated by \( \otimes_{-n-k}^{-n-2} A_2 \cap \{p\} \otimes (\otimes_{-n-1}^n \{1, p\}) \otimes_{n+1}^{n+k-1} A_2 \cap \{p\}' \) and \( \otimes Z A_3 \cap A_2' \) and satisfies that \( \|w - u^* \sigma(u)\| \approx 1/k \). This time we can impose an extra condition that \( u \) commutes with \((pe_-)_{-n-k+i}(pe_+)_{n+1+i}\) for \( i = 0, 1, \ldots, k - 2 \).

Let
\[ D_1 = \text{Ad } u(\otimes_{-n-k}^{n+1} D \otimes (\otimes_{n+1}^{n+k} p)). \]

Note that \( D_1 \) is in the multiplier algebra, \( 1_{D_1} = \otimes_{-n-k}^{n+k} p \), and \( D_1 \supset (\otimes_{-n}^n A_1)1_{D_1} \). Let \( g = \text{Ad } u(1_{(-n-k,n)}) \), which is a projection in the commutant of \( D_1 \) and satisfies that \( g \supset \text{Ad } u(1_{(-n,n)}) = 1_{(-n,n)}. \)
Setting \( v = u\omega(u^*) \), we have a unitary \( z \in D_1 + 1 \) such that \( \text{Ad} v\sigma(p_{-n-k-1}x) = \text{Ad} z(x)p_{n+k+1} \) for \( x \in D_1 \). We should note that if \( x \in D_1g \), then \( \text{Ad} v\sigma((pe_{-})_{-n-k-1}x) = \text{Ad} z(x)(pe_{+})_{n+k+1} \).

For \( \ell = \ell_1 + \ell_2 \) with \( \ell_1 \gg \ell_2 \gg 1 \) and \( \ell = \ell_1 + \ell_2 \), we have an orthogonal family \((f_i)_{i=-\ell}^{\ell}\) of projections in \( \bigotimes_z A_4 \cap A_4' \) and a unitary \( v_1 \in \bigotimes_z A_4 \cap A_4' \) such that \( v_1 \approx 1 \), \( \text{Ad} v_1\sigma(f_i) = f_{i+1} \), and \( [f_i] > 1/(2\ell + 2)[1] \). We define \( F_i = f_i^{(\bigotimes_{-n-k-\ell+1}p)} \) just as in 2.5. Then we should note that \( g \) commutes with \( F_i \) and that \( \text{Ad} v_1z^*vU_\sigma(F_i g) = F_{i+1} \) as well as \( \text{Ad} v_1z^*vU_\sigma(F_i) = F_{i+1} \). We define an almost \( \text{Ad} v_1z^*vU_\sigma \)-invariant projection \( F \in \bigotimes_z A_4 \times _\sigma Z \) in terms of \( (F_i) \) and \( (v_1z^*vU_\sigma)^{2\ell_1+\ell_2} \) just as before. In particular we have that \( F \geq \sum_{i=-\ell_1}^{\ell_1} F_i \equiv e \) and that \( F \leq \bigotimes_{-n-k-1}p \leq 1_{D_1} \). We should notice that \( g \) commutes with \( F, e \) and that \( Fg \) can be defined just in the same way as \( F \) in terms of \( (F_i g) \) and \( (v_1z^*vU_\sigma)^{2\ell_1+\ell_2} \). By the same computation as before we can see that \( F, Fg, \text{Ad} v_1z^*vU_\sigma(F), \text{Ad} v_1z^*vU_\sigma(Fg) \) commutes with \( D_1 \). Since \( \text{Ad} v_1z^*vU_\sigma(F) = \text{Ad} v_1z^*vU_\sigma(Fg) = \text{Ad} v_1z^*vU_\sigma(Fg) \), we get a unitary \( V \) in the commutant of \( D_1 \) in the multiplier algebra with \( V \) such that \( \text{Ad} Vv_1vU_\sigma(F) = F \) and \( \text{Ad} Vv_1vU_\sigma(Fg) = Fg \). Taking \( Vv_1v \) for \( V \) and \( D_1 \) for \( D \), we can check all the other properties.

Finally we indicate how the assertion made after Theorem 3.3 follows. Let \( z \) be a unitary in \( D(\equiv D_1) \) such that \( \text{Ad} VU_\sigma(xF) = \text{Ad} z(x)F \) for \( x \in D \). Then, since \( z^*VU_\sigma \) is in the commutant of \( Dg \) and \( 1_{(-n,n)}1_D \) is a projection in \( Dg \), the \( C^* \)-subalgebra generated by \( 1_{(-n,n)}D_1(-n,n)F_1 \) and \( z^*VU_\sigma F_1(-n,n) \) is isomorphic to \( 1_{(-n,n)}D_1(-n,n) \times C(T) \) (or a quotient of it) and its identity \( 1_{(-n,n)}F \) is close to \( 1_{(-n,n)} \) in the sense that \( [1_{(-n,n)} - 1_{(-n,n)}F] \leq [1_{(-n,n)}(1 - e)] < \varepsilon[1_{(-n,n)}] \) in \( \mathbb{R} \otimes K_0 \otimes Z \). \( (A_k, e_{-}, e_{+}) \). We should also notice that \( F_1(-n,n) \) commutes with \( VU_\sigma 1_{(-n,n-1)}(\approx U_\sigma 1_{(-n,n-1)}) \). ☐

**Lemma 3.6.** Let \( A \) be a unital simple AF \( C^* \)-algebra with a unique tracial state and let \((A_n)\) and \( e_{-}, e_{+} \in A_1\) be as before. For any \( n \in \mathbb{N} \) and \( \varepsilon > 0 \) there exist a projection \( e \in \bigotimes_z A \cap A_1' \), a 2 by 2 matrix \( C^* \)-subalgebra \( C \) of \( \bigotimes_z A \cap A_1' \), a projection \( F \in \bigotimes_z A \times _\sigma Z \), and a unitary \( V \) in the multiplier algebra of \( \bigotimes_z A \times _\sigma Z \) such that

\[
e, F \in C',
\]

\[
[F, 1_{(-n,n)}] = 0,
\]

\[
F \in (\bigotimes_{-n} A_1)',
\]

\[
1_C \geq F \geq e,
\]

\[
||V - 1|| < \varepsilon,
\]

\[
\text{Ad} VU_\sigma | CF = id,
\]
\[
\text{Ad } VU_\sigma(F_1_{(-n,n)}) = F_1_{(-n,n)},
\]

\[\lbrack e \rbrack > (1 - \varepsilon)[1] \text{ in } \mathbb{R} \otimes K_0(\otimes Z A \cap A_1').\]

**Proof.** As in the proof of the previous lemma, we obtain a projection \(p \in A_2 \cap A_1'\) and a full matrix \(C^*\)-subalgebra \(D\) of \(pA_2p\) with \(D \supset A_1p\) and then construct \(D_1, u, v, z, g\) for a large \(k \in \mathbb{N}\) just as there.

Since \(A \cap A_1'\) is approximately divisible, Lemma 2.6 gives, for any \(\varepsilon > 0\), a projection \(q\) and a 2 by 2 matrix \(C^*\)-subalgebra \(C \subset A \cap A_1'\) such that \(q = 1_C\) and \([q] > (1 - \varepsilon)[1]\) in \(\mathbb{R} \otimes K_0(A \cap A_1')\). Here we may replace \(A\) by \(A_4\). We should note here that \([qpe_-] = [qpe_+]\) in \(K_0(A_4)\) since \([pe_-] = [pe_+]\) in \(K_0(A_2)\). Then just as in the proof of the previous lemma (or Lemma 2.5), we define \(C_1 = \text{Ad } u'(\otimes \mathbb{N}_n - k C \otimes \otimes \mathbb{N}_n^{k+1} q)\) for the same \(k\) as above and for some unitary \(u'(\otimes \mathbb{Z} A_6 \cap A_3')\), and also defines unitaries \(v'(\otimes \mathbb{Z} A_5 \cap A_3')\) and \(z'(\otimes \mathbb{C}_1 + 1)\), in particular \(||v' - 1|| < 1/k\), \(v'\) commutes with \(q\) at any point of \(\mathbb{Z}\), and \(A d\ v'z\text{'}vU_\sigma(q_{n-k-1}x) = q_{n+k+1}x\) for \(x \in C_1\). Then it follows that \(A d\ z'z'vU_\sigma((pq)_n - k - 1) = (pq)_{n+k+1}\) for \(x \in C^*(C_1, D_1)\) and \(A d\ z'z'vU_\sigma((pqe_-)_{n+k-1}) = (pqe_+)_{n+k+1}\) for \(x \in C^*(C_1, D_1)\). We should note here that \(1_{(-n,n)}C^*(C_1, D_1)1_{(-n,n)} \subset C^*(C_1, D_1)g\) since \(1_{(-n,n)} \in C^*(C_1, D_1)g\).

Let \(\ell_1, \ell_2 \in \mathbb{N}\) be such that \(\ell_1 \geq \ell_2 \geq 1\) and let \(\ell = \ell_1 + \ell_2\). By the Rohlin property \(\sigma\) on \(\otimes \mathbb{Z} A_6 \cap A_3'\), we obtain an orthogonal family \((f_i)_{n-k-1}^{\ell - 1}\) of projections and a unitary \(v_1 \in \otimes \mathbb{Z} A_6 \cap A_3'\) such that \(v_1 \approx 1\), \(A d\ v_1\sigma(f_i) = f_{i+1}\), and \([f_i] > (2\ell + 2)^{-1}[1]\) in \(\mathbb{R} \otimes K_0(\otimes \mathbb{Z} A_6 \cap A_3')\). We define, for \(i = -\ell, -\ell + 1, \ldots, \ell\),

\[
F_i = f_i(\otimes \mathbb{N}_n^{k+\ell+i} pq).
\]

Note that \(A d\ v_1z'z'vU_\sigma(F_i) = F_{i+1}\) and \(A d\ v_1z'z'vU_\sigma(F_1_{(-n,n)}) = F_{i+1}1_{(-n,n)}\). We can then proceed just as in the proof of the previous lemma (taking \(1_{(-n,n)}\) for \(g\) this time). See also the proof of 2.7. \(\square\)

**Lemma 3.7.** Let \(A\) be a unital simple AF \(C^*\)-algebra with a unique tracial state and let \((A_n), e_-, e_+ \in A_1\) be as usual. For any \(n \in \mathbb{N}\) and \(\varepsilon > 0\), there exists a \(C^*\)-subalgebra \(C\) of \(\otimes \mathbb{Z} (A, e_-, e_+) \times_\sigma \mathbb{Z}\) such that

\[
1_C = 1_{(-n,n)},
\]

\[
C \subset (\otimes \mathbb{N}_n (A_1, e_-, e_+))',
\]

\[
C \cong M_2 \oplus M_3,
\]

\[
||[U_\sigma 1_{(-n,n-1)}, x]|| < \varepsilon ||x|| \quad \text{for} \quad x \in C.
\]
Proof. We use the previous lemma for a sufficiently small $\varepsilon > 0$, where in the conclusion we may replace $A$ by $A_k$ for a sufficiently large $k$. By changing indices we may assume that $k = 2$.

Let $\ell \in \mathbb{N}$ be such that $\ell \gg 1$ and $\ell < (3e)^{-1}$. By using the Rohlin property for $\sigma$ on $\otimes \mathbb{Z} A \cap A_2'$, we find an orthogonal family $(f_i)_{i=0}^{\ell-1}$ of projections and a unitary $v$ in $\otimes \mathbb{Z} A \cap A_2'$ such that $v \equiv 1$, $\text{Ad} v U_\sigma(f_i) = f_{i+1}$, and $[f_i] > (1 + \ell)^{-1}[1]$ in $R \otimes K_0(\otimes \mathbb{Z} A \cap A_2')$. Let $(e_j)$ be a set of matrix units in $C$. We may assume that $[1 - e] \leq [e_{11} e_{11}]$ in $K_0(\otimes \mathbb{Z} A \cap A_1')$ and let $b \in \otimes \mathbb{Z} A \cap A_1'$ be a partial isometry such that $b^* b = 1 - e$ and $bb^* \leq e_{11} e_{11}$. We define

$$y = \frac{1}{\sqrt{\ell}} \sum_{i=0}^{\ell-1} (v U_\sigma)^i b (v U_\sigma)^{-i} (1 - F).$$

Then $y$ belongs to $(\otimes \mathbb{Z} A \cap A_1') (1 - F) \subset (\otimes \mathbb{Z} A \cap A_1')'$, is close to a partial isometry

$$y_1 = \frac{1}{\sqrt{\ell}} \sum_{i=0}^{\ell-1} (v U_\sigma)^i b (v U_\sigma)^{-i} (1 - F)$$

with $y_1^* y_1 = 1 - F$, and is close to

$$y_2 = \frac{1}{\sqrt{\ell}} \sum_{i=0}^{\ell-1} (v U_\sigma)^i b (v U_\sigma)^{-i} (1 - F)$$

with $y_2 = Fe_{11} y_2$. Note also that

$$||\text{Ad} v U_\sigma(y) - y|| \leq 2 / \sqrt{\ell}.$$

Let $v_1$ be the partial isometry obtained by the polar decomposition of $Fe_{11} y_1 1_{(-n,n)} = 1_{(-n,n)} Fe_{11} y \in (\otimes \mathbb{Z} A_1')'$, which is close to a partial isometry. Then it follows that $v_1^* v_1 = 1_{(-n,n)} (1 - F)$, $v_1 v_1^* \leq 1_{(-n,n)} Fe_{11}$, and $[U, v_1] \sim 0$ for $U \equiv U_\sigma 1_{(-n,n-1)} = 1_{(-n+1,n)} U_\sigma$. Then the $C^*$-subalgebra $C_1$ generated by $C1_{(-n,n)} F$ and $v_1$ satisfies that $C_1 \subset (\otimes \mathbb{Z} A_1')'$ and $1_{C_1} = 1_{(-n,n)}$, $C_1 \cong M_2 \oplus M_3$. Moreover it follows that $||[U, x]||$ is small for $x$ in the unit ball of $C_1$. Since $\otimes \mathbb{Z} A_1, e_-, e_+ = (\otimes \mathbb{Z} A_1) 1_{(-n,n)}$, this concludes the proof.

References