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Embedding of a Pseudo-residual Design into a Möbius Plane*

AGNES HUI CHAN

Department of Mathematics, Northeastern University, Boston, Massachusetts 02115

AND

D. K. RAY-CHAUDHURI

Department of Mathematics, The Ohio State University, Columbus, Ohio 43210

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Let $\mathfrak{A}$ be a class of subsets of a finite set X. Elements of $\mathfrak{A}$ are called blocks. Let v, t and λ_i, $0 \leqslant i \leqslant t$, be nonnegative integers, and K be a subset of nonnegative integers such that every member of K is at most v. A pair $(X, \mathfrak{A})$ is called a $(\lambda_0, \lambda_1,..., \lambda_t; K, v)$ *t-design* if (1) $|X| = v$, (2) every i-subset of X is contained in exactly λ_i blocks, $0 \leqslant i \leqslant t$, and (3) for every block A in $\mathfrak{A}$, $|A| \in K$. It is well-known that if K consists of a singleton k, then $\lambda_0,..., \lambda_{t-1}$ can be determined from v, t, k and λ_t. Hence, we shall denote a $(\lambda_0,..., \lambda_t; \{k\}, v)$ t-design by $S_\lambda(t, k, v)$, where $\lambda = \lambda_t$. A Möbius plane M is an $S_1(3, q+1, q^2+1)$, where q is a positive integer. Let A be a fixed block in M. If A is deleted from M together with the points contained in A, then we obtain a residual design M' with parameters $\lambda_0 = q^3 + q - 1$, $\lambda_1 = q^2 + q$, $\lambda_2 = q + 1$, $\lambda_3 = 1$, $K = \{q+1, q, q-1\}$, and $v = q^2 - 1$. We define a design to be a pseudo-block-residual design of order q (abbreviated by PBRD(q)) if it has these parameters. We consider the reconstruction problem of a Möbius plane from a given PBRD(q). Let B and B' be two blocks in a residual design M'. If B and B' are tangent to each other at a point x, and there exists a block C of size $q+1$ such that C is tangent to B at x and is secant to B', then we say B is r-tangent to B' at x. A PBRD(q) is said to satisfy the r-tangency condition if for every block B of size q, and any two points x and y not in B, there exists at most one block which is r-tangent to B and contains x and y. We show that any PBRD(q) D can be uniquely embedded into a Möbius plane if and only if D satisfies the r-tangency condition.

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1. Introduction

Let $(X, \mathfrak{A})$ be an ordered pair, where X is a finite set and $\mathfrak{A}$ is a collection of subsets of X. Members of X are called *points* and elements of $\mathfrak{A}$ are called *blocks*. Let $\mathbb{N}_0$ denote the set of nonnegative integers, and $v, t \in \mathbb{N}_0$ such that $v \geqslant t \geqslant 0$; for every i, $t \geqslant i \geqslant 0$, let $\lambda_i \in \mathbb{N}_0$. Let K be a set of nonnegative integers such that every member of K is smaller than or equal to v. A structure $D = (X, \mathfrak{A})$ is called a $(\lambda_0, \lambda_1, ..., \lambda_t; K, v)$ *t-design*, denoted by $S(\lambda_0, ..., \lambda_t; K, v)$ if and only if (1) every i-subset of X is contained in exactly λ_i blocks, $0 \leqslant i \leqslant t$, (2) for every block A in $\mathfrak{A}$, $|A| \in K$, and (3) $|X| = v$. If $|A| = k$, then A is called a *k-valent* block. In cases where Axiom (1) is only known to be satisfied by $i = t$, the design D will be denoted by $S_\lambda(t, K, v)$, where $\lambda = \lambda_t$. Further, if $\lambda = 1$, we only use $S(t, K, v)$. For simplicity, if K consists of a singleton k, we write D as an $S_\lambda(t, k, v)$ instead of $S_\lambda(t, \{k\}, v)$. Since λ_0 denotes the number of blocks contained in the design D, it is customary to write b in place of λ_0; also, r usually takes the place of λ_1.

Let A_∞ be a fixed block in D. A *block-residual* design of D with respect to A_∞ is a design $D' = (X', \mathfrak{A}')$, where $X' = X - \{\text{points contained in } A_\infty\}$, and $\mathfrak{A}' = \mathfrak{A} - \{A_\infty\}$. If D is an $S_\lambda(t, K, v)$, then D' is an $S_\lambda(t, K', v-k)$ where k is the size of the deleted block A_∞ and every member of K' is not larger than the maximal member of K. A design with parameters equal to those of a block-residual design is called a *pseudo-block-residual* design.

Definition. Let D' be a pseudo-block-residual design. D' is said to be *embeddable* if and only if there exists a design D such that the residual design D'' obtained from D is isomorphic to D'.

Hall and Connor [5] proved the embedding theorem for a pseudo-block-residual design of an $S_2(2, k, v)$. Bose *et al.* [2] and Shrikhande and Singhi [6] extended the result to $S_\lambda(2, k, v)$ for $\lambda \geqslant 3$. In this paper, we prove an embedding theorem for a pseudo-block-residual design of a Möbius plane.

2. Motivation and Statement of Theorem

A *Möbius plane M* is an $S(3, q+1, q^2+1)$, where q is a positive integer. If M' is a block-residual design obtained from M, then M' is an $S(\lambda_0, \lambda_1, \lambda_2, \lambda_3; K, v)$, where

$$\begin{aligned} &\lambda_0 = q^3 + q - 1, \quad &&\lambda_1 = q^2 + q, \quad &&\lambda_2 = q + 1, \\ &\lambda_3 = 1, &&K = \{q+1, q, q-1\}, &&v = q^2 - q. \end{aligned} \tag{1}$$

Any 3-design with parameters as those given in (1) is called a pseudo-block-residual design of order q, abbreviated by PBRD(q).

Let us first study properties possessed by a Möbius plane M. (For a detailed treatment of Möbius planes, Dembowski [4, Chap. 6] is an excellent reference.) A block B is said to be *tangent* to another block B' at a point x if and only if $B \cap B' = \{x\}$. They are said to be *secant* to one another if $|B \cap B'| \geqslant 2$. Let B and B' be two distinct blocks in M that are tangent to a block A at a point x. It can be seen easily that B and B' are mutually tangent at x. A maximal set of blocks which are mutually tangent at a point x is called a *pencil with carrier* x. One can show that every pencil in M consists of q blocks. If the point x is deleted from M, then the blocks in a pencil with carrier x are pairwise disjoint. A set of pairwise disjoint blocks that partition the set of points in a design is called a *parallel class* of blocks. Clearly, a pencil in M with carrier x is a parallel class of blocks in $M - x$.

For every deleted point x in A_∞ there exists a pencil $\mathfrak{A}'$ in M with x as the carrier, which contains A_∞. Clearly, $\mathfrak{A}' - A_\infty$ forms a parallel class of blocks in M'; moreover, each block in $\mathfrak{A}' - A_\infty$ is q-valent. Conversely, given any parallel class of q-valent blocks in M', there corresponds a unique point deleted from A_∞. Thus, in order to embed a PBRD(q) into a Möbius plane M, first of all, we have to establish the $q+1$ parallel classes of q-valent blocks in D.

When the parallel classes are established in D, we still have to find means to "complete" the $(q-1)$-valent blocks to $(q+1)$-valent blocks. Again, we are motivated by examining the $(q-1)$-valent blocks in M'. Let B be any $(q-1)$-valent block in M' and let x and y be the corresponding deleted points in A_∞. Consider the pencil $\mathfrak{A}'$ in M with carrier x; every block in the corresponding parallel class is tangent to B at a point. We define this type of tangency to be *r-tangency*. One can easily check that if B is r-tangent to B' at a point z, then there exists a $(q+1)$-valent block A such that A is tangent to B at z but is secant to B'. On the basis of this, we generalize the definition of r-tangency.

DEFINITION. Let D be an $S_\lambda(t, K, v)$ with $K = \{k+1, k, k-1\}$. A block B in D is said to be *r-tangent* to another block B' at a point x if and only if there exists a $(k+1)$-valent block A such that A is tangent to B at x and is secant to B'.

We define a block B to be an *r-transversal* of a parallel class $\mathfrak{A}$ if and only if B is r-tangent to every block in $\mathfrak{A}$. It is obvious then that any $(q-1)$-valent block in M' is an r-transversal of exactly two parallel classes. Conversely, given any two parallel classes of q-valent blocks in M', there are q common r-transversals, namely, the blocks in M that contain the two corresponding deleted points. Hence if we can show that every $(q-1)$-valent block in a pseudo-block-residual design D' is an r-transversal of exactly two parallel classes, then we can "complete" the $(q-1)$-valent blocks by adjoining their corresponding parallel classes. Our Fundamental Lemma, stated in the next section, shows how these parallel classes and r-transversals lead to the embedding of an $S(v, \{k+1, k, k-1\}, 1)$ 3-design into an $S(v+k+1, k+1, 1)$ 3-design.

From the above discussions, we see that we have to set up the parallel classes in PBRD(q). To do so, we consider the set of blocks that are r-tangent to a given q-valent block. We define a maximal set of blocks which are mutually r-tangent at x and contains at least four blocks to be an *r-pencil with carrier x*. We show that each r-pencil consists of one q-valent block and q $(q-1)$-valent blocks.

A PBRD is said to satisfy the *Tangency Condition* if given two distinct points x, y and a block A with $x \in A$, $y \notin A$, there exists at most one block containing y and tangent to A at x.

This condition is certainly satisfied by a block-residual design of a Möbius plane. If a PBRD(q) satisfies the Tangency Condition, then we can show that the r-tangents of B which contain a common point x are mutually r-tangent. Furthermore, these r-tangents determine a unique r-pencil whose q-valent block is either B itself or disjoint from B. From this, we obtain the $q+1$ parallel classes of q-valent blocks. We also show that every $(q-1)$-valent block that is r-tangent to a q-valent block B, is an r-transversal of the parallel class containing B. Thus, we are able to establish the main theorem as stated below.

THEOREM. *If D is a pseudo-block-residual design of order q and satisfies the Tangency Condition, then D is uniquely embedded into a Möbius plane of order q.*

3. FUNDAMENTAL LEMMA

In this section we shall reconstruct a 3-design from a pseudo-block-residual design by means of parallel classes and r-transversals. Let us first state the result as follows:

LEMMA 1 (The Fundamental Lemma). *Let D be an $S(3, K, v)$, where $K = \{k+1, k, k-1\}$. Suppose D satisfies the following conditions*:

(A1) *The collection of k-valent blocks can be partitioned into $k+1$ parallel classes.*

(A2) *Every $(k-1)$-valent block is the r-transversal of exactly two parallel classes and every two parallel classes have exactly k common $(k-1)$-valent r-transversals which are pairwise disjoint.*

(A3) *Given any two distinct points x and y and a parallel class $\mathfrak{A}$,* *either there exists exactly one block in $\mathfrak{A}$ that contains x and y, or there exists exactly one $(k-1)$-valent block that contains x and y and is an r-transversal of $\mathfrak{A}$. Then D can be uniquely embedded into an $S(3, k+1, v+k+1)$.*

In proving the Fundamental Lemma, we reconstruct the $k+1$ "missing" points and adjoin them to the k-valent and $(k-1)$-valent blocks in D. Finally, we show that it is a 3-design. Before we proceed, let us establish two simple lemmas.

LEMMA 2. *Let D be as defined in the Fundamental Lemma. Then, $v = k^2 - k$.*

Proof. Since every $(k-1)$-valent block in D is an r-transversal of a parallel class $\mathfrak{A}$, $\mathfrak{A}$ consists of $k-1$ k-valent blocks; hence, $v = k(k-1)$.

LEMMA 3. *If $\mathfrak{A}$ and $\mathfrak{A}'$ are two distinct parallel classes in D and x is a point, then there exists a unique $(k-1)$-valent block that contains x and is a common r-transversal of $\mathfrak{A}$ and $\mathfrak{A}'$.*

Proof. Since the k common r-transversals of $\mathfrak{A}$ and $\mathfrak{A}'$ are pairwise disjoint, they partition the $k^2 - k$ points in D. Hence, given any point x in D, there exists a unique common r-transversal of $\mathfrak{A}$ and $\mathfrak{A}'$ that contains x.

Now we can proceed to prove the Fundamental Lemma.

Construction of "new points." Let $\mathfrak{A}_1,\dots, \mathfrak{A}_{k+1}$ be the $k+1$ parallel classes of k-valent blocks in D. Corresponding to every parallel class $\mathfrak{A}_i$, we define a "new" point $\mathfrak{A}_i'$, $1 \leqslant i \leqslant k+1$. Let $\bar{X}$ be the set of points consisting of the points X in D and the $k+1$ new points $\mathfrak{A}_1',\dots, \mathfrak{A}_{k+1}'$.

Construction of "new blocks." Let A be a block in D.

(1) If A is a $(k+1)$-valent block, then we let A' be A.

(2) If A is k-valent, then A is contained in a unique parallel class $\mathfrak{A}_i$ for some i, $1 \leqslant i \leqslant k+1$. We define A' to be a block consisting of all points in A and the new point $\mathfrak{A}_i'$.

(3) If A is $(k-1)$-valent, then there exist exactly two parallel classes $\mathfrak{A}_i$ and $\mathfrak{A}_j$, $i \neq j$, $1 \leqslant i, j \leqslant k+1$, such that A is a common r-transversal of both classes. We extend A to a block A' consisting of all points in A, together with the two new points $\mathfrak{A}_i'$ and $\mathfrak{A}_j'$. Finally, we let A_∞ be a block consisting of the $k+1$ new points and

$$\bar{\mathfrak{A}} = \{A_\infty\} \cup \{A' \mid A \text{ is a block in } D\}.$$

Construction of a 3-design. Let S be the incidence structure $(\bar{X}, \bar{\mathfrak{A}})$. We shall show that S is an $S(3, k+1, v+k+1)$.

From the constructions of new points and new blocks, it is clear that S has $v+k+1$ points and each block in S has $k+1$ points. We only have to show that $\lambda = 1$. Let x, y and z be any three distinct points in X.

Case 1. $x, y, z \in X$. There exists a unique block A in D that contains x, y and z; then the extended block A' in $\bar{\mathfrak{A}}$ is the unique block containing x, y and z.

Case 2. $x, y \in X$ and $z = \mathfrak{A}_i'$ for some i, $1 \leqslant i \leqslant k+1$. Consider the two points x and y and the parallel class $\mathfrak{A}_i$ in D. By Axiom (A3), either there exists a k-valent block A in $\mathfrak{A}_i$ containing x and y, or there exists a $(k-1)$-valent block A that contains x and y and is an r-transversal of $\mathfrak{A}_i$. In either case the extended block A' in $\bar{\mathfrak{A}}$ is the unique block containing x, y and z.

Case 3. $x \in X$, $y = \mathfrak{A}_i$ and $z = \mathfrak{A}_j$ with $1 \leqslant i < j \leqslant k+1$. Let us consider the point x and the two parallel classes $\mathfrak{A}_i$ and $\mathfrak{A}_j$ in D. By Lemma 3, there exists a unique $(k-1)$-valent block A containing x which is a common r-transversal of $\mathfrak{A}_i$ and $\mathfrak{A}_j$. Thus, A' is the unique block in S that contains x, y and z.

Case 4. $x = \mathfrak{A}_i'$, $y = \mathfrak{A}_j'$ and $z = \mathfrak{A}_m'$ with $1 \leqslant i < j < m \leqslant k+1$. The block A_∞ is the unique block that contains x, y and z.

It is clear that if we delete the block A_∞ from S, then the block-residual design S' thus obtained is isomorphic to D. Moreover, since the $k+1$ new points and the extended blocks in S are uniquely determined, D is uniquely embeddable.

By virtue of this lemma, we see that if we can establish the parallel classes and the r-transversals that satisfy Axioms (A1)–(A3), then a PBRD(q) can be uniquely embedded into a Möbius plane.

4. The Three Classes of Blocks

Let $D = (X, \mathfrak{A})$ be any PBRD(q). For $k \in K = \{q+1, q, q-1\}$, we denote the set of k-valent blocks by $\mathfrak{A}(k)$. Clearly, $\mathfrak{A}$ is partitioned into the three classes, $\mathfrak{A}(q+1)$, $\mathfrak{A}(q)$ and $\mathfrak{A}(q-1)$. Throughout this section, we shall use A, B and C to denote members of $\mathfrak{A}(q+1)$, $\mathfrak{A}(q)$ and $\mathfrak{A}(q-1)$, respectively; other letters will be used to denote blocks of various sizes. Let us first compute the order of $\mathfrak{A}(k)$ for each k in K.

Lemma 4. *Let D be a PBRD(q). For $k \in K$, let $b(k)$ denote the number of k-valent blocks contained in D. Then*

$$\begin{aligned} b(q+1) &= \tfrac{1}{2}q(q-1)(q-2), \\ b(q) &= (q+1)(q-1), \\ b(q-1) &= \tfrac{1}{2}q^2(q+1). \end{aligned} \tag{2}$$

Proof. Total number of blocks in D

$$= q^3 + q - 1 = b(q+1) + b(q) + b(q-1). \tag{3}$$

Total number of triples (x, y, z) where x, y and z are distinct points in D

$$= \binom{q^2-q}{3} = \binom{q+1}{3} b(q+1) + \binom{q}{3} b(q) + \binom{q-1}{3} b(q-1). \tag{4}$$

Next, let us count the number of ordered pairs (x, E) where x is a point incident with a block E in D. Fixing a point x, there exists λ_1 choices of E and there are $q^2 - q$ points in D. Hence the total number of ordered pairs equals $(q^2+q)(q^2-q)$. On the other hand, for a fixed k-valent block E, there are k choices of x. Hence

$$(q^2+q)(q^2-q) = (q+1)\,b(q+1) + qb(q) + (q-1)\,b(q-1). \tag{5}$$

Using (3), (4) and (5), the parameters $b(q+1)$, $b(q)$ and $b(q-1)$ are easily computed to be those given in (2).

LEMMA 5. *Let D be a PBRD(q). If for $k \in K$, $r(k)$ denotes the number of k-valent blocks containing a given point in D, then for every point x in D,*

$$\begin{aligned} r(q+1) &= \tfrac{1}{2}(q+1)(q-2), \\ r(q) &= q+1, \\ r(q-1) &= \tfrac{1}{2}q(q+1). \end{aligned} \tag{6}$$

Proof. Let x be any point in D. Since x is contained in $q^2 + q$ blocks, we have

$$r(q+1) + r(q) + r(q-1) = q^2 + q. \tag{7}$$

Next, let us count the number of triples (x, y, z) where y and z are points distinct from x and $y \neq z$. Then,

$$\binom{q}{2} r(q+1) + \binom{q-1}{2} r(q) + \binom{q-2}{2} r(q-1) = \binom{q^2-q-1}{2}. \tag{8}$$

Lastly, we count the number of ordered pairs (y, E) where y is a point distinct from x and both x and y are incident with the block E. For every point y distinct from x, there are $q+1$ blocks containing both x and y. Therefore,

$$qr(q+1)+(q-1)\,r(q)+(q-2)\,r(q-1)=(q^2-q-1)(q+1). \tag{9}$$

Combining (7), (8) and (9), we get the parameters $r(q+1)$, $r(q)$ and $r(q-1)$ given in (6).

From these two lemmas, we observe that for each k in K, $(X, \mathfrak{A}(k))$ is an $S_{r(k)}(1, k, q^2-q)$. Even though none of the three designs is a 2-design, for each k in K, $\lambda_2(k)$ takes only two values.

LEMMA 6. *Let D be a PBRD(q) and for each $k \in K$, let $\lambda_2(k)$ denote the number of k-valent blocks containing two given points in D.*
If $q \equiv 1$ (mod 2), then $\lambda_2(q)=0$ or 2 and $\lambda_2(q+1)=\lambda_2(q-1)$.
If $q \equiv 0$ (mod 2), then $\lambda_2(q)=1$ or $q+1$ and $\lambda_2(q+1)=\lambda_2(q-1)$.

Proof. Let x and y be two distinct points in D. Since D is a 2-design,

$$\lambda_2(q+1)+\lambda_2(q)+\lambda_2(q-1)=\lambda_2=q+1. \tag{10}$$

The blocks containing x and y partition the points of D distinct from x and y. Hence,

$$(q-1)\,\lambda_2(q+1)+(q-2)\,\lambda_2(q)+(q-3)\,\lambda_2(q-1)=q^2-q-2. \tag{11}$$

From (10) and (11), we obtain

$$\lambda_2(q)+2\lambda_2(q-1)=q+1. \tag{12}$$

From (10) and (12), we have

$$\lambda_2(q+1)=\lambda_2(q-1). \tag{12a}$$

Case 1. $q \equiv 1$ (mod 2). Since $q+1 \equiv 0$ (mod 2), from (12), we see that $\lambda_2(q) \equiv 0$ (mod 2).

Suppose $\lambda_2(q) \neq 0$. Then there exists a block $B \in \mathfrak{A}(q)$ such that B contains x and y. For each point y_i in B, $y_i \neq x$, the points x and y_i are contained in B and B is q-valent. Since $\lambda_2(q) \equiv 0$ (mod 2), there exists at least one other q-valent block containing x and y_i; let B_i be such a block. Clearly, for $i \neq j$, $B_i \neq B_j$ and there are $q-1$ such in B_i's. But $r(q)=q+1$ implies that there exists another q-valent block B' containing x. If B' contains x and y, then $\lambda_2(q)=3$ and contradicts that $\lambda_2(q) \equiv 0$ (mod 2). Therefore, $\lambda_2(q)=2$ and B' is not secant to B. In fact, B' is the unique q-valent block that is tangent to B at x.

Case 2. $q \equiv 0$ (mod 2). Since $q+1 \equiv 1$ (mod 2), from (12), we see that $\lambda_2(q) \equiv 1$ (mod 2).

Suppose $\lambda_2(q) \neq q+1$. Then (12) implies that $\lambda_2(q-1) \neq 0$ and by (12a) there exists a $(q+1)$-valent block A containing x and y. Let $B_1,..., B_{\lambda_2(q)}$ be the q-valent blocks containing x and y, and $B'_1,..., B'_n$ be the other q-valent blocks containing x. For every point z in A, distinct from x and y, z is not contained in B_i for any i, $1 \leqslant i \leqslant \lambda_2(q)$. Since $\lambda_2(q) \equiv 1 \pmod 2$, z is contained in at least one B'_i, $1 \leqslant i \leqslant n$. But there are $q-1$ points in A that are distinct from x and y. Hence

$$n \geqslant q-1. \tag{13}$$

Since $r(q) = q+1$, $\lambda_2(q) + n = q+1$, or equivalently,

$$n = q + 1 - \lambda_2(q). \tag{14}$$

From (13) and (14), we have $\lambda_2(q) \leqslant 2$. But $\lambda_2(q) \equiv 1 \pmod 2$; hence $\lambda_2(q) = 1$.

COROLLARY 7. *Let $B \in \mathfrak{A}(q)$ and x be a point in B.*

If $q \equiv 1 \pmod 2$, then there exists a unique q-valent block tangent to B at x.

If $q \equiv 0 \pmod 2$, then there exists no q-valent block tangent to B at x.

Proof. *Case* 1. $q \equiv 1 \pmod 2$. The result is clear from the previous proof.

Case 2. $q \equiv 0 \pmod 2$. If for every point y in B, $y \neq x$, B is the only q-valent block containing x and y, then every other q-valent block containing x is tangent to B; furthermore, they are mutually tangent. This implies that there are q^2 points in D contradicting that $v = q^2 - q$. Hence, there exists a point y in B such that every q-valent block containing x contains x and y. Hence, there exists no q-valent block tangent to B at x.

We shall need these lemmas later. Meanwhile, let us divert our attentions to blocks that are tangent to each other in D.

5. TANGENTS

Let us recall that two blocks E and E' in D are said to be *tangent* to each other if they intersect in exactly one point, and if they intersect in exactly two points, then they are *secant* to one another. Furthermore, if E and E' are tangent at x and there exists a $(q+1)$-valent block A which is tangent to E at x and secant to E', then E is said to be *r-tangent* to E' at x. In this section, we shall establish the existence of r-tangents in D.

LEMMA 8. *Let $i \in \{0, 1, 2\}$ and E be a fixed $(q+1-i)$-valent block in*

D. If x is a point incident with E and y is a point not in E, then there exist exactly $i+1$ blocks which are tangent to E at x and contain y.

Proof. Let z be a point in E which is distinct from x. The three points x, y and z determine a unique block E' in D, and E' is clearly secant to E. Since there are $q-i$ distinct points in E that are different from x, there are $q-i$ blocks in D which contain x and y and are secant to E. This implies that all other bocks containing x and y are tangent to E at x. Since there are $q+1$ blocks containing x and y and $(q+1)-(q-i)=i+1$, the conclusion of the lemma follows.

LEMMA 9. *Let $i \in \{0, 1, 2\}$ and E be a fixed $(q+1-i)$-valent block in D. If x is a point incident with E, then there exist exactly $(i+1)q-1$ blocks which are tangent to E at x.*

Proof. For every point y in E, different from x, the points x and y are contained in q blocks other than E. Hence, there are $q(q-1)$ blocks which contain x and are secant to E. But every block incident with x other than E is either a secant or a tangent of E, and since there are q^2+q-1 blocks incident with x other than E, the number of blocks tangent to E at x is $q^2+q-1-q(q-i)$, or $q(i+1)-1$.

LEMMA 10. *If A is a $(q+1)$-valent block and x is a point in A, then the $q-1$ tangents of A at x are mutually tangent to each other at x.*

Proof. Let E and E' be two distinct tangents of A at x, and suppose E and E' intersect at two distinct points x and y. Since y is a point not in A, by Lemma 8, there exists a unique block containing y which is tangent to A at x. But both E and E' contain y and are tangent to A at x. Hence, E and E' are mutually tangent at x.

By virtue of this lemma, we see that the tangents of A at x together with A constitute a pencil in D with carrier x.

PROPOSITION 11. *If $\mathfrak{A}$ is a pencil with carrier x such that $\mathfrak{A}$ contains a $(q+1)$-valent block A, then $\mathfrak{A}$ contains q blocks and $\mathfrak{A}$ partitions the points in D that are distinct from x.*

Proof. Let A be a $(q+1)$-valent block in $\mathfrak{A}$. For every point y distinct from x, there exists a unique block E in $\mathfrak{A}$ such that E contains y and E is tangent to A at x. Hence, $\mathfrak{A}$ partitions the points that are distinct from x. It is clear from the previous lemma that $\mathfrak{A}$ contains q blocks.

From the above, we see that a $(q+1)$-valent block A cannot be r-tangent to any other block E. The converse is also valid.

LEMMA 12. *Let E be a block in D. If A and A' are two distinct $(q+1)$-*

valent blocks that are tangent to E at a point x, then A and A' are mutually tangent at x.

Proof. Suppose A is not tangent to A' at x. Let x and y be the two points of intersection of A and A'. For every point z in A' such that $z \neq x$ and $z \neq y$, there exists a unique block tangent to A at x which contains z. Hence, there are $q-1$ blocks that are tangent to A at x and are secant to A'. But there are only $q-1$ tangents of A at x of which E is one. This contradicts that E is tangent to A' at x. Hence, A and A' are mutually tangent.

From this, we observe that if A is a $(q+1)$-valent block and A is not r-tangent to E, then E is not r-tangent to A. We shall establish this property for every block E in D. That is, we want to show that the relation, "r-tangency," is a symmetric relation.

LEMMA 13. *Let $q \geqslant 4$ and let E be a block in D such that E is not $(q+1)$-valent. If x is a point in E, then there exists at least one $(q+1)$-valent block A tangent to E at x.*

Proof. Let $|E| = q+1-i$, where $i \in \{1, 2\}$. For $k \in K$, let $t_i(k)$ denote the number of k-valent blocks tangent to E at x. By Lemma 9, there are $(i+1)q-1$ blocks tangent to E at x. Hence

$$t_i(q+1) + t_i(q) + t_i(q-1) = (i+1)q - 1. \tag{15}$$

Next, we count the number of ordered pairs (y, E') such that $y \in E'$, $y \notin E$ and E' is tangent to E at x. For each point y not in E, there are $i+1$ blocks containing y and tangent to E at x. Since there are $v-(q+1-i)$ such points y, we have

$$qt_i(q+1) + (q-1)\,t_i(q) + (q-2)\,t_i(q-1) \\ = (i+1)(q^2 - 2q - 1 + i). \tag{16}$$

Combining (15) and (16), we obtain

$$2t_i(q+1) + t_i(q) = i^2 + q - 3. \tag{17}$$

Case 1. $|E| = q$ (i.e., $i=1$). By Corollary 7, there exists either none or exactly one q-valent block tangent to E at x, depending on whether q is even or odd. Hence, $t_i(q) \leqslant 1$. From (17), we obtain the inequality,

$$2t_i(q+1) \geqslant q - 3.$$

For $q \geqslant 4$, $t_1(q+1) > 0$. Hence, there exists at least one $(q+1)$-valent block tangent to E at x.

Case 2. $|E| = q-1$ (i.e., $i=2$). Suppose there exists no $(q+1)$-valent block tangent to E at x; then $t_2(q+1)=0$ and from (17), $t_2(q)=q+1$. Hence, every q-valent block that contains x is tangent to E; this implies that for every point y in E distinct from x, the points x and y are not contained in any q-valent block. We show that this cannot be the case.

If $q \equiv 0 \pmod 2$, then by Lemma 6 there exists at least one q-valent block containing x and y. Hence, we arrive at a contradiction.

If $q \equiv 1 \pmod 2$, then by Lemma 5 for each y in E distinct from x exactly half of the $q+1$ blocks that contain x and y are $(q-1)$-valent; thus there are $\frac{1}{2}(q+1)(q-2)$ $(q-1)$-valent blocks containing x and secant to E. Since there are $\frac{1}{2}(q+1)q$ $(q-1)$-valent blocks containing x, there are q $(q-1)$-valent blocks tangent to E at x. But if $t_2(q+1)=0$ and $t_2(q)=q+1$, then by Eq. (17), $t_2(q-1)=2q-2$. Hence, $q=2q-2$ or $q=2$. This contradicts that $q \equiv 1 \pmod 2$. Consequently, $t_2(q+1) \neq 0$, and there exists at least one $(q+1)$-valent block tangent to E at x.

LEMMA 14. *Let E and E' be two distinct blocks in D. If E is r-tangent to E' at x, then E' is r-tangent to E at x.*

Proof. If E is r-tangent to E' at x, then there exists a $(q+1)$-valent block A that is tangent to E at x and is secant to E'. Suppose E' is not r-tangent to E at x; then for every $(q+1)$-valent block A' that is tangent to E' at x, A' is also tangent to E. Consider the two $(q+1)$-valent blocks A and A'. Since both A and A' are tangent to E at x, A is tangent to A' at x. But E' is also tangent to A' at x; hence by Lemma 10, E' is tangent to A at x. This contradicts that A is secant to E'. Therefore, E' is r-tangent to E at x.

Thus we see that r-tangency is a symmetric relation. Let us define two distinct blocks E and E' to be *M-tangent* at x if and only if E is tangent to E' but is not r-tangent to E' at x. A pencil with carrier x is called an *M-pencil* if the blocks in the pencil are mutually M-tangent.

LEMMA 15. *If E is M-tangent to E' at x and E' is M-tangent to E'' at x, then E is M-tangent to E'' at x.*

Proof. Let A be a $(q+1)$-valent block in D such that A is tangent to E at x. Since E is not r-tangent to E', A is tangent to E' at x. But E' is M-tangent to E'' at x; hence A is also tangent to E'' at x. Consequently, E is not r-tangent to E'' at x, or equivalent, E is M-tangent to E'' at x.

From this lemma, we observe that any pencil that contains a $(q+1)$-valent block is an M-pencil. Next, we shall study blocks that are tangent to a given q-valent block B.

LEMMA 16. *Let B be a fixed q-valent block in D and x be a point incident with B. If A is a $(q+1)$-valent block tangent to B at x, then there exist exactly q blocks which are tangent to B at x and are secant to A.*

Proof. For $i = 1, 2$, let

$$T_i = \{E \mid E \text{ is a block in } D \text{ tangent to } B \text{ at } x, E \neq A \text{ and } |E \cap A| = i\}.$$

The two sets T_1 and T_2 partition the set of tangents of B other than A at x. By Lemma 9,

$$|T_1| + |T_2| = 2q - 2. \tag{18}$$

Next, we count the number of ordered pairs (y, E) where E is tangent to B at x, $E \neq A$ and $y \in E \cap A$. If y is distinct from x, then by Lemma 9, there exist two blocks containing y and tangent to B at x, of which one is A. Hence, there exists a unique choice of E. If $y = x$, then by Lemma 9, there are $2q - 2$ choices of E. Since there are q points in A that are distinct from x, the number of ordered pairs (y, E) is $q + 2q - 2$. On the other hand, if E is tangent to B at x and $|E \cap A| = i$, $i = 1, 2$, then there are i choices of y. Thus,

$$|T_1| + 2|T_2| = q + 2q - 2. \tag{19}$$

Using (18) and (19), we obtain $|T_2| = q$. Therefore, there are q blocks tangent to B at x and secant to A.

It should be noted that B is r-tangent to these q blocks at x. Next, we show that they are mutually tangent at x.

LEMMA 17. *Let B be a q-valent block and A be a $(q+1)$-valent block tangent to B at a point x. If E and E' are two distinct blocks tangent to B at x and secant to A, then E and E' are mutually tangent at x.*

Proof. Suppose E and E' intersect each other at two points x and y. If $y \in A$, then E, E' and A are three blocks containing y and tangent to B at x. This contradicts the fact that there exist only two such tangents (by Lemma 8). Thus, the point y is not contained in A.

Since $y \notin A$ and A is $(q+1)$-valent, by Lemma 8, there exists a unique block E'' containing y and tangent to A at x. But both E'' and B are tangent to A at x, by Lemma 8, E'' and B are mutually tangent at x. Hence, E, E' and E'' are three blocks tangent to B at x and containing y. This contradicts that there are only two such blocks.

Therefore, E and E' cannot intersect at two points and hence, they are mutually tangent at x.

LEMMA 18. *Let A and B be $(q+1)$-valent and q-valent blocks, respectively, such that A is tangent to B at x. If C is tangent to B at x and secant to A, then C is a $(q-1)$-valent block.*

Proof. Let $C_1, \ldots, C_q$ be the q blocks that are tangent to B at x and secant to A. Since for $1 \leqslant i < j \leqslant q$, C_i and C_j are mutually tangent at x and each C_i has at least $q-2$ points other than x, we have

$$q(q-2) \leqslant \sum_{i=1}^{q} (|C_i| - 1) \leqslant q^2 - q - |B|.$$

But B is q-valent; hence $\sum_{i=1}^{q}(|C_i| - 1) = q(q-2)$. Consequently, each C_i is a $(q-1)$-valent block.

From the proof, we also observe that the blocks $C_1, \ldots, C_q$ and B partition the points distinct from x. In fact, we shall show that they form an r-pencil in D with carrier x.

LEMMA 19. *Let $B, C_1, \ldots, C_q$ be as defined in the previous lemma and x be their common point. Let $T = \{B, C_1, \ldots, C_q\}$. If E and E' are two distinct blocks in T and A is a $(q+1)$-valent block tangent to E at x, then A is secant to E'.*

Proof. Consider a point y in A distinct from x. Since the blocks in T partition the points distinct from x, there exists a unique block E_y in T such that E_y contains x and y. Thus, there are q blocks in T that are secant to A.

But T consists of $q+1$ blocks; hence there exists a unique block in T that is tangent to A at x, and E is such a block. Thus, $|A \cap E'| = 2$ and the conclusion of the lemma follows.

PROPOSITION 20. *Let B be a q-valent block in D and x be a point in B. The pair (x, B) determines a unique r-pencil in D with carrier x, denoted by $\mathfrak{P}(x, B)$. Furthermore, the r-pencil $\mathfrak{P}(x, B)$ consists of $q+1$ blocks and they partition the points distinct from x. We shall call B the* carrier block *of $\mathfrak{P}(x, B)$.*

Proof. Let $T = \{B, C_1, \ldots, C_q\}$ be defined as above. For every block in T, there exists a $(q+1)$-valent block tangent to E at x. Hence, by the previous lemma E is r-tangent to every other block in T. Thus, they form an r-pencil at x.

Next, we count the number of points contained in T. Since the q $(q-1)$-valent blocks and the q-valent block B in T are mutually tangent at x, we have $q(q-2) + q = q^2 - q$ points in T. Thus, the blocks in T partition the points distinct from x.

Since for every $(q+1)$-valent block A that is tangent to B at x, A is secant to C_i in T, $1 \leqslant i \leqslant q$, and by Lemma 16, $C_1, \ldots, C_q$ are the only blocks that are r-tangent to B at x. Thus, $\mathfrak{P}(x, B)$ is the unique r-pencil with carrier x and carrier block B.

COROLLARY 21. *Let B be a q-valent block in D. B is r-tangent to exactly q^2 $(q-1)$-valent blocks in D.*

COROLLARY 22. *Let B be a q-valent block in D. If x is a point not in B, then B is r-tangent to exactly q blocks in D that contain x. Furthermore, these q blocks are $(q-1)$-valent.*

Proof. Let y be a point in B. $\mathfrak{P}(y, B)$ partitions the points distinct from y; hence there exists a unique block C_y containing x that is r-tangent to B at y. Since there are q points in B, there are q blocks containing x that are r-tangent to B. Clearly, these are the only r-tangents of B that contain x.

Eventually, we would like to show that these q $(q-1)$-valent blocks that are r-tangent to B and contain x determine an r-pencil in D with carrier x. Let us first conclude our discussions on q-valent blocks in the following theorem.

THEOREM 23. *Let B be a q-valent block. If x is a point in B, then there exist exactly one r-pencil and one M-pencil with carrier x that contain B.*

Proof. Since B is q-valent, by Lemma 9 there are $2q-1$ blocks which are tangent to B at x. By Proposition 20, q of these tangents together with B form an r-pencil $\mathfrak{P}(x, B)$ with carrier x. Among the remaining $q-1$ tangents of B, there is a $(q+1)$-valent block A. Since every tangent of B at x is not r-tangent to A at x, these $q-2$ tangents of B together with A and B form an M-pencil with carrier x.

Finally, we shall study the tangents of a $(q-1)$-valent block C at a point x.

LEMMA 24. *Let C be a $(q-1)$-valent block in D and x be a point in C. If $g(C, x)$ denotes the number of 2-valent blocks r-tangent to C at x, then* $\operatorname{ave} g(\cdot, x) = 2$.

Proof. We count the number of ordered pairs (B, C) where B and C are q-valent and $(q-1)$-valent blocks, respectively, and B is r-tangent to C at x. For every q-valent block B containing x, there are q choices of C. Since there are $q+1$ q-valent blocks containing x, there are $q(q+1)$ ordered pairs (B, C). On the other hand, if C is a $(q-1)$-valent block containing x, then there are $g(C, x)$ choices of B. Hence, $\sum_C g(C, x) = q(q+1)$. But there are $\frac{1}{2}q(q+1)$ $(q-1)$-valent blocks containing x; thus $\operatorname{ave} g(\cdot, x) = 2$.

PROPOSITION 25. *Let C be a $(q-1)$-valent block in D. If x is a point in C, then there exist exactly two q-valent blocks that are r-tangent to C at x.*

Proof. We shall show that there exist at most two q-valent blocks that are r-tangent to C at x, then using the previous lemma, we obtain the conclusion of the proposition.

Let us recall that for $k \in K$, $t_2(k)$ denotes the number of k-valent blocks tangent to C at x. From (17) we have

$$2t_2(q+1) + t_2(q) = q + 1. \tag{20}$$

For $k \in K$, let $t'(k)$ denote the number of k-valent blocks M-tangent to C at x. From Lemma 12, $t'(q+1) = t_2(q+1)$. Let A be a $(q+1)$-valent block M-tangent to C at x, and let $\mathfrak{P}$ denote the M-pencil with carrier x that contains A. C is a block in $\mathfrak{P}$. Since for every block E in $\mathfrak{P}$, $E \neq C$, E is M-tangent to C at x and they are the only M-tangents of C at x.

$$t_2(q+1) + t'(q) + t'(q-1) + 1 = |\mathfrak{P}| = q. \tag{21}$$

On the other hand, the blocks in $\mathfrak{P}$ partition the points that are distinct from x; we have

$$qt_2(q+1) + (q-1)\,t'(q) + (q-2)(t'(q)+1) + 1 = q^2 - q. \tag{22}$$

Using (21) and (22), we obtain

$$2t_2(q+1) + t'(q) = q - 1. \tag{23}$$

Combining (20) and (23), we get

$$t_2(q) - t'(q) = 2.$$

Thus, there exists at most two q-valent blocks r-tangent to C at x and the proof is complete.

PROPOSITION 26. *Let C be a $(q-1)$-valent block in D. If x is a point in C, then C is contained in exactly one M-pencil with carrier x. Furthermore, there are at least two r-pencils with carrier x that contain C.*

Proof. From the proof of the previous lemma, if A is a $(q+1)$-valent block tangent to C at x, then the M-pencil with carrier x that contains A is the unique M-pencil with carrier x that contains C.

Since there are exactly two q-valent blocks r-tangent to C at x and each of these two blocks determines a unique r-pencil with carrier x, C is contained in at least two r-pencils.

We shall show that C is contained in exactly two r-pencils with carrier x.

LEMMA 27. *Let $\mathfrak{P}(x, B)$ and $\mathfrak{P}(x, B')$ be two distinct r-pencils with carrier x and carrier block B and B', respectively. If $c(B, B')$ denotes the number of common blocks in $\mathfrak{P}(x, B)$ and $\mathfrak{P}(x, B')$, then* $\operatorname{ave} c(\cdot, \cdot) = 1$, *where average runs over all pairs of distinct q-valent blocks containing x.*

Proof. Let us count the number of triples $(\mathfrak{P}(x, B), \mathfrak{P}(x, B'), C)$ where $\mathfrak{P}(x, B)$ and $\mathfrak{P}(x, B')$ are distinct r-pencils with carrier x and C is a common block in $\mathfrak{P}(x, B)$ and $\mathfrak{P}(x, B')$. For every $(q-1)$-valent block C that contains x, there are exactly two q-valent blocks B and B' that contain x and are r-tangent to C. Hence, there are exactly two r-pencils with carrier x that contain a q-valent block and C. Thus, there are two choices for the pair $(\mathfrak{P}(x, B), \mathfrak{P}(x, B'))$. But there are $\frac{1}{2}(q+1)q$ choices of C that contain x, so the total number of triples is $(q+1)q$. On the other hand, for every distinct pair $(\mathfrak{P}(x, B), \mathfrak{P}(x, B'))$, there are $c(B, B')$ choices of C. Hence,

$$\sum c(B, B') = q(q+1),$$

where the summation runs over all pairs $(\mathfrak{P}(x, B), \mathfrak{P}(x, B'))$. But for every q-valent block B containing x, there corresponds a unique r-pencil $\mathfrak{P}(x, B)$; hence there are $(q+1)q$ distinct pairs $(\mathfrak{P}(x, B), \mathfrak{P}(x, B'))$. Thus,

$$(q+1)q \operatorname{ave} c(\cdot, \cdot) = q(q+1),$$

or equivalently, $\operatorname{ave} c(\cdot, \cdot) = 1$.

LEMMA 28. *If $\mathfrak{P}(x, B)$ and $\mathfrak{P}(y, B')$ are two distinct r-pencils with carriers x and y, and carrier blocks B and B', respectively, then $|\mathfrak{P}(x, B) \cap \mathfrak{P}(y, B')| \leqslant 1$. In particular, if $x = y$, then $|\mathfrak{P}(x, B) \cap \mathfrak{P}(x, B')| = 1$.*

Proof. *Case* 1. $x \neq y$ and $B = B'$. Clearly $\mathfrak{P}(x, B) \cap \mathfrak{P}(y, B') = B$.

Case 2. $x \neq y$ and $B \neq B'$. Suppose $|\mathfrak{P}(x, B) \cap \mathfrak{P}(y, B') \geqslant 2$ and let C, C' be two common r-tangents of B and B' at x and y, respectively. Since by Proposition 20, the blocks in $\mathfrak{P}(x, B)$ partition the points distinct from x, there exists a unique block E in $\mathfrak{P}(x, B)$ that contains y. But this contradicts that both C and C' contain y. Hence, $|\mathfrak{P}(x, B) \cap \mathfrak{P}(y, B')| \leqslant 1$.

Case 3. $x = y$ and $B \neq B'$. For every point z in B', $z \neq x$, there exists a unique block in $\mathfrak{P}(x, B)$ that contains z. Since there are $q-1$ points in B' that are distinct from x, there are $q-1$ blocks in $\mathfrak{P}(x, B)$ that are secant to B'. But $\mathfrak{P}(x, B)$ consists of $q+1$ blocks; hence there are two blocks in $\mathfrak{P}(x, B)$ that are tangent to B' at x. If both these blocks are M-tangent to B', then they are contained in the M-pencil with carrier x that contains B'; hence they are mutually M-tangent to each other. This contradicts that they are both in $\mathfrak{P}(x, B)$. Thus, there exists at least one block in $\mathfrak{P}(x, B)$ that is r-tangent to B' at x. By the previous lemma, we obtain that there exists exactly one, that is, $|\mathfrak{P}(x, B) \cap \mathfrak{P}(x, B')| = 1$.

LEMMA 29. *Let C be a $(q-1)$-valent block and x be a point in C. Let $\mathfrak{P}(x, B)$ and $\mathfrak{P}(x, B')$ be the two r-pencils with carrier x and carrier blocks B*

and B' that contain C. If C' is a $(q-1)$-valent block that is r-tangent at x, then C' is contained in exactly one of the two r-pencils $\mathfrak{P}(x, B)$ and $\mathfrak{P}(x, B')$.

Proof. Since each of the two r-pencils consists of $q+1$ blocks and C is a common block in $\mathfrak{P}(x, B)$ and $\mathfrak{P}(x, B')$, there are $2q$ blocks that are r-tangent to C at x and are contained in $\mathfrak{P}(x, B)$ and $\mathfrak{P}(x, B')$. By Lemma 9, there are $3q-1$ tangents of C at x of which $q-1$ are contained in the unique M-pencil with carrier x that contains C. Thus, there are only $2q$ blocks that are r-tangent to C at x, and they are contained in either $\mathfrak{P}(x, B)$ or $\mathfrak{P}(x, B')$. Clearly, an r-tangent C' of C cannot be contained in both $\mathfrak{P}(x, B)$ and $\mathfrak{P}(x, B')$; otherwise C and C' are two common blocks of the r-pencils and this contradicts the previous lemma.

LEMMA 30. *Let C and C' be two distinct $(q-1)$-valent blocks. If C is r-tangent to C' at x, then there exists a unique block which is r-tangent to C' and M-tangent to C at x.*

Proof. Let $\mathfrak{P}$ denote the M-pencil with carrier x that contains C. For every point y in C', $y \neq x$, there exists a unique block E in $\mathfrak{P}$ such that E contains x and y. Hence, there are $q-2$ blocks in $\mathfrak{P}$ that are M-tangent to C but are secant to C'. But $\mathfrak{P}$ consists of q blocks of which one is C; hence there exists a unique block E M-tangent to C at x which is tangent to C'. Clearly, E is not M-tangent to C' at x; otherwise, $C' \in \mathfrak{P}$; this contradicts that C is r-tangent to C'. Hence E is r-tangent to C' at x.

LEMMA 31. *Let C and C' be two distinct $(q-1)$-valent blocks containing x. If C is r-tangent to C' at x, then there exist exactly q blocks r-tangent to both C and C' at x.*

Proof. Let $\mathfrak{P}(x, B)$ be an r-pencil with carrier x and carrier block B that contains both C and C'. Since $\mathfrak{P}(x, B)$ contains $q+1$ blocks, there are at least $q-1$ blocks that are r-tangent to both C and C'.

Let $\mathfrak{P}(x, B')$ be the other r-pencil containing C. If E is a block r-tangent to C at x and $E \notin \mathfrak{P}(x, B)$, then $E \in \mathfrak{P}(x, B')$. We shall show that there exists exactly one block E in $\mathfrak{P}(x, B')$ such that E is r-tangent to both C and C' at x.

For every point y in C', $y \neq x$, there exists a unique block E in $\mathfrak{P}(x, B')$ such that E contains x and y. Hence, there are $q-1$ r-tangents of C at x that are secant to C'. But $\mathfrak{P}(x, B')$ contains $q+1$ blocks, of which one is C; hence there are two blocks in $\mathfrak{P}(x, B')$ that are tangent to C' at x. By the previous lemma, one of these two blocks is M-tangent to C' at x. Therefore, there exists a unique block E in $\mathfrak{P}(x, B')$ such that E is r-tangent to C' at x.

Consequently, there are exactly q blocks r-tangent to both C and C' at x.

PROPOSITION 32. *Let C be a $(q-1)$-valent block. If x is a point in C, then C is contained in exactly two r-pencils with carrier x. Moreover, every r-pencil in D contains a unique q-valent carrier block.*

Proof. Suppose $\mathfrak{P}$ is another r-pencil containing C such that $\mathfrak{P}$ is distinct from $\mathfrak{P}(x, B)$ and $\mathfrak{P}(x, B')$. Let C' be a block in $\mathfrak{P}$ and in $\mathfrak{P}(x, B)$. Since $\mathfrak{P} \neq \mathfrak{P}(x B)$, there exists a block C'' in $\mathfrak{P}$ such that $C'' \notin \mathfrak{P}(x, B')$. By the previous lemma, C'' is the only other block in $\mathfrak{P}$. Thus $|\mathfrak{P}| = 3$. But by the definition of an r-pencil, $|\mathfrak{P}| \geqslant 4$; hence $\mathfrak{P}(x, B)$ and $\mathfrak{P}(x, B'')$ are the only two r-pencils with carrier x that contain C.

PROPOSITION 33. *Let E_1, E_2, E_3 be mutually tangent at a point x such that they are not contained in any M-pencil or any r-pencil with carrier x. If $\mathfrak{P}$ is a maximal set of mutually tangent blocks containing x such that $\mathfrak{P}$ contains E_1, E_2 and E_3, then $\mathfrak{P}$ contains at most four blocks.*

Proof. Since E_1, E_2 and E_3 are not contained in any M-pencil or any r-pencil, either they are mutually r-tangent and are contained in two distinct r-pencils at x, or E_1 is M-tangent to E_2.

Case 1. E_1, E_2 and E_3 are mutually r-tangent. From the previous proposition, there exists no other block that is r-tangent to all E_i's at x. Let E be a block in $\mathfrak{P}$, $E \neq E_i$, $i = 1, 2, 3$. Without loss of generality, E is M-tangent to E_1 at x, then E is not M-tangent to E_2 and E_3. Otherwise, E_1 and E_2 are mutually M-tangent and this contradicts our assumption. Hence, E_1 is r-tangent to E_2 at x. By Lemma 30, E is the unique block that is M-tangent to E_1 and r-tangent to E_2 at x. Thus $|\mathfrak{P}| \leqslant 4$.

Case 2. E_1 is M-tangent to E_2. By Lemma 15, E_3 is r-tangent to both E_1 and E_2 at x. Moreover, E_1, E_3 and E_2, E_3 are contained in distinct r-pencils at x. Let $E \in \mathfrak{P}$, $E \neq E_i$, $i = 1, 2, 3$.

Subcase 2.1. E is r-tangent to E_i, $i = 1, 2, 3$. Since the pairs (E_1, E_3) and (E_2, E_3) are in distinct r-pencils at x, either E, E_1, E_3 or E, E_2, E_3 are three mutually r-tangent blocks at x that are contained in distinct r-pencils. By Case 1, $|\mathfrak{P}| \leqslant 4$.

Subcase 2.2. E is M-tangent to E_i for some i, $1 \leqslant i \leqslant 3$. Suppose E is M-tangent to E_1 at x; then E is not M-tangent to E_3; otherwise, E_1 and E_3 are mutually M-tangent at x. But then E_2 and E are two blocks that are M-tangent to E_1 and r-tangent to E_3; this contradicts that there exists such a unique block. Hence, E is r-tangent to E_1 at x. Similarly, E is r-tangent to E_2 at x. Thus, E is M-tangent to E_3 at x.

Suppose $E' \in \mathfrak{P}$, $E' \neq E$, E_i, $i = 1, 2, 3$. Using the same arguments as above, E' is r-tangent to both E_1 and E_2 at x. If E' is also r-tangent to E_3, then by Subcase 2.1, $|\mathfrak{P}| \leqslant 4$. This contradicts that $E' \neq E$. Hence, E' is M-

tangent to E_3 at x. But then E and E' are two distinct blocks that are M-tangent to E_3 and r-tangent to E_1 at x. This contradicts that there exists only one such block. Hence E' does not exist, and $|\mathfrak{P}| \leqslant 4$.

6. Parallel Classes

In this section, we shall establish the parallel classes of q-valent blocks by looking at the r-pencils in D. First let us state

The Tangency Condition. Let B be a q-valent block. If x and y are two distinct points not in B, then there exists at most one block containing x and y which is r-tangent to B.

Let D be a PBRD(q) such that D satisfies the Tangency Condition. Let B be a q-valent block in D and let x be a point in B. By Corollary 22, there are q $(q-1)$-valent blocks that contain x and are r-tangent to B. We shall show that the Tangency Condition implies that these q blocks are mutually r-tangent at x.

Lemma 34. *Let B be a q-valent block in D. If x is a point not in B, then the q blocks that contain x and are r-tangent to B are mutually tangent to each other.*

Proof. Suppose C and C' are two r-tangents of B that contain x and intersect at two points x and y. Clearly, x and y are two distinct points not in B. But this contradicts the Tangency Condition. Hence, C and C' are mutually tangent.

Proposition 35. *Let $q \geqslant 5$ and B be a q-valent block in D. If x is a point not in B, then the q blocks containing x and r-tangent to B are mutually r-tangent to each other.*

Proof. Let $C_1,..., C_q$ be the $(q-1)$-valent blocks r-tangent to B which contain x. If $C_1,..., C_q$ are mutually M-tangent, then there exists a $(q+1)$-valent block A containing x which is tangent to $C_1,..., C_q$. But this contradicts that every M-pencil contains only q blocks. Suppose C_1 is M-tangent to C and C_1 is r-tangent to C_3; then by Proposition 33, a maximal set $\mathfrak{P}$ of mutually tangent blocks that contain C_1, C_2 and C_3 contains at most four blocks. But $q \geqslant 5$; hence, $C_1,..., C_q$ are mutually r-tangent at x.

Corollary 36. *The blocks $C_1,..., C_q$ determine a unique r-pencil $\mathfrak{P}(x, B')$ with carrier x.*

Proof. By Proposition 33, $C_1, \ldots, C_q$ are contained in the same r-pencil $\mathfrak{P}(x, B')$ with carrier x.

PROPOSITION 37. *Let $q \geqslant 5$ and B be a q-valent block in D. Let x be a point not in B. If $\mathfrak{P}(x, B')$ is the r-pencil with carrier x such that each $(q-1)$-valent block in $\mathfrak{P}(x, B')$ is r-tangent to B, then B' is disjoint from B.*

Proof. Since there are q $(q-1)$-valent blocks in $\mathfrak{P}(x, B')$, for each point y in B, there exists a unique $(q-1)$-valent block r-tangent to B at y. The blocks in $\mathfrak{P}(x, B')$ partition the points distinct from x; hence, B and B' are disjoint.

COROLLARY 38. *Every r-tangent of B is an r-tangent of B', and vice versa.*

Proof. For every point x in B' there exist q $(q-1)$-valent blocks r-tangent to B which contain x. These q blocks, together with B', form an r-pencil $\mathfrak{P}(x, B')$. Hence, they are also r-tangents of B'. Since there are q points in B', there are q^2 blocks that are r-tangents of both B and B'. But by Corollary 21, B has only q^2 r-tangents. Thus, every r-tangent of B is an r-tangent of B'.

Next, we shall construct the parallel classes.

DEFINITION. Let B and B' be two q-valent blocks in D. B is said to be parallel to B' if and only if either $B = B'$, or B is disjoint from B' and every r-tangent of B is an r-tangent of B' and vice versa. We shall denote them by $B // B'$.

PROPOSITION 39. *If $B // B'$ and $B' // B''$, then $B // B''$.*

Proof. Suppose $x \in B \cap B''$. Consider the q $(q-1)$-valent blocks that contain x and are r-tangent to B'; these q blocks determine a unique r-pencil $\mathfrak{P}(x, B)$ with carrier x. Hence, $B = B''$.

Suppose $B \neq B''$, then every r-tangent of B is an r-tangent of B', which, in turn, is an r-tangent of B''. Thus every r-tangent of B is an r-tangent of B'' and $B \cap B'' = \varnothing$, so $B // B''$.

PROPOSITION 40. *Each q-valent block B is contained in a parallel class $\mathfrak{P}(B)$, and $\mathfrak{P}(B)$ consists of $q-1$ blocks.*

Proof. Let us count the number of ordered pairs (x, B') such that $x \in B'$ and $B // B'$. For every point x in D, there are q $(q-1)$-valent blocks r-tangent to B and containing x. They determine a unique q-valent block B' parallel to

B. Hence, there are $q^2 - q$ pairs. On the other hand, for every block parallel to B, there are q choices of x; hence,

$$q \cdot (\text{number of blocks parallel to } B) = q^2 - q,$$

or

$$\text{number of blocks parallel to } B = q - 1.$$

Since parallelism is a transitive relation, these $q - 1$ blocks are mutually parallel to each other. Furthermore, they partition the points in D; hence, they form a parallel class of $\mathfrak{P}(B)$.

COROLLARY 41. *There are $q + 1$ parallel classes in D.*

Proof. Since each parallel class contains $q - 1$ blocks and there are $q^2 - 1$ q-valent blocks in D, there are $q + 1$ parallel classes in D.

7. PROOF OF THE MAIN THEOREM FOR $q \geqslant 5$

From the previous section, we have found the $q + 1$ parallel classes in D. Next we have to establish the r-transversals of these parallel classes.

LEMMA 42. *Let C be a $(q - 1)$-valent block in D. C is an r-transversal of exactly two parallel classes in D.*

Proof. Let x be a fixed point in C. There exist two q-valent blocks B and B' containing x and r-tangent to C. Clearly, B and B' are in different parallel classes $\mathfrak{P}(B)$ and $\mathfrak{P}(B')$. Since C is an r-tangent of B, C is an r-tangent of every block in $\mathfrak{P}(B)$, that is, C is an r-transversal of $\mathfrak{P}(B)$. Similarly, C is an r-transversal of $\mathfrak{P}(B')$. Clearly, $\mathfrak{P}(B)$ and $\mathfrak{P}(B')$ are the only two parallel classes for which C is an r-transversal.

Next, we show that there are q common r-transversals for every two distinct parallel classes.

LEMMA 43. *Every two distinct parallel classes have exactly q common r-transversals and they are disjoint.*

Proof. Let $\mathfrak{P}(B)$ and $\mathfrak{P}(B')$ be two distinct parallel classes. We first show that any two common r-transversals of $\mathfrak{P}(B)$ and $\mathfrak{P}(B')$ are disjoint. Suppose C and C' are two common r-transversals such that $x \in C \cap C'$. Let B and B' be the q-valent blocks in $\mathfrak{P}(B)$ and $\mathfrak{P}(B')$, respectively, such that B and B' contain x. Since C and C' are both r-tangents of B and B' at x, $|\mathfrak{P}(x, B) \cap \mathfrak{P}(x, B')| \geqslant 2$. This contradicts Lemma 28 that there exists a unique block r-tangent to both B and B' at x. Thus, the common r-transversals of $\mathfrak{P}(B)$ and $\mathfrak{P}(B')$ are pairwise disjoint.

Let us count the number of triples $(C, \mathfrak{P}(B), \mathfrak{P}(B'))$ where C is a common r-transversal of $\mathfrak{P}(B)$ and $\mathfrak{P}(B')$. For every $(q-1)$-valent block C in D, there exist exactly two parallel classes of which C is an r-transversal. Since there are $\frac{1}{2}q^2(q+1)$ $(q-1)$-valent blocks,

$$\sum \text{number of common } r\text{-transversals of } \mathfrak{P}(B) \text{ and } \mathfrak{P}(B')$$
$$= \tfrac{1}{2}q^2(q+1) \cdot 2 \cdot 1 = q^2(q+1),$$

where the sum runs over all pairs $(\mathfrak{P}(B), \mathfrak{P}(B'))$. But there are $q+1$ distinct parallel classes; hence

average number of common r-transversals of two distinct parallel classes $= (q^2(q+1))/(q+1)q = q$.

Since the common r-transversals are pairwise disjoint and there are $q^2 - q$ points in D, there are at most q common r-transversals of $\mathfrak{P}(B)$ and $\mathfrak{P}(B')$. Thus, every two distinct parallel classes have exactly q common r-transversals.

Thus far we see that D is a PBRD(q) that satisfies axioms (A1) and (A2) in the Fundamental Lemma. Next, we shall establish axiom (A3).

LEMMA 44. *Let x and y be two distinct points in D. If $\mathfrak{P}(B)$ is a parallel class in D, then either there exists a q-valent block in $\mathfrak{P}(B)$ containing x and y, or there exists exactly one r-transversal of $\mathfrak{P}(B)$ containing x and y.*

Proof. Let B be the block in $\mathfrak{P}(B)$ containing x. If $y \in B$, then we are done. If $y \notin B$, then there exists a unique block C containing y and r-tangent to B at x. Since C is an r-tangent of B, C is clearly an r-transversal of $\mathfrak{P}(B)$. The proof is thus complete.

From the lemmas, we see that D satisfies axioms (A1)–(A3) in the Fundamental Lemma; hence D is embeddable into a Möbius plane. Thus we conclude,

THEOREM 45. *Let $q \geqslant 5$. If D is a PBRD(q) that satisfies the Tangency Condition, then D is uniquely embeddable into a Möbius plane of order q.*

8. PROOF OF THE THEOREM

The block-residual design of a Möbius plane obviously satisfies the Tangency Condition. Let D be a PBRD(q) that satisfies the r-tangency condition. If $q \geqslant 5$, then by Theorem 45, D is uniquely embeddable. Next, we consider $q = 1$, 2, 3 and 4.

For $q = 1$, the design PBRD(q) is a null design and is trivially embeddable.

For $q = 2$, PBRD(2) consists of two points and nine blocks. Let the points be $\{1, 2\}$. Since PBRD(2) is a 1-design and by Lemma 5, there are no 3-valent blocks, three 2-valent blocks and six 1-valent blocks. The blocks of PBRD(2) are,

1 2,	1,	2,
1 2,	1,	2,
1 2,	1,	2.

To complete this design to a Möbius plane of order 2, we adjoin the new points $\{3, 4, 5\}$ to the blocks and form

1 2 3,	1 3 4,	2 3 4,
1 2 4,	1 3 5,	2 3 5,
1 2 5,	1 4 5,	2 4 5,
3 4 5.		

Hence, PBRD(2) can be uniquely embedded into a Möbius plane.

For $q = 3$, there are six points in PBRD(3). Let them be $\{1, 2, 3, 4, 5, 6\}$. Using Lemma 4 and the fact that it is a 2-design, one can check that the blocks of PBRD(3) are isomorphic to the following:

1 2 3 4	1 3 5	2 3 5	1 2	2 3	3 5
1 2 5 6	1 3 6	2 3 6	1 2	2 4	3 6
3 4 5 6	1 4 5	2 4 5	1 3	2 5	4 5
	1 4 6	2 4 6	1 4	2 6	4 6
			1 5	3 4	5 6
			1 6	3 4	5 6.

If we define

$$\mathfrak{A}_1 = \{1\ 3\ 5,\quad 2\ 4\ 6\},$$

$$\mathfrak{A}_2 = \{1\ 3\ 6,\quad 2\ 4\ 5\},$$

$$\mathfrak{A}_3 = \{1\ 4\ 5,\quad 2\ 3\ 6\},$$

$$\mathfrak{A}_4 = \{1\ 4\ 6,\quad 2\ 3\ 5\}$$

then they are the four parallel classes of 3-valent blocks. It can be easily checked that every 2-valent block is an r-transversal of exactly two parallel classes, and they satisfy axioms (A1)–(A3) in the Fundamental Lemma. Hence, it is uniquely embeddable into the Möbius plane.

For $q = 4$, there are 12 points in PBRD(4). Let them be $\{1, 2, \ldots, 11, 12\}$. Using Lemmas 4, 5 and 6, one can see that the blocks of PBRD(4) are isomorphic to the following:

1 3 5 10 11	1 3 6 7 12	1 4 6 8 10	
1 4 7 9 11	1 5 8 9 12	2 3 5 7 9	
2 3 8 10 12	2 4 5 8 11	2 4 6 9 12	
2 6 7 10 11	3 6 8 9 11	4 5 7 10 12	
1 2 9 10	3 4 5 6	7 8 9 10	
1 2 11 12	3 4 9 10	5 6 7 8	
1 2 5 6	3 4 7 8	9 10 11 12	
1 2 7 8	3 4 11 12	5 6 9 10	
1 2 3 4	5 6 11 12	7 8 9 10	
1 3 8	1 3 9	1 4 5	1 4 12
1 5 7	1 6 9	1 6 11	1 7 10
1 8 11	1 10 12	2 3 6	2 3 11
2 4 7	2 4 10	2 5 10	2 5 12
2 6 8	2 7 12	2 8 9	2 9 11
3 5 8	3 5 12	3 6 10	3 7 10
3 7 11	3 9 12	4 5 9	4 6 7
4 6 11	4 8 9	4 8 12	4 10 11
5 7 11	5 8 10	5 9 11	6 7 9
6 8 12	6 10 12	7 9 12	8 10 11.

If we define

$$\mathfrak{A}_1 = \{1\ \ 2\ \ 9\ \ 10,\ \ 3\ \ 4\ \ 5\ \ 6,\ \ 7\ \ 8\ \ 11\ \ 12\},$$

$$\mathfrak{A}_2 = \{1\ \ 2\ \ 11\ \ 12,\ \ 3\ \ 4\ \ 9\ \ 10,\ \ 5\ \ 6\ \ 7\ \ 8\},$$

$$\mathfrak{A}_3 = \{1\ \ 2\ \ 5\ \ 6,\ \ 3\ \ 4\ \ 7\ \ 8,\ \ 9\ \ 10\ \ 11\ \ 12\},$$

$$\mathfrak{A}_4 = \{1\ \ 2\ \ 7\ \ 8,\ \ 3\ \ 4\ \ 11\ \ 12,\ \ 5\ \ 6\ \ 9\ \ 10\},$$

$$\mathfrak{A}_5 = \{1\ \ 2\ \ 3\ \ 4,\ \ 5\ \ 6\ \ 11\ \ 12,\ \ 7\ \ 8\ \ 9\ \ 10\}$$

then they are the five parallel classes of 4-valent blocks. It can be easily checked that the above blocks satisfy axioms (A1)–(A3) in the Fundamental Lemma. Hence, it is uniquely embeddable into the Möbius Plane.

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