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Fixed point theorems in $\mathbb{R}$-trees with applications to graph theory

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Abstract

It is proved that a commutative family of nonexpansive mappings of a complete $\mathbb{R}$-tree X into itself always has a nonempty common fixed point set if X does not contain a geodesic ray. As a consequence of this, we show that any commuting family of edge preserving mappings of a connected reflexive graph G that contains no cycles or infinite paths always has at least one common fixed edge. This approach provides a new proof of the classical fixed edge theorem of Nowakowski and Rival. Several related results are also obtained.
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1. Introduction

The fixed edge theorem in graph theory [15] asserts that an edge preserving mapping defined on a connected graph which has no cycles or infinite paths always leaves some edge of the graph fixed. We will give precise definitions later. The object of this paper is to prove

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some new fixed point theorems for nonexpansive mappings defined on $\mathbb{R}$-trees and apply these results to obtain some new results in graph theory. Among other things, we show that under the above assumptions on a graph, any commuting family of edge preserving mappings has either a unique fixed edge or a common fixed vertex. (These outcomes are not mutually exclusive.) When the family consists of a single mapping this reduces to the fixed edge theorem. We also show that an edge preserving mapping defined on a connected graph which has no cycles always leaves some edge fixed if it has bounded orbits.

Our results on $\mathbb{R}$-trees should be of independent interest as well. These spaces arise in a variety of contexts and have been studied intensively in recent years; see, e.g., [3] and references therein.

2. Preliminary definitions

The general framework we work in is the class of geodesic spaces. Let (X, d) be a metric space. Recall that a *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a *geodesic* (or *metric*) *segment* joining x and y. When it is unique we denote this geodesic $[x, y]$. The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A *geodesic ray* in X is a subset of X isometric to the half-line $[0, \infty) \subset \mathbb{R}$.

A subset $Y \subseteq X$ is said to be *convex* if Y includes every geodesic segment joining any two of its points, and Y is said to be *gated* [5] if for any point $x \notin Y$ there exists a unique point $x_Y \in Y$ (called the *gate* of x in Y) such that for any $z \in Y$,

$$d(x, z) = d(x, x_Y) + d(x_Y, z).$$

Obviously gated sets in a complete geodesic space are always closed and convex.

Recall that a mapping f of a metric space X into itself is *nonexpansive* if $d(f(x), f(y)) \leqslant d(x, y)$ for all $x, y \in X$. It is known [5] that gated subsets of a complete geodesic space X are proximal nonexpansive retracts of X. Specifically, if A is a gated subset of X, then the mapping that associates with each point x in X its gate in A (i.e., the gate-map, or 'nearest point map') is nonexpansive. Several other properties of gated sets can be found, for example, in [21] (see p. 98). In particular it can be easily shown by induction that the family of gated sets in a complete geodesic space X has the *Helly property*. Thus if $S_1, \dots, S_n$ is a collection of pairwise intersecting gated sets in X then $\bigcap_{i=1}^{n} S_i \neq \emptyset$. Another property we use below is the transitive law for gated sets: Suppose $A \subseteq B \subseteq X$. If B is gated in X and if A is gated in B, then A is gated in X.

A metric space Y is said to be *hyperconvex* [1] if every family $\{B(y_\alpha; r_\alpha)\}$ of closed balls centered at $y_\alpha \in Y$ with radii $r_\alpha \geqslant 0$ has nonempty intersection whenever

$$d(y_\alpha, y_\beta) \leqslant r_\alpha + r_\beta.$$

In particular, a complete geodesic space X is hyperconvex if it has the binary ball intersection property (that is, any family of closed balls in X has nonempty intersection whenever each two members of the family intersect). Such spaces include the classical L_∞ spaces of analysis [12]. It is known that compact hyperconvex spaces (often called *Helly spaces*) are contractible and locally contractible; hence they have the fixed point property for continuous mappings (see [16]). The fact that bounded hyperconvex spaces have the fixed point property for nonexpansive mappings is basically due, independently, to Sine [17] and Soardi [19]. Subsequently Baillon [2] extended this result to commuting families of nonexpansive mappings.

We now turn to a concept introduced by Tits in [20].

Definition 2.1. An $\mathbb{R}$-*tree* is a metric space T such that:

(i) there is a unique geodesic segment (denoted by $[x,y]$) joining each pair of points $x,y \in T$;
(ii) if $[y,x] \cap [x,z] = \{x\}$, then $[y,x] \cup [x,z] = [y,z]$.

From (i) and (ii) it is easy to deduce:

(iii) If $p,q,r \in T$, then $[p,q] \cap [p,r] = [p,w]$ for some $w \in M$.

The facts linking the preceding notions are these.

(1) A metric space is a complete $\mathbb{R}$-tree if and only if it is hyperconvex and has unique geodesic segments.
(2) The gated subsets of an $\mathbb{R}$-tree are precisely its closed and convex subsets.

The first fact has been known for some time in the compact case. A detailed proof in the general case is given in [11]. (Also see [13, Theorem B] for another proof that a complete $\mathbb{R}$-tree is hyperconvex.) The second fact is an immediate consequence of the definitions.

The fact that compact R-trees have the fixed point property for continuous maps goes back to Young [22]; also see the discussion in [14].

3. Gated sets

We begin with two observations about gated sets that we will use later.

Proposition 3.1. *Let (X,d) be a complete geodesic space, and let $\{H_\alpha\}_{\alpha\in\Lambda}$ be a collection of nonempty gated subsets of X which is directed downward by set inclusion. If X (or more generally, some H_α) does not contain a geodesic ray, then $\bigcap_{\alpha\in\Lambda} H_\alpha \neq \emptyset$.*

Proof. Let $H_0 \in \{H_\alpha\}_{\alpha\in\Lambda}$, select $x_0 \in H_0$ and let

$$r_0 = \sup\bigl\{\operatorname{dist}(x_0, H_0 \cap H_\alpha):\ \alpha \in \Lambda\bigr\}.$$

If $x_0 \in \bigcap_{\alpha\in\Lambda} H_\alpha$ we are finished. Otherwise choose $H_1 \in \{H_\alpha\}_{\alpha\in\Lambda}$ so that, $H_1 \subset H_0$, $x_0 \notin H_1$, and

$$\operatorname{dist}(x_0, H_1) \geqslant \begin{cases} r_0 - 1 & \text{if } r_0 < \infty; \\ 1 & \text{if } r_0 = \infty. \end{cases}$$

Now take x_1 to be the gate of x_0 in H_1. Having defined x_n, let

$$r_n = \sup\bigl\{\operatorname{dist}(x_n, H_n \cap H_\alpha)\colon \alpha \in \Lambda\bigr\}.$$

Now choose $H_{n+1} \in \{H_\alpha\}_{\alpha\in\Lambda}$ so that $x_n \notin H_{n+1}$ (if possible), $H_{n+1} \subset H_n$, and

$$\operatorname{dist}(x_n, H_{n+1}) \geqslant \begin{cases} r_n - \frac{1}{n} & \text{if } r_n < \infty; \\ 1 & \text{if } r_n = \infty. \end{cases}$$

Now take x_{n+1} to be the gate of x_n in H_{n+1}. Either this process terminates after a finite number of steps, yielding a point $x_n \in \bigcap_{\alpha\in\Lambda} H_\alpha$, or we have sequences $\{x_n\}$, $\{H_n\}$ for which $i < j \Longrightarrow x_j$ is the gate of x_i in H_j. Since X does not contain a geodesic ray, it must be the case that $r_n < \infty$ for some n (and hence for all n). By transitivity of gated sets the sequence $\{x_n\}$ is linear and thus lies on a geodesic in X. Since X does not contain a geodesic ray, the sequence $\{x_n\}$ must in fact be Cauchy. Let $x_\infty = \lim_n x_n$. Since each of the sets H_n is closed, clearly $x_\infty \in \bigcap_{n=1}^\infty H_n$. Also $\sum_{n=1}^\infty r_n < \infty$, so $\lim_n r_n = 0$.

Now let P_α, $\alpha \in \Lambda$, be the nearest point projection of X onto H_α, and for each $n \in \mathbb{N}$, let $y_n = P_\alpha(x_n)$. Then $d(y_n, x_n) \leqslant r_n$, and since P_α is nonexpansive, for any $m, n \in \mathbb{N}$, $d(y_n, y_m) \leqslant d(x_n, x_m)$. It follows that $P_\alpha(x_\infty) = x_\infty$ for each $\alpha \in \Lambda$. Therefore $x_\infty \in \bigcap_{\alpha\in\Lambda} H_\alpha$. □

Proposition 3.2. *Let (X, d) be a complete geodesic space, and let $\{H_n\}$ be a descending sequence of nonempty gated subsets of X. If $\{H_n\}$ has a bounded selection, then $\bigcap_{n=1}^\infty H_n \neq \emptyset$.*

Proof. Here we simply describe the step-by-step procedure. Let $\{z_n\}$ be a bounded selection for $\{H_n\}$. Let $x_0 = z_0$. Then let n_1 be the smallest integer such that $x_0 \notin H_{n_1}$. Let x_1 be the gate of x_0 in H_{n_1} and take $x_2 = z_{n_1}$. Now take n_2 to be the smallest integer such that $x_2 \notin H_{n_2}$ and take x_3 to be the gate of x_2 in H_{n_2}. Continuing this procedure inductively it is clear that one generates a sequence $\{x_n\}$ which is isometric to an increasing sequence of positive numbers on the real line. Since $\{x_{2n}\}$ is a subsequence of the bounded sequence $\{z_n\}$ it must be the case that $\{x_n\}$ is also bounded. Therefore $\lim_n x_n$ exists and lies in $\bigcap_{n=1}^\infty H_n$. □

Remark. Proposition 3.2 holds for arbitrary descending chains; however the above is sufficient for our purposes (cf., the proof of Theorem 4.5).

4. Hyperconvexity and $\mathbb{R}$-trees

For our next result we need the following fact about hyperconvex spaces.

Theorem 4.1 [18]. *Let (X, d) be a hyperconvex metric space and suppose $T : X \to X$ is nonexpansive. Then for each $\varepsilon > 0$ the set*

$$F_\varepsilon(T) := \{x \in X \colon d(x, T(x)) \leqslant \varepsilon\}$$

is also hyperconvex.

We will also need a result of Kirk [10]. Suppose (X, d) is a geodesic space with unique geodesics. We say that the metric on d is *convex* if the following holds: Given $p, x, y \in X$ and $t \in (0, 1)$, let m be the point of $[x, y]$ satisfying

$$d(x, m) = td(x, y) \quad \text{and} \quad d(y, m) = (1 - t)d(x, y).$$

Then

$$d(p, m) \leqslant (1 - t)d(p, x) + td(p, y).$$

The following is a special case of Theorem 3 of [10].

Lemma 4.2. *Suppose (X, d) is uniquely geodesic with a convex metric, suppose $T : X \to X$ is nonexpansive, and suppose $x_0 \in X$ satisfies*

$$d(x_0, T(x_0)) = \inf\{d(x, T(x)) \colon x \in X\} > 0.$$

Then the sequence $\{T^n(x_0)\}$ is unbounded and lies on a geodesic ray.

The preceding observations combined with Proposition 3.1 give an easy proof of the following fact. The significance of this result is the fact that K itself is not assumed to be bounded. (This result might also be compared with Theorem 32.2 of [7] where it is shown that the complex Hilbert ball with a hyperbolic metric has the fixed point property for nonexpansive mappings if and only if it is geodesically bounded.)

Theorem 4.3. *Let (X, d) be a complete $\mathbb{R}$-tree, and suppose K is a closed convex subset of X which does not contain a geodesic ray. Then every commuting family $\mathfrak{F}$ of nonexpansive mappings of $K \to K$ has a nonempty common fixed point set.*

Proof. Let $T \in \mathfrak{F}$. We first show that the fixed point set of T is nonempty. Let $d = \inf\{d(x, T(x)) \colon x \in K\}$ and let

$$F_n := \left\{x \in K \colon d(x, T(x)) \leqslant d + \frac{1}{n}\right\}.$$

Since K is a closed convex subset of a complete $\mathbb{R}$-tree, K itself is hyperconvex, so by Theorem 4.1 $\{F_n\}$ is a descending sequence of nonempty closed convex (hence gated) subsets of K. Since K does not contain a geodesic ray, Proposition 3.1 implies $F := \bigcap_{n=1}^\infty F_n \neq \emptyset$. Therefore there exists $z \in K$ such that

$$d(z, T(z)) = d.$$

Since K does not contain a geodesic ray, in view of Lemma 4.2, $d = 0$.

Because $\mathbb{R}$-trees are uniquely geodesic, the fixed point set F of T is closed and convex, and hence again an $\mathbb{R}$-tree. Now suppose $G \in \mathfrak{F}$. Since G and T commute it follows that

$G : F \to F$, and by applying the preceding argument to G and F we conclude that G has a nonempty fixed point set in F. In particular the fixed point set of T and the fixed point set of G intersect. Since these are gated sets in X, by the Helly property of gated sets we conclude that every finite subcollection of $\mathfrak{F}$ has a nonempty common fixed point set (which is itself gated). Now let $\mathcal{A}$ be the collection of all finite subcollections of $\mathfrak{F}$, and for $\alpha \in \mathcal{A}$, let H_α be the common fixed point set of α. Then given $\alpha, \beta \in \mathcal{A}$, $H_{\alpha \cup \beta} \subseteq H_\alpha \cap H_\beta$, so clearly the family $\{H_\alpha\}_{\alpha \in \mathcal{A}}$ is directed downward by set inclusion. Since these are all gated sets, we again apply Proposition 3.1 to conclude that $\bigcap_{\alpha \in \mathcal{A}} H_\alpha \neq \emptyset$, and thus that $\mathfrak{F}$ has a nonempty common fixed point set. □

A nonlinear nonexpansive semigroup on a metric space (X, d) is a family $S(t) : X \to X$, $t \geqslant 0$, of nonexpansive mappings satisfying $S(0)$ is the identity and $S(t_1 + t_2) = S(t_1) \circ S(t_2)$. A nonexpansive mapping T (respectively, a semigroup $S(t)_{t \geqslant 0}$ of nonexpansive mappings) defined on a metric space X is said to have *bounded orbits* (respectively, to be *bounded*) if for each $x \in X$ there is a number $M(x)$ such that $d(x, T^n(x)) \leqslant M(x)$ for all $n \geqslant 1$ (respectively, $d(x, S(t)x) \leqslant M(x)$ for all $t \geqslant 0$).

For our next result we apply the following theorem due to Khamsi and Reich.

Theorem 4.4 [9]. *For a hyperconvex metric space X the following are equivalent*:

(A) *Any nonexpansive mapping of $X \to X$ with bounded orbits has a fixed point.*
(B) *Any bounded nonexpansive semigroup on X has a nonempty common fixed point set.*
(C) *Any decreasing sequence of hyperconvex subspaces of X with a bounded selection has nonempty intersection.*

Theorem 4.5. *Let (X, d) be a complete $\mathbb{R}$-tree. Then*:

(i) *any nonexpansive mapping of $X \to X$ with bounded orbits has a fixed point*; *and*
(ii) *any bounded nonexpansive semigroup on X has a nonempty common fixed point set.*

Proof. Proposition 3.2 implies that condition (C) of Theorem 4.4 holds. Thus conditions (A) and (B) also hold. □

Finally, we have a result about iteration. The following is a consequence of Proposition 1 of [10].

Proposition 4.6. *Suppose (X, d) is uniquely geodesic with a convex metric in the sense defined above, and suppose $T : X \to X$ is nonexpansive. Fix $t \in (0, 1)$ and define $f : X \to X$ by taking $f(x)$ to be the point of $[x, T(x)]$ such that*

$$d\bigl(x, f(x)\bigr) = t d\bigl(x, T(x)\bigr), \quad x \in X.$$

If $d(f^n(x), f^{n+1}(x)) \equiv r > 0$ for some $x \in X$, then the sequence $\{f^n(x)\}$ lies on a geodesic ray.

This fact yields the following result.

Theorem 4.7. *Let (X,d) be a complete $\mathbb{R}$-tree, suppose K is a closed convex subset of X that does not contain a geodesic ray, and suppose $T:K\to K$ is nonexpansive. Fix $t\in(0,1)$ and define $f_t:K\to K$ by taking $f_t(x)$ to be the point of $[x,T(x)]$ such that*

$$d\big(x, f_t(x)\big) = t\,d\big(x, T(x)\big), \quad x \in K.$$

Then $\{f_t^n(x)\}$ converges to a fixed point of T for each $x\in K$.

Proof. Let $x\in K$. Since $\{d(f_t^n(x), f_t^{n+1}(x))\}$ is nonincreasing

$$d := \inf_n \big\{d\big(f_t^n(x), f_t^{n+1}(x)\big)\big\} = \lim_n d\big(f_t^n(x), f_t^{n+1}(x)\big). \tag{1}$$

For each $n\in\mathbb{N}$ let

$$F_n := \left\{u \in K:\ d\big(u, f_t(u)\big) \leqslant d + \frac{1}{n}\right\}.$$

As in the proof of Theorem 4.3, F_n is gated for each n, and $F := \bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

Let $x_0 = x$ and let n_1 be the smallest integer such that $x_0 \notin F_{n_1}$. Now let x_1 be the gate of x_0 in F_{n_1} and let $x_2 = f_t^{n_2}(x)$ where n_2 is chosen so that $f_t^{n_2}(x)\in F_{n_1}$. Next let n_3 be the smallest integer such that $x_2\notin F_{n_3}$ and let x_3 be the gate of x_2 in F_{n_3}. Proceed in this way step-by-step. If the process terminates we clearly have $f_t^n(x)\in F$ for n sufficiently large. In this case, in view of Proposition 4.6, we conclude $d=0$ and $f_t^n(x)$ is a fixed point of f_t (hence also of T). Otherwise we obtain a sequence $\{x_n\}$ which lies on a geodesic and for which $\{x_{2n}\}$ is a subsequence of $\{f_t^n(x)\}$. Since K is linearly bounded, $\{x_n\}$ must be a Cauchy sequence. This proves that a subsequence of $\{f_t^{m_i}(x)\}_{i=1}^{\infty}$ converges, say to a point $z\in F$. In view of (1) this implies $d(f_t^n(z), f_t^{n+1}(z)) = d$, $n=0,1,2,\ldots.$ By Proposition 4.6 it must be the case that $d=0$. But this implies $z=f_t(z)$ and hence that $\{f_t^n(x)\}$ itself converges to z. □

5. Applications to graph theory

A *graph* is an ordered pair (V,E) where V is a set and E is a binary relation on V ($E\subseteq V\times V$). Elements of E are called *edges*. We are concerned here with (undirected) graphs that have a "loop" at every vertex (i.e., $(a,a)\in E$ for each $a\in V$) and no "multiple" edges. Such graphs are called *reflexive*. In this case $E\subseteq V\times V$ corresponds to a reflexive (and symmetric) binary relation on V.

Given a graph $G=(V,E)$, a path of G is a sequence $a_0, a_1, \ldots, a_{n-1}, \ldots$ with $(a_{i+1}, a_i)\in E$ for each $i=0,1,2,\ldots.$ A *cycle* is a finite path $(a_0, a_1, \ldots, a_{n-1})$ with $(a_0, a_{n-1})\in E$. A graph is *connected* if there is a finite path joining any two of its vertices. A finite path $(a_0, a_1, \ldots, a_{n-1})$ is said to have *length* n. Finally, a *tree* is a connected graph with no cycles.

For a graph $G=(V,E)$ a map $f:V\to V$ is *edge-preserving* if $(a,b)\in E \Longrightarrow (f(a), f(b))\in E$. For such a mapping we simply write $f:G\to G$. There is a standard way of *metrizing* connected graphs; let each edge have length one and take distance $d(a,b)$ between two vertices a and b to be the length of the shortest path joining them. With this metric edge preserving mappings become precisely the *nonexpansive* mappings. (Keep in mind that in a reflexive graph an edge-preserving map may collapse edges between distinct points since loops are allowed.)

We now show how the classical fixed edge theorem of Nowakowski and Rival is a consequence of results of the preceding section.

Theorem 5.1 [15]. *Let G be a reflexive graph that is connected, contains no cycles, and contains no infinite paths. Then every edge-preserving map of G into itself fixes an edge.*

Proof. Suppose $f : G \to G$ is edge preserving. Since a connected graph with no cycles is a tree, one can construct from the graph G an $\mathbb{R}$-tree T by identifying each (nontrivial) edge with a unit interval of the real line and assigning the shortest path distance to any two points of T. It is easy to see that with this metric T is complete. One can now extend f affinely on each edge to the corresponding unit interval, and the resulting mapping $\tilde{f}$ is a nonexpansive mapping of $T \to T$. Thus $\tilde{f}$ has a fixed point z by Theorem 4.3. Either z is a vertex of G, or z lies properly on a unit interval of T in which case f must leave the corresponding edge fixed. □

An application of Theorem 4.3 in its full generality tells us much more.

Theorem 5.2. *Let G be a reflexive graph that is connected, contains no cycles, and contains no infinite paths. Suppose $\mathfrak{F}$ is a commuting family of edge-preserving mappings of G into itself. Then either*:

(a) *there is a unique edge in G that is left fixed by each member of $\mathfrak{F}$*; *or*
(b) *some vertex of G is left fixed by each member of $\mathfrak{F}$.*

Proof. Embed G in an $\mathbb{R}$-tree T as in the preceding proof, and extend the mappings $f \in \mathfrak{F}$ affinely to T to obtain a commuting family $\widetilde{\mathfrak{F}} = \{\tilde{f}\colon\ f \in \mathfrak{F}\}$ of nonexpansive mappings of $T \to T$. In view of Theorem 4.3 there is a point $z \in T$ that is a fixed point of each member of $\widetilde{\mathfrak{F}}$. Either z is a vertex of G, or z properly lies on an interval of T whose corresponding edge is a common fixed edge of $\mathfrak{F}$. However the only way $\mathfrak{F}$ can fail to have a common fixed vertex is if z is the midpoint of some interval of T corresponding to an edge (a, b) for which $f(a) = b$ and $f(b) = a$ for some $f \in \mathfrak{F}$. Since fixed point sets of nonexpansive mappings in T are convex, this would imply that z is the unique fixed point of $\tilde{f}$. Since any mapping commuting with $\tilde{f}$ maps the fixed point set of $\tilde{f}$ into itself, z is a fixed point of *every* member of $\widetilde{\mathfrak{F}}$. Therefore (a, b) is the *unique* edge left fixed by each member of $\mathfrak{F}$. □

Theorem 4.5(i) also has a graph theory counterpart. In this context, a bounded orbit means that given $x \in G$ there exists $M \in \mathbb{N}$ such that each two points of $\{T^n(x)\}$ can be joined by a path of length at most M.

Theorem 5.3. *Let G be a reflexive graph that is connected and contains no cycles. Then every edge-preserving mapping T of G into itself which has bounded orbits fixes an edge.*

The following algorithm is a consequence of Theorem 4.7 (although it can be proved directly as well).

Theorem 5.4. *Let G be a reflexive graph that is connected, contains no cycles, and contains no infinite paths, and let $T : G \to G$ be an edge-preserving map. Fix $x_0 \in G$, and having defined x_n choose x_{n+1} so that (x_n, x_{n+1}) is a nontrivial edge on the path joining x_n to $T(x_n)$ if such an edge exists; otherwise set $x_{n+1} = x_n$. Then there exists an integer n for which either*

(i) $x_{n+1} = x_n$ (*i.e., x_n is a fixed vertex*), *or*
(ii) $x_{n+2} = x_n$ (*and (x_n, x_{n+1}) is a fixed edge*).

Proof. Again embed the problem in an $\mathbb{R}$-tree and consider the mapping f_t from Theorem 4.7, where $t \in (0, 1)$ is chosen so small that $f_t(x_0)$ lies on the interval of the $\mathbb{R}$-tree path corresponding to the edge (x_0, x_1). The desired integer n is the smallest integer for which $d(f_t^{n+1}(x), f_t^{n+2}(x)) < d(f_t^n(x), f_t^{n+1}(x))$. Such an integer must exist because $\{f_t^n(x)\}$ converges. □

Remarks. (1) Metric graphs are the spaces obtained by taking a connected graph and metrizing the nontrivial edges of the graph as bounded intervals of the real line. Such a graph is an $\mathbb{R}$-tree if the corresponding metric graph is connected and simply connected. However in general $\mathbb{R}$-trees are much more complicated than metric graphs. For example, consider the set $[0, \infty) \times [0, \infty)$ with the distance between two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ defined by

$$d(x, y) = \begin{cases} x_1 + y_1 + |x_2 - y_2| & \text{if } x_2 \neq y_2; \\ |x_1 - y_1| & \text{if } x_2 = y_2. \end{cases}$$

The asymptotic cone $\mathbb{H}^n_{\mathcal{U}}$ of the classical hyperbolic n-space $\mathbb{H}^n$ provides another non-simplicial example of an $\mathbb{R}$-tree. In this case, the complement of every point in the $\mathbb{R}$-tree has infinitely many connected components. (The *asymptotic cone* of $\mathbb{H}^n$ is the ultraproduct $\prod X_n$ over some nontrivial ultrafilter $\mathcal{U}$, where $X_n = (\mathbb{H}^n, \frac{1}{n}d)$. For a discussion, see [4]. Also, for some explicit constructions of $\mathbb{R}$-trees related to the asymptotic geometry of hyperbolic metric spaces see [6].)

(2) Straightforward examples show that the sufficient conditions for a fixed edge in Theorem 5.1 (connectedness, no cycles, and no infinite paths) are also necessary. See [15].

(3) See [8] for an example showing that Theorem 4.5 does not hold in an arbitrary hyperconvex space.

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References

[1] N. Aronszajn, P. Panitchpakdi, Extensions of uniformly continuous transformations and hyperconvex metric spaces, Pacific J. Math. 6 (1956) 405–439.
[2] J.B. Baillon, Nonexpansive mappings and hyperconvex spaces, in: Contemp. Math., vol. 72, Amer. Math. Soc., Providence, RI, 1986, pp. 11–22.
[3] M. Bestivina, $\mathbb{R}$-trees in topology, geometry, and group theory, in: Handbook of Geometric Topology, North-Holland, Amsterdam, 2002, pp. 55–91.
[4] M. Bridson, A. Haefliger, Metric Spaces of Non-Positive Curvature, Springer, Berlin, 1999.
[5] A. Dress, R. Scharlau, Gated sets in metric spaces, Aequationes Math. 34 (1987) 112–120.
[6] A. Dyubina, I. Polterovich, Explicit constructions of universal $\mathbb{R}$-trees and asymptotic geometry of hyperbolic spaces, Bull. London Math. Soc. 33 (2001) 727–734.
[7] K. Goebel, S. Reich, Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings, Marcel Dekker, New York, 1984.
[8] M.A. Khamsi, M. Lin, R. Sine, On the fixed points of commuting nonexpansive maps in hyperconvex spaces, J. Math. Anal. Appl. 168 (1992) 372–380.
[9] M.A. Khamsi, S. Reich, Nonexpansive mappings and semigroups in hyperconvex spaces, Math. Japon. 35 (1990) 467–471.
[10] W.A. Kirk, Krasnoselskii's iteration process in hyperbolic space, Numer. Funct. Anal. Optimiz. 4 (1981–82) 371–381.
[11] W.A. Kirk, Hyperconvexity of $\mathbb{R}$-trees, Fund. Math. 156 (1998) 67–72.
[12] H.E. Lacey, The Isometric Theory of Classical Banach Spaces, Springer, Berlin, 1974.
[13] U. Lang, V. Schroeder, Kirszbraun's theorem and metric spaces of bounded curvature, Geom. Funct. Anal. 7 (1997) 535–560.
[14] S.B. Nadler, Continuum Theory, Marcel Dekker, New York, 1992.
[15] R. Nowakowski, I. Rival, Fixed-edge theorem for graphs with loops, J. Graph Theory 3 (1979) 339–350.
[16] A. Quilliot, On the Helly property working as a compactness criterion on graphs, J. Combin. Theory Ser. A 40 (1985) 186–193.
[17] R. Sine, On nonlinear contractions in sup norm spaces, Nonlinear Anal. 3 (1979) 885–890.
[18] R. Sine, Hyperconvexity approximate fixed points, Nonlinear Anal. 13 (1989) 863–869.
[19] P. Soardi, Existence of fixed points of nonexpansive mappings in certain Banach lattices, Proc. Amer. Math. Soc. 73 (1979) 25–29.
[20] J. Tits, A "Theorem of Lie–Kolchin" for trees, in: Contributions to Algebra: a Collection of Papers Dedicated to Ellis Kolchin, Academic Press, New York, 1977.
[21] M. van de Vel, Theory of Convex Structures, North-Holland, Amsterdam, 1993.
[22] G.S. Young, The introduction of local connectivity by change of topology, Amer. J. Math. 68 (1946) 479–494.