JOURNAL OF DIFFERENTIAL EQUATIONS 52, 378–406 (1984)

Nonexistence of Global Solutions to Semilinear Wave Equations in High Dimensions

THOMAS C. SIDERIS

Department of Mathematics, Purdue University, West Lafayette, Indiana 47907

Received March 30, 1982; revised November 1, 1982

Consider the Cauchy problem for the semilinear wave equation

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) u(x, t) = |u(x, t)|^p \qquad (p > 1),$$

$$u(x, 0) = f(x), \qquad u_t(x, 0) = g(x) \tag{1}$$

with $f, g \in C_0^\infty(\mathbb{R}^n)$. It is well known that this problem does not admit a global solution (i.e., one defined for all $(x, t) \in \mathbb{R}^n \times [0, \infty)$) for any $p > 1$ when the initial values f and g are large in some sense (cf. [3, 9, 10]). On the other hand, John [6] has recently shown that in three space dimensions global solutions exist if $p > 1 + \sqrt{2}$ and the initial data is suitably small, and moreover, that global solutions do not exist when $1 < p < 1 + \sqrt{2}$ for any (nontrivial) choice of f and g. Interestingly, Strauss discovered the same number as the root of a dimension dependent polynomial in his work on low energy scattering for the nonlinear Klein–Gordon equation [18] (see also [7]). This led him to conjecture that the critical value, $p_0(n)$, generalizing John's result to n dimensions, should be the positive root of $(n-1)x^2 - (n+1)x - 2 = 0$. Glassey [4, 5] subsequently verified the conjecture in two dimensions by showing $p_0(2) = \frac{1}{2}(3 + \sqrt{17})$.

In this paper, one half of this question is resolved in dimensions $n > 3$. Namely, if $p_0(n)$ is the postive root of the quadratic above, then global solutions of (1) do not exist when $1 < p < p_0(n)$, provided that the initial data is compactly supported and satisfies a certain positivity condition (Theorem 2).

The main technical difficulty with the higher dimensional problem lies in the fact that the Riemann function for the wave equation is no longer a positive operator when $n > 3$. Consequently, the pointwise lower bounds of the solution which were essential in showing the solution "blows up" in two and three dimensions are not valid. Nevertheless, in Section 4 it is shown that by averaging the Riemann function in time, a positive operator results.

0022-0396/84 $3.00

When combined with a positive nonlinearity such as $|u|^p$, this yields a useful lower bound for time averages of the solution. The introduction of time averages is paid for, however, in the subsequent asymptotic analysis of space–time averages of the free solution, which involves some rather lengthy special function calculations.

The question of global existence of small solutions of (1) when $p > p_0(n)$ and $n > 3$ is still open, due again to the more complicated form of the Riemann function. Some simplification of the problem is achieved by imposing radial symmetry on the Cauchy data. Then, while it would still be technically involved, proving the analogue of John and Glassey's wieghted L^∞ estimate [6, 5] is at least conceivable. In fact, Glassey (unpublished) has done this for $n = 5$. Without radial symmetry the problem requires some new idea.

Section one is devoted to the construction of local solutions of (1) in $L^q(\mathbb{R}^n)$ (Theorem 1). A more classical space cannot be used since in higher dimensions the usual energy methods are not adequate when the nonlinearity is not smooth. The nonexistence argument of the following sections applies to weak solutions, so there actually do exist solutions of (1) which are small initially and which break down in finite time, provided p lies in the critical range. In contrast to this, the dispersive equation

$$u_{tt} - \Delta u = -|u|^{p-1}u$$

possesses a positive definite energy form which can be used to extablish the existence of global weak solutions for all n and $p > 1$ [17]. Together, the two examples illustrate the importance of both the strength and dispersive character of the nonlinearity in determining the existence of global solutions.

The present work improves, in part, a result of Kato [8] which showed that global solutions of (1) do not exist in n space dimensions when $1 < p \leqslant (n+1)/(n-1)$ $(<p_0(n))$, provided the initial data is compactly supported and g is positive on the average. Actually, Kato's theorem applies to more general hyperbolic equations. The proof of his result, for the case at hand, will be outlined in Section 2 along with a brief description of the work of Glassey and John.

It should be mentioned that the corresponding problem for the parabolic equation

$$u_t - \Delta u = u^\alpha \qquad \text{on } \mathbb{R}^n$$

was solved earlier by Fujita [20]. Here the critical power is $\alpha_0(n) = 1 + 2/n$. Weissler [19] has shown that the critical power $\alpha_0(n)$ belongs to the blow up case. For the hyperbolic equation (1), this has recently been done in three dimensions by Schaeffer [15].

1. Existence of Local Solutions

In this section, local solutions of the Cauchy problem for the semilinear wave equation

$$u_{tt} - \Delta u = |u|^p$$

are constructed in $L^q(\mathbb{R}^n)$, $n \geq 2$. Although the approach used here is the standard contraction argument, the details are given in order to demonstrate the particular continuity and support properties of the solution required in the next section.

First, a few lemmas concerning the homogeneous wave equation are necessary. Define the tempered distributions $R(t)$ and $R'(t)$ by

$$\langle R(t), \phi\rangle = \int_{\mathbb{R}^n} |\xi|^{-1} \sin t\,|\xi|\, \hat{\phi}(\xi)\, d\xi$$

and

$$\langle R'(t), \phi\rangle = \int_{\mathbb{R}^n} \cos t\,|\xi|\, \hat{\phi}(\xi)\, d\xi,$$

where $\phi \in \mathscr{S}$. The solution in the sense of distributions of the homogeneous wave equation with initial data f and g in $\mathscr{S}'$ can be expressed as

$$u^0(t) = R'(t) * f + R(t) * g.$$

The following lemma is well known (cf. [14, p. 309]).

Lemma 1. *If $f, g \in \mathscr{S}'$ and* $\operatorname{supp} f, g \subset \{|x| < k\}$ *for some $k > 0$, then for every $t > 0$* $\operatorname{supp} u^0(t) \subset \{|x| < k + t\}$.

Throughout this paper, fix the indices

$$q = \frac{2(n+1)}{n-1} \qquad \text{and} \qquad r = \frac{2(n+1)}{n+3}.$$

Lemma 2. *If $g \in L^r(\mathbb{R}^n)$, $n \geq 2$, then $R(t) * g \in C((0, \infty); L^q(\mathbb{R}^n)$ and $\|R(t) * g\|_{L^q} \leq Ct^{-(n-1)/(n+1)} \|g\|_{L^r}$, for $t > 0$.*

Proof. The inequality is due to Strichartz [16], and the continuity follows from it at once.

Lemma 3. *If $f \in H^1(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n)$, $n \geq 2$, then $u^0(t) \in C(\mathbb{R}; L^q(\mathbb{R}^n))$.*

Proof. $H^1(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$ by the Sobolev imbedding theorem, since

$$\frac{1}{q} \geqslant \frac{1}{2} - \frac{1}{n}.$$

Thus,

$$\begin{aligned}
&\|[R'(t+\Delta t) - R'(t)] * f\|_{L^q} \\
&\quad \leqslant C \|[R'(t+\Delta t) - R'(t)] * f\|_{H^1} \\
&\quad = C \|(1+|\xi|^2)^{1/2} [\cos(t+\Delta t)|\xi| - \cos t|\xi|] \hat{f}(\xi)\|_{L^2} \\
&\quad \to 0, \quad \text{as } \Delta t \to 0.
\end{aligned}$$

The continuity of $R(t) * g$ follows from the inequality

$$\begin{aligned}
&\|[R(t+\Delta t) - R(t)] * g\|_{L^q}^2 \\
&\quad \leqslant C \left\| (1+|\xi|^2)^{1/2} \left[\frac{\sin(t+\Delta t)}{|\xi|} - \frac{\sin t|\xi|}{|\xi|} \right] \hat{g}(\xi) \right\|_{L^2}^2 \\
&\quad \leqslant C|\Delta t| \int_{|\xi|<1} (1+|\xi|^2) |\hat{g}(\xi)|^2 \, d\xi \\
&\qquad + C \int_{|\xi|>1} [\sin(t+\Delta t)|\xi| - \sin t|\xi|]^2 |\hat{g}(\xi)|^2 \, d\xi. \quad ∎
\end{aligned}$$

As is standard practice, the following local existence theorem is formulated in terms of the integrated form of the equation. The lack of smoothness in the nonlinearity makes it convenient to work with strong solutions rather than classical ones.

THEOREM 1. *Let $f \in H^1(\mathbb{R}^n)$, $g \in L^2(\mathbb{R}^n)$, $n \geqslant 2$, with* $\operatorname{supp} f$, $g \subset \{|x| < k\}$, *and suppose $1 \leqslant p \leqslant (n+3)/(n-1)$. Then there exists a $T > 0$ and a unique solution $u(t) \in C([0, T]; L^q(\mathbb{R}^n))$ of the integral equation*

$$u(t) = u^0(t) + \int_0^t R(t-\tau) * |u(\tau)|^p \, d\tau,$$

with $\operatorname{supp} u(t) \subset \{|x| < k + t\}$.

Proof. Let $X(T) = \{u \in C([0, T];\ L^q(\mathbb{R}^n))\colon\ \operatorname{supp} u(t) \subset \{|x| < k + t\}\}$; $X(T)$ is a Banach space with the norm $\|u\| = \sup_{0 \leqslant t \leqslant T} \|u(t)\|_{L^q}$. Define the operator

$$Au(t) = u^0(t) + \int_0^t R(t-\tau) * |u(\tau)|^p \, dt$$

on $X(T)$.

Lemmas 1, 3 show that $u^0 \in X(T)$ and, by Lemmas 1, 2, the integral term maps $X(T)$ into itself. If T is sufficiently small, A is a contraction on $\{\|u-u^0\| \leqslant 1\}$. Thus, the contraction principle guarantees the existence of a unique local solution. ∎

It is possible to construct local solutions for $1 \leqslant p \leqslant (n+3)/(n-1)$ when the initial data is not compactly supported. To do this, the indices q and r of Lemma 2 must be chosen so that $rp = q$. Strichartz's estimates do not permit this when p is close to one, however, the more comprehensive estimates of Marshall, Strauss, and Wainger [12] make this choice possible.

The case $p > (n+3)/(n+1)$ requires a different approach, and since such results are not needed here they have been left to be considered along with the still open question of global existence. Pecher [13] has also recently used the L^q estimates of Strichartz in connection with semilinear wave equations.

2. Nonexistence of Global Solutions

This section contains a statement of the main result, as well as the essential steps in its proof. Several important technical lemmas will be used which are proved in the later sections. Before proceeding, let us sketch briefly the proofs of Kato and John's blowup theorems so as to set the stage for the forthcoming arguments.

Suppose $u(x, t)$ is a smooth solution of

$$u_{tt} - \Delta u = |u|^p \tag{2}$$

on $\mathbb{R}^n \times [0, T]$, with $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$. Assume that supp f, $g \subset \{|x| < k\}$. By Theorem 1, $\operatorname{supp} u(t) \subset \{|x| < k + t\}$. If Eq. (2) is integrated with respect to the spatial variables, one obtains

$$\frac{d^2}{dt^2} \int_{\mathbb{R}^n} u(x, t)\, dx = \int_{\mathbb{R}^n} |u(x, t)|^p\, dx, \tag{3}$$

since by the divergence theorem, $\int_{\mathbb{R}^n} \Delta u(x, t)\, dx = 0$. Let

$$F(t) = \int_{\mathbb{R}^n} u(x, t)\, dx. \tag{4}$$

Using the compact support of $u(\cdot, t)$ and Hölder's inequality, it follows from (3) and (4) that

$$\ddot{F}(t) \geqslant (k+t)^{-n(p-1)} |F(t)|^p, \qquad 0 \leqslant t \leqslant T. \tag{5}$$

Thus, $\ddot{F}(t) \geqslant 0$, and so $F(t) \geqslant \dot{F}(0)t + F(0)$. Now $\dot{F}(0) = \int_{\mathbb{R}^n} u_t(x, 0)\, dx = \int_{\mathbb{R}^n} g(x)\, dx \equiv C_g$. If $C_g > 0$, then

$$F(t) \geqslant (\text{pos. const.})\, t, \qquad t \quad \text{large}. \tag{6}$$

Lemma 4 will show that any function satisfying (5) and (6) cannot remain finite if $1 < p < (n+1)/(n-1)$. Hence, $T < \infty$. This is a special case of Kato's theorem [8].

Glassey has observed in [4] that by improving the lower bound (6) one can obtain sharper results. For example, the solution of (2) satisfies the integral equation

$$u(t) = u^0(t) + \int_0^t R(t-\tau) * |u(\tau)|^p\, d\tau. \tag{7}$$

In three or fewer dimensions the Riemann function $R(t-\tau)*$ is a positive operator. Therefore, for $n = 3$, (7) implies that

$$u(x, t) \geqslant u^0(x, t). \tag{8}$$

Since $u^0_{tt} - \Delta u^0 = 0$, one obtains upon integration, that $(d^2/dt^2) \int_{\mathbb{R}^3} u^0(x, t)\, dx = 0$. Hence,

$$\int_{\mathbb{R}^3} u^0(x, t)\, dx = C_g t + C_f, \tag{9}$$

where $C_g = \int g\, dx$ and $C_f = \int f\, dx$.

In three dimensions, the strong Huygen's principle states that

$$\operatorname{supp} u^0(x, t) \subset \{t - k < |x| < t + k\}, \qquad t > k. \tag{10}$$

Combining (8)–(10), one has

$$\begin{aligned} C_g t + C_f &= \int_{\mathbb{R}^n} u^0(x, t)\, dx = \int_{\{t-k<|x|<t+k\}} u^0(x, t)\, dx \\ &\leqslant \int_{\{t-k<|x|<t+k\}} u(x, t)\, dx \\ &\leqslant \operatorname{vol}\{t - k < |x| < t + k\}^{(p-1)/p} \left(\int_{\mathbb{R}^3} |u(x, t)|^p\, dx \right)^{1/p} \\ &\leqslant C(t+k)^{2(p-1)/p} \left(\int_{\mathbb{R}^3} |u(x, t)|^p\, dx \right)^{1/p}. \end{aligned} \tag{11}$$

Assuming that $C_g > 0$ again, (4) and (11) show that

$$\ddot{F}(t) \geqslant (\text{pos. const.})\, t^{2-p}, \qquad t \text{ large.}$$

Integrating twice, one has

$$F(t) \geqslant (\text{pos. const.})\, t^{4-p}, \qquad t \text{ large.} \tag{12}$$

This is the desired improvement of (6) in three dimensions. Now (5) and (12) imply, via Lemma 4, that $T < \infty$, provided $1 < p < 1 + \sqrt{2}$. This is essentially John's result [6] except for the additional assumption $C_g > 0$.

Glassey has found the corresponding argument in two dimensions. The estimates are considerably more difficult, due to the lack of a strong Huygen's principle in two dimensions. The key feature in both cases is the use of the positivity of the Riemann function $R(t)$. This positivity no longer obtains in higher dimensions. However, the difficulty can be circumvented by further averaging with respect to the time variable.

Let us now turn to the main result. $p_0(n)$ will denote the positive root of the quadratic $(n-1)x^2 - (n+1)x - 2 = 0$. Note that $1 < p_0(n) < (n+3)/(n-1)$. Hence, for $1 < p < p_0(n)$, let $u(t)$ be the local solution of the integral equation

$$u(t) = u^0(t) + \int_0^t R(t-\tau) * |u(\tau)|^p \, d\tau$$

in $C([0, T_0); L^q(\mathbb{R}^n))$ with initial values $f, g \in C_0^\infty(\mathbb{R}^n)$, as constructed in Theorem 1. Assume that T_0 is maximal, in the sense that the solution $u(t)$ cannot be defined on any interval containing $[0, T_0)$.

Define $\eta = \eta(n)$ to be 0 if n is odd and $\frac{1}{2}$ if n is even.

THEOREM 2. *Suppose $n > 3$ and $1 < p < p_0(n)$. If $\int_{\mathbb{R}^n} |x|^{n-1} f(x)\, dx$ and $\int_{\mathbb{R}^n} |x|^n g(x)\, dx$ are positive, then T_0 is necessarily finite.*

Proof. The first step is to obtain a differential inequality. Although the functional $F(t)$ will be different from the one used in the previous examples, let us begin by integrating the equation.

Let $\phi \in \mathscr{S}$. Since $u(t)$ and $R(t-\tau) * |u(\tau)|^p$ $(0 \leqslant \tau \leqslant t)$ both lie in $L^q(\mathbb{R}^n)$ and are supported in $\{|x| < k + t\}$,

$$\langle u(t), \phi\rangle = \langle u^0(t), \phi\rangle + \int_0^t \langle R(t-\tau) * |u(\tau)|^p, \phi > d\tau. \tag{13}$$

It follows from (13) that $\langle u(t), \phi\rangle \in C^2[0, T_0)$ and

$$\frac{d^2}{dt^2}\langle u(t), \phi\rangle = \langle u(t), \Delta\phi\rangle + \langle |u(t)|^p, \phi\rangle. \tag{14}$$

Fix $0 < T_1 < T_0$ and take $\phi_0 \in \mathscr{S}$ with $\phi_0 \equiv 1$ on $\{|x| < k + T_1\}$. Then for all $0 \leqslant t \leqslant T_1$, $\langle u(t), \phi_0 \rangle = \int_{\mathbb{R}^n} u(t)\, dx$, and (14) implies

$$\frac{d^2}{dt^2} \int u(x, t)\, dx = \int |u(x, t)|^p\, dx, \tag{15}$$

since $\Delta\phi \equiv 0$ on $\{|x| < k + T_1\}$. Equation (15) actually holds on $[0, T_0)$ since T_1 was arbitrary. (In fact, (15) is also valid for the weak solutions of [17].)

Now, for $k \leqslant t < T_0$ (if $T_0 \leqslant k$ there is nothing to prove) define

$$F(T) = \int_{T-k}^{T} (T-t)^m \int_{\mathbb{R}^n} u(x, t)\, dx\, dt,$$

where

$$\begin{aligned} m &= \tfrac{1}{2}(n-5) \qquad \text{if } n \text{ is odd,} \\ &= \tfrac{1}{2}(n-4) \qquad \text{if } n \text{ is even.} \end{aligned}$$

(The reason for this choice will be made apparent in Section 4.) Since $a(t) \equiv \int_{\mathbb{R}^n} u(x, t)\, dx$ is a C^2 function of t, it follows from integrating by parts that

$$\begin{aligned} F(t) = {} & \frac{k^{m+1}}{m+1} a(T-k) + \frac{k^{m+2}}{(m+1)(m+2)} \dot{a}(T-k) \\ & + \frac{1}{(m+1)(m+2)} \int_{T-k}^{T} (T-t)^{m+2} \ddot{a}(t)\, dt. \end{aligned}$$

Differentiating this twice and using (15), one sees that

$$\ddot{F}(T) = \int_{T-k}^{T} (T-t)^m \int_{\mathbb{R}^n} |u(x, t)|^p\, dx\, dt.$$

Therefore, by Hölder's inequality and the compact support of $u(\cdot, t)$

$$\ddot{F}(T) \geqslant C(k+T)^{-n(p-1)} |F(T)|^p, \tag{16}$$

for some $C > 0$. In Section 5, it will be shown that

$$F(T) \geqslant \text{pos. const. } (k+T)^{n+1-p(n-1)/2}, \tag{17}$$

for T large. However, once this is done, (16), (17), and Lemma 4 imply that $T_0 < \infty$, provided $1 < p < p_0(n)$. ∎

3. An Ordinary Differential Inequality

Lemma 4, required for the proof of Theorem 2, appeared implicitly in [4].

LEMMA 4. *Suppose $F(t) \in C^2[a, b)$, and for $a \leqslant t < b$,*

$$F(t) \geqslant C_0(k+t)^{r}, \tag{18}$$

$$\ddot{F}(t) \geqslant C_1(k+t)^{-q} F(t)^p, \tag{19}$$

where C_0, C_1, and $k > 0$. If $p > 1$, $r \geqslant 1$, and $(p-1)r > q-2$, then b must be finite.

Proof. Since $(p-1)r > q-2$ and $r \geqslant 1$,

$$rp - q > r - 2 \geqslant -1. \tag{20}$$

By (18) and (19),

$$\ddot{F}(t) \geqslant C(k+t)^{pr-q}$$

on $[a, b)$. Upon integration, one has

$$\dot{F}(t) - \dot{F}(a) \geqslant C \int_a^t (k+s)^{pr-q}\, ds. \tag{21}$$

From (20), $pr - q \geqslant -1$ and so (21) implies that unless b is finite, $\dot{F}(t)$ must eventually be positive. Thus, one may assume there exists an a_0 such that $a < a_0 < b$ and

$$\dot{F}(t) > 0, \tag{22}$$

for all $a_0 \leqslant t < b$. It also follows from the assumptions on p, q, and r that there is a $\theta \in (0, 1)$ such that

$$\frac{1}{p} < \theta < 1 - \frac{q-2}{pr}. \tag{23}$$

Thus, interpolating between (18) and (19), one has

$$\ddot{F}(t) \geqslant C(k+t)^{(1-\theta)pr-q} F(t)^{\theta p}. \tag{24}$$

Let $\alpha = \theta p$ and $\beta = q - (1-\theta)pr$. By (23), $\alpha > 1$ and $\beta < 2$. Without loss of generality, $\beta \geqslant 0$.

By (22), one can multiply (24) by $\dot{F}(t)$ and integrate

$$\begin{aligned}
\tfrac{1}{2}[\dot{F}(t)^2 - \dot{F}(a_0)^2] &\geqslant C \int_{a_0}^{t} (k+s)^{-\beta} F(s)^{\alpha} \dot{F}(s)\, ds \\
&\geqslant C(k+t)^{-\beta} \int_{a_0}^{t} F(s)^{\alpha} \dot{F}(s)\, ds \qquad (25)\\
&\geqslant C(k+t)^{-\beta}[F(t)^{1+\alpha} - F(a_0)^{1+\alpha}].
\end{aligned}$$

Choose the constant C in (25) small enough so that

$$\tfrac{1}{2}\dot{F}(a_0)^2 \geqslant C(k+a_0)^{-\beta} F(a_0)^{1+\alpha}.$$

It then follows from (25) that

$$\dot{F}(t) \geqslant C(k+t)^{\beta/2} F(t)^{(1+\alpha)/2}$$

for $a_0 \leqslant t < b$. One final integration yields

$$F(a_0)^{(1-\alpha)/2} - F(t)^{(1-\alpha)/2} \geqslant C[(k+t)^{1-\beta/2} - (k+a_0)^{1-\beta/2}].$$

Since $1-(\beta/2) > 0$, it is clear that t cannot be arbitrarily large. ∎

4. The Riemann Function in High Dimensions

Although the Riemann function for the wave equation is not positive in more than three space dimensions, the following result shows that certain of its time averages are.

Lemma 5. *Suppose* $v \in C([0, T_1]; L^r(\mathbb{R}^n))$, $n \geqslant 4$, *and* $v(t) \geqslant 0$ *a.e.* $[x]$, *for every* $0 \leqslant t \leqslant T_1$. *Let*

$$w_n(x, T) = \int_0^T (T-t)^m \int_0^t R(t-\tau) * v(\tau)\, d\tau\, dt,$$

where

$$\begin{aligned}
m &= \tfrac{1}{2}(n-5) \qquad \textit{if } n \textit{ is odd},\\
&= \tfrac{1}{2}(n-4) \qquad \textit{if } n \textit{ is even}.
\end{aligned}$$

Then $w_n(\cdot, T) \in L^q(\mathbb{R}^n)$ *and* $w_n(x, T) \geqslant 0$ *a.e.* $[x]$ *for every* $0 \leqslant T \leqslant T_1$.

(Recall that $r = 2(n+1)/(n+3)$ and $q = 2(n+1)/(n-1)$.)

Proof. First assume that $v(x, t)$ is a bounded C^∞ function defined on $\mathbb{R}^n \times [0, T_1]$. $R(t-\tau) * v(\tau)$ then makes sense in $\mathscr{S}'$ and is a C^∞ function of its variables. Since

$$\phi_n(x, t) \equiv \int_0^t R(t-\tau) * v(\tau)\, d\tau$$

solves $((\partial^2/\partial t^2) - \Delta)\phi_n = v$, with zero initial data, the classical formula (cf. [1, pp. 691–692]) gives

$$\phi_n(x, t) = \frac{1}{(n-2)!} \int_0^t \left(\frac{\partial}{\partial t}\right)^{n-2} \int_0^{t-\tau} [(t-\tau)^2 - r^2]^{(n-3)/2} r Q_n(x, r; \tau)\, dr\, d\tau,$$

where where $Q_n(x, r; \tau) = (1/\omega_n) \int_{\Omega_n} v(x + r\omega, \tau)\, d\omega_n$; $d\omega_n$ being surface measure on the unit sphere $\Omega_n = \{\omega \in \mathbb{R}^n : |\omega| = 1\}$.

Suppose n is odd, and write $n = 2k + 3$. Define

$$G_n(t, \tau) \equiv \left(\frac{\partial}{\partial t}\right)^k \int_0^{t-\tau} [(t-\tau)^2 - r^2]^k r Q_n\, dr.$$

Thus,

$$\phi_n(x, t) = \frac{1}{(2k+1)!} \int_0^t \left(\frac{\partial}{\partial t}\right)^{k+1} G_n(t, \tau)\, d\tau.$$

So now, inverting the order of integration, and then integrating by parts,

$$\begin{aligned}\int_0^T (T-t)^k \phi_n(x, t)\, dt &= \int_0^T (T-t)^k \left\{\frac{1}{(2k+1)!} \int_0^t \left(\frac{\partial}{\partial t}\right)^{k+1} G_n(t, \tau)\, d\tau\right\} dt \\ &= \frac{k!}{(2k+1)!} \int_0^T G_n(T, \tau)\, d\tau \\ &\quad + \sum_{j=0}^k \frac{k!}{(2k+1)!} \int_0^T (T-\tau)^j \left[\left(\frac{\partial}{\partial t}\right)^j G_n(t, \tau)\right]_{t=\tau} d\tau. \end{aligned} \tag{26}$$

The claim is that $[(\partial/\partial t)^j G_n(t, \tau)]_{t=\tau} = 0$, for $j = 0, 1, \ldots, k$. Since $[(t-\tau)^2 - r^2]^k = (t-\tau-r)^k (t-\tau+r)^k$, it follows from Leibnitz' rule that

$$G_n(t, \tau) = \int_0^{t-\tau} \sum_{i=0}^k C_i (t-\tau-r)^i (t-\tau+r)^{k-i} r Q_n\, dr, \tag{27}$$

with $C_i > 0$. Let $p_n(s, r) = \sum_{i=1}^k C_i (s-r)^i (s+r)^{k-i}$ and $p_n^{(j)}(s, r) = (\partial/\partial s)^j p_n(s, r)$. Then from (27),

$$\left(\frac{\partial}{\partial t}\right)^j G_n(t,\tau) = \int_0^t p_n^{(j)}(t-\tau, r)\, rQ_n(x, r; \tau)\, dr$$
$$+ \sum_{l=0}^{j-1} \left(\frac{\partial}{\partial t}\right)^l [p_n^{(j-l)}(t-\tau, t-\tau)\cdot(t-\tau)\cdot Q_n(x, t-\tau; \tau)]. \tag{28}$$

Now, $(t-\tau)\, p_n^{(j-l)}(t-\tau, t-\tau)$ is homogeneous of degree $k-(j-l)+1$ in $(t-\tau)$, so

$$\left(\frac{\partial}{\partial t}\right)^l [p_n^{(j-l)}(t-\tau, t-\tau)\cdot(t-\tau)\cdot Q_n(x, t-\tau; \tau)]$$

is homogeneous of degree $k-j+1$ in $(t-\tau)$. Since $j \leqslant k$, it follows from (28) that

$$\left| \left(\frac{\partial}{\partial t}\right)^j G_n(t,\tau) \right|_{t=\tau} = 0.$$

Hence, from (26)

$$\int_0^T (T-t)^k\, \phi_n(x,t)\, dt = \frac{k!}{(2k+1)!} \int_0^T G_n(T,\tau)\, d\tau.$$

One differentiation of this with respect to T yields

$$w_n(x,t) \equiv \int_0^T (T-t)^{k-1}\, \phi_n(x,t)\, dt$$
$$= \frac{(k-1)!}{(2k+1)!} \left\{ \int_0^T \int_0^{T-\tau} p_n^{(1)}(T-\tau, r)\, rQ_n(x, r; \tau)\, dr\, d\tau \right\}$$
$$+ \int_0^T p_n(T-\tau, T-\tau)\cdot(T-\tau)\cdot Q_n(x, T-\tau; \tau)\, d\tau.$$

Therefore, if $v \geqslant 0$, then $w_n \geqslant 0$.

If n is even, reverse the descent method (cf. [1, p. 686]).

$$\phi_n(x,t) = \frac{1}{(n-2)!} \int_0^t \left(\frac{\partial}{\partial t}\right)^{n-2} \int_0^{t-\tau} [(t-\tau)^2 - r^2]^{(n-3)/2} rQ_n(x, r; \tau)\, dr\, d\tau$$
$$= \frac{1}{(n-1)!} \int_0^t \left(\frac{\partial}{\partial t}\right)^{n-1} \int_0^{t-\tau} [(t-\tau)^2 - r^2]^{(n-2)/2} rQ_{n+1}(x, r; \tau)\, dr\, d\tau$$
$$= \phi_{n+1}(x,t).$$

By what was just proved for odd dimensions, one has that

$$\begin{aligned} w_n(x, t) &\equiv \int_0^T (T-t)^{(n-4)/2} \phi_n(x, t)\, dt \\ &= \int_0^T (T-t)^{((n+1)-5)/2} \phi_n + 1(x, t)\, dt \\ &= w_{n+1}(x, t), \end{aligned}$$

so that $v \geqslant 0$ implies $w_n \geqslant 0$ again.

Now suppose $v \in X \equiv C([0, T_1]; L^r(\mathbb{R}^n))$ and $v \geqslant 0$. Lemma 2 shows that the map

$$v \longmapsto w(\cdot, T)$$

is bounded from X into $L^q(\mathbb{R}^n)$. Let $\{v_j\}$ be a sequence of positive C^∞ functions in X which converge to v in X. The corresponding $w_j(\cdot, T)$ converge to $w(\cdot, T)$ in L^q, and so some subsequence $w_i(x, T)$ converges to $w(x, T)$, a.e. $[x]$ for every $0 \leqslant T \leqslant T_1$. By the preceding argument, $w_i(x, T) \geqslant 0$, so the lemma follows. ∎

Again let us point out that this portion of the argument is valid for the weak solutions of [17] since they are obtained as limits is strong solutions.

5. Asymptotics

The key step remaining in the proof of Theorem 2 is the estimate

$$F(T) \geqslant \text{pos. const. } (k + T)^{n+1-p(n-1)/2} \tag{29}$$

for T large. The result of Section 4 reduces the proof of (29) to the following concrete statement concerning $u^0(x, t)$, be solution of the homogeneous equation.

Lemma 6. *Suppose $u^0(x, t)$ is the solution of the homogeneous wave equation with initia data satisfying the hypotheses of Theorem 2. Then*

$$\int_{T-k}^T (T-t)^m \int_{|x|>T} u^0(x, t)\, dx\, dt \geqslant \text{pos. const. } (k + T)^{(n-1)/2}$$

for large T.

Before proving this lemma, let us use it together with Lemma 5 to deduce (29). Since the solution $u(x, t)$ satisfies the integral equation

$$u(t) = u^0(t) + \int_0^t R(t-\tau) * |u(\tau)|^p \, d\tau$$

on $[0, T_0)$, and since $|u(t)|^p \in C([0, T_1); L^r)$, Lemma 5 implies that

$$\int_0^T (T-t)^m u(x, t)\, dt \geqslant \int_0^T (T-t)^m u^0(x, t)\, dt, \tag{30}$$

a.e. $[x]$. Recalling that both $u(t)$ and $u^0(t)$ are supported in $\{|x| < k+t\}$, one may integrate both sides of (30) over the set $\{|x| > T\}$ and then invert the order of integration to obtain

$$\int_0^T (T-t)^m \int_{|x|>T} u(x, t)\, dx\, dt \geqslant \int_0^T (T-t)^m \int_{|x|>T} u^0(x, t)\, dx\, dt.$$

Because of the special support property, these integrals are actually equivalent to

$$\int_{T-k}^T (T-t)^m \int_{|x|>T} u(x, t)\, dx\, dt \geqslant \int_{T-k}^T (T-t)^m \int_{|x|>T} u^0(x, t)\, dx\, dt, \tag{31}$$

provided $T > k$. Using Hölder's inequality on the left of (31) and the lower bound of Lemma 6 on the right, one has

$$\begin{aligned} C_1(k+T)^{(n-1)(p-1)/p} &\left(\int_{T-k}^T (T-t)^m \int |u(x, t)|^p\, dx\, dt\right)^{1/p} \\ &\geqslant C_2(k+T)^{(n-1)/2}, \end{aligned}$$

for T large. This can be rewritten as

$$\int_{T-k}^T (T-t)^m \int |u(x, t)|^p\, dx\, dt \geqslant C(k+T)^{n-1-p(n-1)/2},$$

for T large. From (16), this last integral is $\ddot{F}(T)$, so after two integrations, one obtains

$$F(T) \geqslant C(k+T)^{n+1-p(n-1)/2} + AT + B,$$

for large T. However, the linear term can be neglected, for large T, since in the present case $(n+1) - p((n-1)/2) > 1$. Thus, (29) is proved.

Proof of Lemma 6. Begin by writing

$$u^0 = v^{(1)} + v_t^{(2)}, \tag{32}$$

where $v_{tt}^{(j)} - \Delta v^{(j)} = 0$, $v^{(j)}(0) = 0$, and $v_t^{(1)}(0) = g$, $v_t^{(2)}(0) = f$. Since the $v^{(j)}$ satisfy the homogeneous wave equation,

$$\int_{\mathbb{R}^n} v^{(1)}(x, t)\, dx = C_g t \qquad \text{and} \qquad \int_{\mathbb{R}^n} v_t^{(2)}(x, t)\, dx = C_f, \tag{33}$$

where $C_g = \int_{\mathbb{R}^n} g\, dx$ and $C_f = \int_{\mathbb{R}^n} f\, dx$. Let

$$I \equiv \int_{T-k}^{T} (T-t)^m \int_{r>T} u^0(x, t)\, dx\, dt,$$

where $r = |x|$. Substituting the expression in (32) for u^0, this becomes

$$I = \int_{T-k}^{T} (T-t)^m \int_{r>T} v^{(1)}(x, t)\, dx\, dt + \int_{T-k}^{T} (T-t)^m \int_{r>T} v_t^{(2)}(x, t)\, dx\, dt. \tag{34}$$

Suppose first that $n > 5$. Then $m \geqslant 1$, and the second integral above can be integrated by parts. The boundary term vanishes since $\operatorname{supp} v^{(j)} \subset \{|x| < k + t\}$, and so,

$$I = \int_{T-k}^{T} (T-t)^m \int_{r>T} v^{(1)}(x, t)\, dx\, dt + m \int_{T-k}^{T} (T-t)^{m-1} \int_{r>T} v^{(2)}(x, t)\, dx\, dt.$$

Making use of (33) and the fact that

$$\int_{r>T} v^{(j)}(x, t)\, dx = \int_{\mathbb{R}^n} v^{(j)}(x, t)\, dx - \int_{r<T} v^{(j)}(x, t)\, dx,$$

I can be written as

$$\begin{aligned} I &= O(T) - \int_{T-k}^{T} (T-t)^m \int_{r<T} v^{(1)}(x, t)\, dx\, dt \\ &\quad - m \int_{T-k}^{T} (T-t)^{m-1} \int_{r<T} v^{(2)}(x, t)\, dx\, dt \\ &= O(T) - I_1 - m I_2. \end{aligned}$$

When $n = 4$ or 5, $m = 0$. So from (34),

$$I = \int_{T-k}^{T} \int_{r>T} v^{(1)}(x, t)\, dx\, dt + \int_{r>T} v^{(2)}(x, T)\, dx - \int_{r>T} v^{(2)}(x, T-k)\, dx.$$

This last integral vanishes by the support properties of $v^{(2)}$. Thus, using (33) again, one has

$$\begin{aligned} I &= O(T) - \int_{T-k}^{T} \int_{r<T} v^{(1)}(x,t)\,dx\,dt - \int_{r<T} v^{(2)}(x,t)\,dx \\ &\equiv O(T) - I_1 - I_2^0, \end{aligned} \tag{36}$$

when $n = 4$ or 5.

After this preliminary reduction, the next step is to study the integrals I_1, I_2, and I_2^0 by means of the Fourier transform.

Since

$$v^{(1)}(x,t) = (2\pi)^{-n/2} \int e^{ix\cdot\xi} \frac{\sin t\rho}{\rho}\, \hat{g}(\xi)\, d\xi, \qquad \rho = |\xi|,$$

one has that

$$\int_{r<T} v^{(1)}(x,t)\,dx = (2\pi)^{-n/2} \int \hat{g}(\xi) \frac{\sin t\rho}{\rho} \left(\int_{r<T} e^{ix\cdot\xi}\,dx \right) d\xi.$$

Switching to polar coordinates and performing the angular integration, one obtains

$$\int_{r<T} e^{ix\cdot\xi}\,dx = \gamma_0 \int_0^T (r\rho)^{1-n/2} J_{n/2-1}(r\rho)\, r^{n-1}\,dr,$$

for some positive constant γ_0 (cf. [11, p. 79]). Because of the relation $(d/ds)[s^\nu J_\nu(s)] = s^\nu J_{\nu-1}(s)$, this last integral can be computed. Thus,

$$\int_{r<T} e^{ix\cdot\xi}\,dx = \gamma_0 \left(\frac{T}{\rho}\right)^{n/2} J_{n/2}(T\rho).$$

Hence,

$$I_1 = \gamma_1 T^{n/2} \int \hat{g}(\xi) \frac{J_{n/2}(T\rho)}{\rho^{n/2+1}} \left(\int_{T-k}^{T} (T-t)^m \sin t\rho\,dt \right) d\rho, \tag{37}$$

where $\gamma_1 = (2\pi)^{-n/2}\gamma_0$. The interchange of the order of integration is justified since the integral converges absolutely. Similarly,

$$I_2 = \gamma_1 T^{n/2} \int \hat{f}(\xi) \frac{J_{n/2}(T\rho)}{\rho^{n/2+1}} \left(\int_{T-k}^{T} (T-t)^{m-1} \sin t\rho\,dt \right) d\xi, \qquad n = 6, 7, 8, \ldots, \tag{38}$$

and

$$I_2^0 = \gamma_1 T^{n/2} \int \hat{f}(\xi) \frac{J_{n/2}(T\rho)}{\rho^{n/2+1}} \sin T\rho \, d\xi, \qquad n = 4, 5. \tag{39}$$

First, consider I_1. Define

$$\alpha(T, \rho) = \int_{T-k}^{T} (T-t)^m \sin t\rho \, dt. \tag{40}$$

Note that α is uniformly bounded. Now, substitute the asymptotic formula (cf. [11, p. 139])

$$J_{n/2}(z) = \gamma_2 \frac{\cos(z - \theta_n)}{z^{1/2}} + \frac{\phi_n(z)}{z^{1/2}}, \qquad z > 0, \tag{41}$$

where $\theta_n = ((n+1)/4)\pi$, $\gamma_2 > 0$, and $|\phi_n(z)| \leqslant C/(1+z)$, into (37) to obtain

$$\begin{aligned} I_1 &= \gamma_3 T^{(n-1)/2} \left\{ \int \hat{g}(\xi) \frac{\cos(T\rho - \theta_n)}{\rho^{(n+3)/2}} \alpha(T, \rho) \, d\xi \right. \\ &\quad \left. + \int \hat{g}(\xi) \frac{\phi_n(T\rho)}{\rho^{(n+3)/2}} \alpha(T, \rho) \, d\xi \right\} \\ &\equiv \gamma_3 T^{(n-1)/2} \{ I_{11} + I_{12} \}, \end{aligned} \tag{42}$$

where $\gamma_3 = \gamma_1 \gamma_2$. I_{12} can be estimated as

$$\begin{aligned} |I_{12}| &\leqslant C \int |\hat{g}(\xi)| \rho^{-(n+3)/2} (1 + T\rho)^{-1} \, d\xi \\ &\leqslant CT^{-\varepsilon} \int |\hat{g}(\xi)| \rho^{-((n+3)/2 + \varepsilon)} (1 + \rho)^{-1+\varepsilon} d\xi. \end{aligned}$$

This last integral is convergent for $n \geqslant 4$, so $I_{12} \leqslant CT^{-\varepsilon}$.

Make the change of variables $\lambda = T - t$ in (40) and use the double angle formula. Then,

$$\alpha(T, \rho) = \sin T\rho \int_0^k \lambda^m \cos \lambda\rho \, d\lambda - \cos T\rho \int_0^k \lambda^m \sin \lambda\rho \, d\lambda,$$

and so,

$$\begin{aligned} I_{11} &= \int \rho^{-(n+3)/2} \hat{g}(\xi) \cos(T\rho - \theta_n) \sin T\rho \int_0^k \lambda^m \cos \lambda\rho \, d\lambda \, d\xi \\ &\quad - \int \rho^{-(n+3)/2} \hat{g}(\xi) \cos(T\rho - \theta_n) \cos T\rho \int_0^k \lambda^m \sin \lambda\rho \, d\lambda \, d\xi \end{aligned}$$

$$= \int \rho^{-(n+3)/2} \hat{g}(\xi) \sin(2T\rho - \theta_n) \int_0^k \lambda^m \cos \lambda\rho \, d\lambda \, d\xi$$

$$+ \int \rho^{-(n+3)/2} \hat{g}(\xi) \sin \theta_n \int_0^k \lambda^m \cos \lambda\rho \, d\lambda \, d\xi$$

$$- \int \rho^{-(n+3)/2} \hat{g}(\xi) \cos(2T\rho - \theta_n) \int_0^k \lambda^m \sin \lambda\rho \, d\lambda \, d\xi$$

$$- \int \rho^{-(n+3)/2} \hat{g}(\xi) \cos \theta_n \int_0^k \lambda^m \sin \lambda\rho \, d\lambda \, d\xi$$

$$\equiv I_{111} + I_{112} - I_{113} - I_{114}.$$

The integrals I_{111} and I_{113} are $o(1)$. For example, let $Q_n(\rho) = \int_{\Omega_n} \hat{g}(\rho\omega) \, d\omega$ and $\mu(\rho) = \int_0^k \lambda^m \cos \lambda\rho \, d\lambda$. Using the double angle formula, one obtains

$$I_{111} = \cos \theta_n \int_0^\infty \rho^{(n-5)/2} Q_n(\rho) \, \mu(\rho) \sin(2T\rho) \, d\rho$$

$$- \sin \theta_n \int_0^\infty \rho^{(n-5)/2} Q_n(\rho) \, \mu(\rho) \cos(2T\rho) \, d\rho.$$

Since $\hat{g} \in \mathscr{S}$, Q_n is small at infinity. Also, $(n-5)/2 > -1$ because $n > 3$. Thus, $\rho^{(n-5)/2} Q_n(\rho) \, \mu(\rho) \in L^1(\mathbb{R}^1)$, and the Riemann–Lebesgue lemma implies that both integrals are $o(1)$. An identical argument applies to I_{113}. Hence,

$$I_{11} = I_{112} - I_{114} + o(1).$$

Let us now study the integral I_{112}. This integral converges absolutely, so the dominated convergence theorem justifies writing

$$I_{112} = \sin \theta_n \lim_{R \to \infty} \int_{\rho < R} \rho^{-(n+3)/2} \, \hat{g}(\xi) \int_0^k \lambda^m \cos \lambda\rho \, d\lambda \, d\xi.$$

Label the truncated integral I_{112}^R. Replacing $\hat{g}(\xi)$ by its definition, one has

$$I_{112}^R = (2\pi)^{-n/2} \int_{\rho < R} \rho^{-(n+3)/2} \left(\int e^{-i\xi \cdot y} g(y) \, dy \right) \int_0^k \lambda^m \cos \lambda\rho \, d\lambda \, d\xi.$$

Now, because of the truncation, the iterated integral converges absolutely, and Fubini's theorem yields

$$I_{112}^R = (2\pi)^{-n/2} \int g(y) \int_0^k \lambda^m \int_{\rho < R} \rho^{-(n+3)/2} \cos \lambda\rho \, e^{-i\xi \cdot y} \, d\xi \, d\lambda \, dy.$$

Change to polar coordinates in the inner integral, and perform the angular integration. As before,

$$I_{112}^{R} = (2\pi)^{-n/2}\gamma_0 \int g(y) \int_0^k \lambda^m \int_0^R \rho^{-(n+3)/2} \cos \lambda\rho \frac{J_{n/2-1}(r\rho)}{(r\rho)^{n/2-1}} \rho^{n-1}\, d\rho\, d\lambda\, dy,$$

where $r = |y|$.

Denote the innermost integral by $F_R(r, \lambda)$. Then,

$$\begin{aligned} |F_R(r,\lambda)| &= \left| \int_0^R \rho^{(n-5)/2} \cos \lambda\rho \frac{J_{n/2-1}(r\rho)}{(r\rho)^{n/2-1}}\, d\rho \right| \\ &\leqslant \int_0^1 \rho^{(n-5)/2} \left| \frac{J_{n/2-1}(r\rho)}{(r\rho)^{n/2-1}} \right| d\rho + r^{-n/2+1} \int_1^R \rho^{-3/2} |J_{n/2-1}(r\rho)|\, d\rho \\ &\leqslant C(1 + r^{1-n/2}), \end{aligned}$$

since $|J_\nu(s)/s^\nu| \leqslant C$, provided $\nu \geqslant 0$. Now since $|g(y) \int_0^k \lambda^m F_R(r,\lambda)\, d\lambda|$ is bounded by the L^1 function $C(1 + r^{1-n/2})\,|g(y)|$, the dominated convergence theorem implies

$$\begin{aligned} I_{112} &= (2\pi)^{-n/2}\gamma_0 \sin\theta_n \lim_{R\to\infty} \int g(y) \int_0^k \lambda^m F_R(r,\lambda)\, d\lambda\, dy \\ &= \gamma_1 \sin\theta_n \int g(y) \lim_{R\to\infty} \int_0^k \lambda^m F_R(r,\lambda)\, d\lambda\, dy. \end{aligned}$$

$F_R(r, \lambda)$ is uniformly bounded in λ so, in fact,

$$I_{112} = \gamma_1 \sin\theta_n \int g(y) \int_0^k \rho^{(n-5)/2} \cos\lambda\rho \frac{J_{n/2-1}(r\rho)}{(r\rho)^{n/2-1}}\, d\rho\, d\lambda\, dy.$$

Recalling that $J_{-1/2}(z) = (2/\pi z)^{1/2} \cos z$, one may write

$$I_{112} = \gamma_4 \sin\theta_n \int r^{1-n/2} g(y) \int_0^k \lambda^{m+1/2} \int_0^\infty \rho^{-1} J_{-1/2}(\lambda\rho)\, J_{n/2-1}(r\rho)\, d\rho\, d\lambda\, dy.$$

where $\gamma_4 = (2/\pi)^{1/2}\gamma_1$. Let us now invoke the following useful formula attributed to Weber and Schaftheitlin (cf. [11, p. 99]).

$$\int_0^\infty \rho^{-\sigma} J_\mu(a\rho)\, J_\nu(b\rho)\, d\rho$$
$$= a^\mu 2^{-\sigma} b^{c-\mu-1} \Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu-\tfrac{1}{2}\sigma)/\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}\sigma-\tfrac{1}{2}\mu)\,\Gamma(1+\mu) \tag{43}$$
$$\times\, {}_2F_1(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu-\tfrac{1}{2}\sigma, \tfrac{1}{2}+\tfrac{1}{2}\mu-\tfrac{1}{2}\nu-\tfrac{1}{2}\sigma; \mu+1; a^2/b^2)$$

valid for

$$\operatorname{Re}(\nu+\mu-\sigma+1)>0, \qquad \operatorname{Re}\sigma>-1, \quad 0<a<b.$$

Here, Γ represents the gamma function and ${}_2F_1$ is the hypergeometric function. If either $\frac{1}{2}(1+\nu+\sigma-\mu)$ or $1+\mu$ is a nonpositive integer (i.e., a pole of Γ), then the integral vanishes. Of course, if $0<b<a$, the formula holds with the indices μ and ν reversed.

Thus, since $r<k$,

$$I_{112} = \gamma_4 \sin\theta_n \left\{ \left(\Gamma\left(\frac{n-3}{4}\right)\middle/ 2\Gamma\left(\frac{n+3}{4}\right)\Gamma\left(\frac{1}{2}\right)\right) \int r^{(3-n)/2} g(y) \right.$$
$$\times \int_0^r \lambda^{m+(3-n)/2}\, {}_2F_1\left(\frac{n-3}{4}, \frac{1-n}{4}; \frac{1}{2}; \frac{\lambda^2}{r^2}\right) d\lambda\, dy$$
$$+ \left(\Gamma\left(\frac{n-3}{4}\right)\middle/ 2\Gamma\left(\frac{5-n}{4}\right)\Gamma\left(\frac{n}{2}\right)\right) \int g(y) \int_r^k \lambda^{m+(3-n)/2}$$
$$\left. \times\, {}_2F_1\left(\frac{n-3}{4}, \frac{n-1}{4}; \frac{n}{2}; \frac{r^2}{\lambda^2}\right) \cdot d\lambda\, dy \right\}.$$

An analogous calculation shows that

$$I_{114} = \gamma_4 \cos\theta_n \int r^{1-n/2} g(y) \int_0^k \lambda^{m+1/2} \int_0^\infty \rho^{-1} J_{1/2}(\lambda\rho)\, J_{n/2-1}(r\rho)\, d\rho\, d\lambda\, dy$$
$$= \gamma_4 \cos\theta_n \left\{ \left(\Gamma\left(\frac{n-1}{4}\right)\middle/ 2\Gamma\left(\frac{n+1}{4}\right)\Gamma\left(\frac{3}{2}\right)\right) \int r^{(1-n)/2} g(y) \right.$$
$$\times \int_0^r \lambda^{m+1}\, {}_2F_1\left(\frac{n-1}{4}, \frac{3-n}{4}; \frac{3}{2}; \frac{\lambda^2}{r^2}\right) d\lambda\, dy$$
$$+ \left(\Gamma\left(\frac{n-1}{4}\right)\middle/ 2\Gamma\left(\frac{7-n}{4}\right)\Gamma\left(\frac{n}{2}\right)\right) \int g(y) \int_r^k \lambda^{m+(3-n)/2}$$
$$\left. \times\, {}_2F_1\left(\frac{n-1}{4}, \frac{n-3}{4}; \frac{n}{2}; \frac{r^2}{\lambda^2}\right) \cdot d\lambda\, dy \right\},$$

where γ_4 is the same constant for both I_{112} and I_{114}.

After making the change of variables $\lambda = rz$ in the first terms of I_{112} and I_{114}, one has

$$\begin{aligned} I_{11} &\equiv I_{112} - I_{114} + o(1) \\ &= \gamma_4 \left\{ A_n \int r^{(5-n)/2+m} g(y)\, dy \right. \qquad (44) \\ &\quad \left. + B_n \int g(y) \int_r^k \lambda^{(3-n)/2}\, {}_2F_1\left(\frac{n-1}{4}, \frac{n-3}{4}; \frac{n}{2}; \frac{r^2}{\lambda^2}\right) d\lambda\, dy \right\} + o(1), \end{aligned}$$

where

$$\begin{aligned} A_n = &\left(\sin\theta_n \Gamma\left(\frac{n-3}{4}\right) \Big/ 2\Gamma\left(\frac{n+3}{4}\right) \Gamma\left(\frac{1}{2}\right)\right) \\ &\times \int_0^1 z^m\, {}_2F_1\left(\frac{n-3}{4}, \frac{1-n}{4}; \frac{1}{2}; z^2\right) dz \\ &- \left(\cos\theta_n \Gamma\left(\frac{n-1}{4}\right) \Big/ 2\Gamma\left(\frac{n+1}{4}\right) \Gamma\left(\frac{3}{2}\right)\right) \\ &\times \int_0^1 z^{m+1}\, {}_2F_1\left(\frac{n-1}{4}, \frac{3-n}{4}; \frac{3}{2}; z^2\right) dz \end{aligned}$$

and

$$\begin{aligned} B_n = &\left(\sin\theta_n \Gamma\left(\frac{n-3}{4}\right) \Big/ 2\Gamma\left(\frac{5-n}{4}\right) \Gamma\left(\frac{n}{2}\right)\right) \\ &- \left(\cos\theta_n \Gamma\left(\frac{n-1}{4}\right) \Big/ 2\Gamma\left(\frac{7-n}{4}\right) \Gamma\left(\frac{n}{2}\right)\right). \end{aligned}$$

In combining the second terms of I_{112} and I_{114}, the simple fact that ${}_2F_1(a, b; c; z) = {}_2F_1(b, a; c; z)$ has been used.

The claim is that $B_n = 0$. Indeed, if $n = 4j + 1$, $j = 1, 2,\ldots$, $\cos\theta_n = \cos(j\pi + (\pi/2)) = 0$, and $(5-n)/4 = 1 - j$ is a pole of Γ, so both terms vanish.

If $n = 4j + 3$, $j = 1, 2,\ldots$, $\sin\theta_n = \sin(j+1)\pi = 0$, and $(7-n)/4 = -j$ is a pole of Γ.

Finally, if $n = 2j$, $j = 2, 3,\ldots$, neither $(5-n)/4$ nor $(7-n)/4$ are poles of Γ. Using the functional equations

$$\Gamma(1+z) = z\Gamma(z) \qquad \text{and} \qquad \Gamma(z)\,\Gamma(-z) = \frac{-\pi}{z \sin(\pi z)}, \tag{45}$$

one has

$$\Gamma\left(\frac{5-n}{4}\right)\Gamma\left(\frac{n-1}{4}\right)=\left(\frac{1-n}{4}\right)\Gamma\left(\frac{1-n}{4}\right)\Gamma\left(\frac{n-1}{4}\right)$$

$$=-\pi\left(\frac{1-n}{4}\right)\Big/\left(\frac{n-1}{4}\right)\sin\left(\frac{n-1}{4}\right)\pi$$

$$=-\pi\Big/\cos\left(\frac{n+1}{4}\right)\pi$$

$$=-\pi/\cos\theta_n.$$

Similarly, $\Gamma((n-3)/4)\,\Gamma((7-n)/4)=-\pi/\sin\theta_n$. It follows that $B_n=0$ in this case as well.

The next task is to show that the constant A_n is strictly negative. Again, the argument is given in three cases. First, suppose $n=4j+1$, $j=1,2,\ldots.$ Then $\sin\theta_n=(-1)^j$ and $\cos\theta_n=0$. So from (44)

$$A_n=((-1)^j\,\Gamma(j-\tfrac{1}{2})/2\Gamma(\tfrac{1}{2})\,\Gamma(j+1))\int_0^1 z^{2j-2}\,{}_2F_1(j-\tfrac{1}{2},-j;\tfrac{1}{2};z^2)\,dz.$$

Let $C_j=\Gamma(j-\tfrac{1}{2})/2\Gamma(\tfrac{1}{2})\,\Gamma(j+1)$ $(C_j>0)$, and make the change of variables $\zeta=z^2$. Then

$$A_n=((-1)^j C^j/2)\int_0^1 \zeta^{j-3/2}\,{}_2F_1(j-\tfrac{1}{2},-j;\tfrac{1}{2};\zeta)\,d\zeta.$$

Because the second index is negative, ${}_2F_1(j-\tfrac{1}{2},-j;\tfrac{1}{2};\zeta)$ is a polynomial, so one may write

$$A_n=((-1)^j C_j/2)\lim_{\varepsilon\to 0+}\int_0^1 \zeta^{-1+\varepsilon}[\zeta^{j-1/2}\,{}_2F_1(j-\tfrac{1}{2},-j;\tfrac{1}{2};\zeta)]\,d\zeta$$

$$=((-1)^j C_j/2)\lim_{\varepsilon\to 0+}\left\{\frac{\zeta^\varepsilon}{\varepsilon}\zeta^{j-1/2}\,{}_2F_1(j-\tfrac{1}{2},-j;\tfrac{1}{2};\zeta)\Big|_0^1\right.$$

$$\left.-\frac{1}{\varepsilon}\int_0^1 \zeta^\varepsilon\frac{d}{d\zeta}[\zeta^{j-1/2}\,{}_2F_1(j-\tfrac{1}{2},-j;\tfrac{1}{2};\zeta)]\,d\zeta\right\}.$$

The boundary term vanishes since ${}_2F_1(j-\tfrac{1}{2},-j;\tfrac{1}{2};1)=0$ (cf. [11, p. 40]). Using the fact that

$$\frac{d}{d\zeta}[\zeta^a\,{}_2F_1(a,b;c;\zeta)]=a\zeta^{a-1}\,{}_2F_1(a+1,b;c,\zeta)$$

(cf. [11, p. 41]), one has

$$A_n = ((-1)^{j-1} C_j(j-\tfrac{1}{2})/2) \lim_{\varepsilon\to 0+} \frac{1}{\varepsilon} \int_0^1 \zeta^{j-3/2+\varepsilon} {}_2F_1(j+\tfrac{1}{2}, -j; \tfrac{1}{2}; \zeta)\, d\zeta$$

Now change variables again, let $z^2 = \zeta$, and use the fact that

$$F(j+\tfrac{1}{2}, -j; \tfrac{1}{2}; z^2) = (-1)^j 2^{2j} \binom{2j}{j}^{-1} P_{2j}(z),$$

where P_ν denotes the Legendre polynomial of order ν (cf. [11, p. 229]). Then

$$A_n = -C_j \left(j + \frac{1}{2}\right) 2^{2j} \binom{2j}{j}^{-1} \lim_{\varepsilon\to 0+} \frac{1}{\varepsilon} \int_0^1 z^{2(j-2+\varepsilon)} P_{2j}(z)\, dz.$$

This last integral can be evaluated explicitly (cf. [11, p. 231]),

$$\begin{aligned} A_n = -C_j \left(j + \frac{1}{2}\right) 2^{2j} \binom{2j}{j}^{-1} \times \lim_{\varepsilon\to 0+} \frac{1}{\varepsilon} \\ \cdot \frac{(2j-2+2\varepsilon)(2j-4+2\varepsilon)\cdots(2+2\varepsilon)(2\varepsilon)}{(2j-1+2\varepsilon)(2j+1+2\varepsilon)\cdots(4j-1+2\varepsilon)}. \end{aligned}$$

Therefore, $A_n < 0$ in this case.

The proof that $A_n < 0$ when $n = 4j+3$, $j = 1, 2, \ldots$, is similar to the case above, so the details have been omitted.

The remaining case is $n = 2j$, $j = 2, 3, \ldots$. Note that neither $(3-n)/4$ nor $(1-n)/4$ are poles of Γ. Using (45) and the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, one has

$$\begin{aligned} &\sin\theta_n \Gamma\left(\frac{n-3}{4}\right) \Big/ \Gamma\left(\frac{n+3}{4}\right) \Gamma\left(\frac{1}{2}\right) \\ &= \sin\left(\frac{n+1}{4}\right)\pi \left[\frac{-\pi}{\left(\frac{n-3}{4}\right)\sin\left(\frac{n-3}{4}\right)} \cdot \frac{1}{\Gamma\left(\frac{3-n}{4}\right)}\right] \Big/ \\ &\quad \left[\left(\frac{n-1}{4}\right)\Gamma\left(\frac{n-1}{4}\right)\right] \cdot \sqrt{\pi} \\ &= \left(\frac{4}{n-1}\right)\left(\frac{4}{n-3}\right)\Gamma\left(\frac{1}{2}\right) \Big/ \Gamma\left(\frac{3-n}{4}\right)\Gamma\left(\frac{n-1}{4}\right). \end{aligned}$$

Similarly,

$$-\cos\theta_n \Gamma\left(\frac{n-1}{4}\right)\Big/\Gamma\left(\frac{n+1}{4}\right)\Gamma\left(\frac{3}{2}\right)$$
$$=\left(\frac{4}{n-1}\right)\left(\frac{4}{n-3}\right)\Gamma\left(-\frac{1}{2}\right)\Big/\Gamma\left(\frac{n-3}{4}\right)\Gamma\left(\frac{1-n}{4}\right).$$

Therefore,

$$\begin{aligned}
\Lambda \equiv{}& \left(\sin\theta_n \Gamma\left(\frac{n-3}{4}\right)\Big/\Gamma\left(\frac{n+3}{4}\right)\Gamma\left(\frac{1}{2}\right)\right) {}_2F_1\left(\frac{n-3}{4},\frac{1-n}{4};\frac{1}{2};z^2\right)\\
&-\left(\cos\theta_n \Gamma\left(\frac{n-1}{4}\right)\Big/\Gamma\left(\frac{n+1}{4}\right)\Gamma\left(\frac{3}{2}\right)\right) z\, {}_2F_1\left(\frac{n-1}{4},\frac{3-n}{4};\frac{3}{2};z^2\right)\\
={}&\left(\frac{4}{n-1}\right)\left(\frac{4}{n-3}\right)\left[\left(\Gamma\left(\frac{1}{2}\right)\Big/\Gamma\left(\frac{n-1}{4}\right)\Gamma\left(\frac{3-n}{4}\right)\right)\right.\\
&\times {}_2F_1\left(\frac{n-3}{4},\frac{1-n}{4};\frac{1}{2};z^2\right)\\
&\left.+\left(\Gamma\left(-\frac{1}{2}\right)\Big/\Gamma\left(\frac{n-3}{4}\right)\Gamma\left(\frac{1-n}{4}\right)\right) z\, {}_2F_1\left(\frac{n-1}{4},\frac{3-n}{4};\frac{3}{2};z^2\right)\right].
\end{aligned}$$

From Erdelyi [2, p. 65],

$$\begin{aligned}
&\left(\Gamma\left(\frac{1}{2}\right)\Big/\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(b+\frac{1}{2}\right)\right) {}_2F_1\left(a,b;\frac{1}{2};z^2\right)\\
&\quad+\left(\Gamma\left(-\frac{1}{2}\right)\Big/\Gamma(a)\,\Gamma(b)\right) z\, {}_2F_1\left(a+\frac{1}{2},b+\frac{1}{2};\frac{3}{2};z^2\right)\\
&\quad=\left(1\Big/\Gamma\left(a+b+\frac{1}{2}\right)\right) {}_2F_1\left(2a,2b;a+b+\frac{1}{2};\frac{1}{2}(1-z)\right),
\end{aligned}$$

provided $a+b+\frac{1}{2}\neq 0, -1, -2, \ldots$. In the present case $a+b+\frac{1}{2}=((n-3)/4)+((1-n)/4)+\frac{1}{2}=0$. However,

$$\lim_{a+b+1/2\to 0}(1/\Gamma(a+b+\tfrac{1}{2}))\,{}_2F_1(2a,2b;a+b+\tfrac{1}{2};\zeta)$$
$$=(2a)(2b)\zeta\,{}_2F_1(2a+1,2b+1;2;\zeta)$$

(cf. [11, p. 38]). Thus,

$$A=\left(\frac{4}{n-1}\right)\left(\frac{4}{n-3}\right)\left[\left(\frac{n-3}{2}\right)\left(\frac{1-n}{2}\right)\frac{1}{2}(1-z)\right.$$
$$\left.\times\,{}_2F_1\left(\frac{n-1}{2},\frac{3-n}{2};2;\frac{1}{2}(1-z)\right)\right]$$
$$=-2(1-z)\,{}_2F_1\left(\frac{n-1}{2},\frac{3-n}{2};2;\frac{1}{2}(1-z)\right).$$

Hence,

$$A_n=\tfrac{1}{2}\int_0^1 z^m A\,dz$$
$$=\int_0^1 z^{j-2}(1-z)\,{}_2F_1(j-\tfrac{1}{2},\tfrac{3}{2}-j;2;\tfrac{1}{2}(1-z))\,dz \qquad (j=2,4,\ldots)$$
$$=-4\int_0^{1/2}(1-2\zeta)^{j-2}\zeta\,{}_2F_1(j-\tfrac{1}{2},\tfrac{3}{2}-j;2;\zeta)\,d\zeta,$$

where $\zeta=\frac{1}{2}(1-z)$. Integrating by parts, one obtains

$$A_n=-4\left\{-\frac{(1-2\zeta)^{j-1}}{2(j-1)}\zeta\,{}_2F_1\left(j-\frac{1}{2},\frac{3}{2}-j;2;\zeta\right)\Big|_0^{1/2}\right.$$
$$\left.+\frac{1}{2(j-1)}\int_0^{1/2}(1-2\zeta)^{j-1}\frac{d}{d\zeta}\left[\zeta\,{}_2F_1\left(j-\frac{1}{2},\frac{3}{2}-j;2;\zeta\right)\right]d\zeta\right\}.$$

The boundary terms vanish, so according to the formula

$$\frac{d}{d\zeta}[\zeta\,{}_2F_1(a,b;c;\zeta)]={}_2F_1(a,b;c-1;\zeta)$$

(cf. [11, p. 41]), one has

$$A_n=-\frac{2}{j-1}\int_0^{1/2}(1-2\zeta)^{j-1}\,{}_2F_1\left(j-\frac{1}{2},\frac{3}{2}-j;1;\zeta\right)d\zeta.$$

Switching to $z=(1-2\zeta)$ again, one has

$$A_n=-\frac{1}{j-1}\int_0^1 z^{j-1}\,{}_2F_1\left(j-\frac{1}{2},\frac{3}{2}-j;1;\frac{1}{2}(1-z)\right)dz.$$

The hypergeometric function in the integrand is equal to the Legendre function $P^0_{j-3/2}(z)$ (cf. [11, p. 174]). The integral can then be evaluated, with the result

$$A_n = -\frac{1}{j-1}\left[\Gamma\left(\frac{j}{2}\right)\Gamma\left(\frac{j+1}{2}\right)\Big/ 2\Gamma\left(\frac{5}{4}\right)\Gamma\left(j+\frac{1}{4}\right)\right]$$

(cf. [11, p. 192]). This shows $A_n < 0$ in the final case of even n.

Therefore, from (42) and (44), and since $B_n = 0$,

$$I_1 = \gamma_3\gamma_4 A_n T^{(n-1)/2} \int |y|^\eta g(y)\, dy + o(T^{(n-1)/2}), \tag{46}$$

where

$$\begin{aligned} \eta &= 0, & &n \text{ odd} \quad \text{and } A_n < 0, \\ &= \tfrac{1}{2}, & &n \text{ odd} \quad \text{and } A_n < 0. \end{aligned}$$

It follows from (38) by arguments identical to those for I_1, that

$$I_2 = \gamma_3\gamma_4 A_n' T^{(n-1)/4} \int |y|^{\eta-1} f(y) dy + o(T^{(n-1)/2}), \quad n \geqslant 6, \tag{47}$$

where

$$\begin{aligned} A_n' = &\left(\sin\theta_n \Gamma\left(\frac{n-3}{4}\right)\Big/\Gamma\left(\frac{n+3}{4}\right)\Gamma\left(\frac{1}{2}\right)\right) \\ &\times \int_0^1 z^{m-1}\, {}_2F_1\left(\frac{n-3}{4}, \frac{1-n}{4}; \frac{1}{2}; z^2\right) dz \\ &- \left(\cos\theta_n \Gamma\left(\frac{n-1}{4}\right)\Big/\Gamma\left(\frac{n+1}{4}\right)\Gamma\left(\frac{3}{2}\right)\right) \\ &\times \int_0^1 z^m\, {}_2F_1\left(\frac{n-1}{4}, \frac{3-n}{4}; \frac{3}{2}; z^2\right) dz. \end{aligned}$$

That the constants A_n' are strictly negative can be proved as was done for the A_n. In fact, the argument is simpler since the ε's introduced in evaluating A_n for odd dimension are unnecessary.

It remains to examine I_2^0 ($n = 4, 5$). Again, one uses the asymptotic expansion (41), this time in (39),

$$\begin{aligned} I_2^0 = \gamma_3 T^{(n-1)/2} &\left\{\int \rho^{-(n+3)/2} \hat{f}(\xi) \sin T\rho \cos(T\rho - \theta_n)\, d\xi \right. \\ &\left. + \int \rho^{-(n+3)/2} \hat{f}(\xi) \sin T\rho\, \phi(T\rho)\, d\xi \right\} \\ \equiv \gamma_3 T^{(n-1)/2} &\{I_{21}^0 + I_{22}^0\}. \end{aligned}$$

As before, $I_{22}^0 = o(1)$. By an obvious trigonometric identity, I_{21}^0 can be rewritten as

$$I_{21}^0 = \int \rho^{-(n+3)/2} \hat{f}(\xi) \sin(T\rho + \theta_n)\, d\xi$$

$$+ \int \rho^{-(n+3)/2} \hat{f}(\xi) \sin \theta_n \, d\xi$$

$$\equiv I_{211}^0 + I_{212}^0.$$

I_{211}^0 is $o(1)$ as $T \to \infty$, by the Riemann–Lebesgue lemma. I_{212}^0 is treated as was I_{112} earlier:

$$I_{212}^0 = \gamma_1 \sin \theta_n \int r^{1-n/2} f(y) \int_0^\infty \rho^{-3/2} J_{n/2-1}(r\rho)\, d\rho\, dy$$

$$= \gamma_1 \sin \theta_n \int r^{3/2-n/2} f(y) \int_0^\infty \sigma^{-3/2} J_{n/2-1}(\sigma)\, d\sigma\, dy,$$

where $\sigma = r\rho$. The inner integral can be evaluated explicitly (cf. [11, p. 91]),

$$\int_0^\infty \sigma^{-3/2} J_{n/2-1}(\sigma)\, d\sigma = \frac{\Gamma((n-3)/4)}{2^{3/2}\Gamma((n+3)/4)} \equiv E_n.$$

Hence,

$$I_2^0 = CB_n^0 T^{(n-1)/2} \int r^{n+1} f(y)\, dy + o(T^{(n-1)/2}), \tag{48}$$

where $B_n^0 = \sin \theta_n E_n < 0$, since $n = 4$ or 5.

Combining (46)–(48) with (35) and (36), one has, because of the assumptions concerning f and g, that

$$I = (\text{pos. const.})\, T^{(n-1)/2} + o(T^{(n-1)/2}).$$

The lemma follows from this. ∎

An interesting feature of this result is that while the integral $\int_{T-k}^T (T-t)^m \int u^0(x,t)\, dx\, dt$ behaves like T at infinity, an integral over a smaller set, $\int_{T-k}^T (T-t)^m \int_{r>T} u^0(x,t)\, dx\, dt$, behaves like $T^{(n-1)/2}$. Thus, there is some cancellation in the first integral which indicates that the free solution $u^0(x,t)$ oscillates to some degree. Of course, this is not surprising for a solution to the wave equation.

Let us also remark that the growth rate $T^{(n-1)/2}$ for I is optimal in view of the standard decay estimate

$$\|u^0(\cdot, t)\|_{L^\infty} \leqslant Ct^{-(n-1)/2}.$$

Thus,

$$I \leqslant CT^{-(n-1)/2} \int_{T-k}^{T} (T-t)^m \int_{T<r<k+T} dx\, dt$$
$$\leqslant CT^{(n-1)/2}.$$

It is possible to weaken slightly the hypothesis that $\int |y|^n g(y)\, dy$ and $\int |y|^{n-1} f(y)\, dy$ both be positive. An examination of the proof shows that, in fact, only a certain linear combination

$$C_1 \int |y|^n g(y)\, dy + C_2 \int |y|^{n-1} f(y)\, dy \qquad (C_1, C_2 > 0)$$

need be positive.

Finally, let us mention that the only quantities which actually blow up are the L^q norm (for which there is local existence) and anything which can be bounded below by the L^q norm. This does not include the quantity $F(T)$.

Acknowledgment

It is a pleasure to acknowledge the invaluable assistance of Professor R. T. Glassey, under whose direction this research was carried out, as part of the author's doctoral dissertation at Indiana University.

References

1. R. Courant and D. Hilbert, "Methods of Mathematical Physics, II," Interscience, New York, 1962.
2. A. Erdelyi, "Higher Transcendental Functions," Vol. 1, McGraw–Hill, New York, 1953.
3. R. T. Glassey, Blow-up theorems for nonlinear wave equations, *Math. Z.* **132** (1973), 183–203.
4. R. T. Glassey, Finite-time blow-up for solutions of nonlinear wave equations, *Math. Z.* **177** (1981), 323–340.
5. R. T. Glassey, Existence in the large for $\Box u = F(u)$ in two space dimensions, *Math. Z.* **178** (1981), 233–261.
6. F. John, Blow-up of solutions of nonlinear wave equations in three space dimensions, *Manuscripta Math.* **28** (1979), 235–268.
7. S. Kadekawa, "On the Nonlinear Schrödinger Equation," Ph.D. thesis, Indiana University, 1980.
8. T. Kato, Blow-up of solutions of some nonlinear hyperbolic equations, *Comm. Pure Appl. Math.* **33** (1980), 501–505.
9. J. Keller, On solutions of nonlinear wave equations, *Comm. Pure Appl. Math.* **10** (1957), 523–532.
10. H. Levine, Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = -Au + F(u)$, *Trans. Amer. Math. Soc.* **192** (1974), 1–21.

11. W. MAGNUS, F. OBERHETTINGER, AND R. P. SONI, "Formulas and Theorems for the Special Functions of Mathematical Physics," Springer-Verlag, New York/Berlin, 1966.
12. B. MARSHALL, W. STRAUSS, AND S. WAINGER, L^p–L^q estimates for the Klein–Gordon equation, *J. Math. Pures Appl.* **59** (1980), 417–440.
13. H. PECHER, L^p-Abschätzungen und Klassische Lösungen für nichtlineare Wellengleichungen, I, *Math. Z.* **150** (1976), 159–183.
14. M. REED AND B. SIMON, "Methods of Modern Mathematical Physics, II: Fourier Analysis, Self Adjointness," Academic Press, New York, 1975.
15. J. SCHAEFFER, Ph.D. thesis, Indiana University, Bloomington, 1983.
16. R. STRICHARTZ, Convolutions with kernels having singularities on a sphere, *Trans. Amer. Math. Soc.* **148** (1970), 461–471.
17. W. STRAUSS, On weak solutions of semilinear hyperbolic equations, *An. Acad. Brasil Cienc.* **42** (1970), 645–651.
18. W. STRAUSS, Nonlinear scattering theory at low energy, *J. Funct. Anal.* **41** (1981), 110–133.
19. F. WEISSLER, Existence and non-existence of global solutions for a semilinear head equation, *Israel J. Math.* **38** (1981), 29–40.
20. H. FUJITA, On the blowing up of solutions to the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **13** (1966), 109–124.