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Two Inequalities for Differentiable Mappings and Applications to Special Means of Real Numbers and to Trapezoidal Formula

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Abstract—In this paper, we shall offer two inequalities for differentiable convex mappings which are connected with the celebrated Hermite-Hadamard's integral inequality holding for convex functions. Some natural applications to special means of real numbers are given. Finally, some error estimates for the trapezoidal formula are also addressed. © 1998 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

is well known in the literature as Hermite-Hadamard's inequality for convex functions [1].

The aim of this paper is to establish some results connected with the right part of (1.1) as well as to apply them for some elementary inequalities for real numbers and in numerical integration.

For several recent results concerning Hermite-Hadamard's inequality (1.1), see [2–6] where further references are listed.

2. MAIN RESULTS

We begin with the following lemma.

LEMMA 2.1. *Let $f : I^o \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o, $a, b \in I^o$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)\,dt. \tag{2.1}$$

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PROOF. It suffices to note that

$$\begin{aligned} I &= \int_0^1 (1-2t) f'(ta+(1-t)b)\,dt \\ &= \frac{f(ta+(1-t)b)}{a-b}(1-2t)\bigg|_0^1 + 2\int_0^1 \frac{f(ta+(1-t)b)}{a-b}\,dt \\ &= \frac{f(a)+f(b)}{b-a} - \frac{2}{b-a}\cdot\frac{1}{b-a}\int_a^b f(x)\,dx. \end{aligned}$$

REMARK 2.1. On using the change of the variable $x = ta+(1-t)b$, $t \in [0,1]$, equality (2.1) can be written as

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\int_a^b \left(x - \frac{a+b}{2}\right) f'(x)\,dx. \tag{2.2}$$

Some applications of identity (2.2) connected with Hermite-Hadamard's integral inequality for convex functions have been presented in [7]. Here we shall offer some more, which are very interesting.

THEOREM 2.2. *Let $f : I^o \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o, $a, b \in I^o$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)\left(|f'(a)|+|f'(b)|\right)}{8}. \tag{2.3}$$

PROOF. Using Lemma 2.1, it follows that

$$\begin{aligned} \left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| &= \left|\frac{b-a}{2}\int_0^1 (1-2t) f'(ta+(1-t)b)\,dt\right| \\ &\le \frac{b-a}{2}\int_0^1 |1-2t|\,|f'(ta+(1-t)b)|\,dt \\ &\le \frac{b-a}{2}\int_0^1 |1-2t|\,[t|f'(a)| + (1-t)|f'(b)|]\,dt \\ &= \frac{(b-a)\left(|f'(a)|+|f'(b)|\right)}{2}\int_0^1 |1-2t|\,t\,dt \\ &= \frac{(b-a)\left(|f'(a)|+|f'(b)|\right)}{8}, \end{aligned}$$

where we have used the fact that

$$\int_0^1 |1-2t|\,(1-t)\,dt = \int_0^1 |1-2t|\,t\,dt = \int_0^{1/2}(1-2t)t\,dt + \int_{1/2}^1 (2t-1)t\,dt = \frac{1}{4}.$$

Another similar result is embodied in the following theorem.

THEOREM 2.3. *Let $f : I^o \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I^o, $a, b \in I^o$ with $a < b$, and let $p > 1$. If the new mapping $|f'|^{p/(p-1)}$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)}{2(p+1)^{1/p}}\left[\frac{|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}}{2}\right]^{(p-1)/p}. \tag{2.4}$$

PROOF. Using Lemma 2.1 and Hölder's integral inequality, we find

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| &\le \frac{b-a}{2}\int_0^1 |1-2t|\,|f'(ta+(1-t)b)|\,dt \\ &\le \frac{b-a}{2}\left(\int_0^1 |1-2t|^p\,dt\right)^{1/p}\left(\int_0^1 |f'(ta+(1-t)b)|^q\,dt\right)^{1/q},\end{aligned} \tag{2.5}$$

where $1/p+1/q=1$.

Using the convexity of $|f'|^q$, we have

$$\int_0^1 |f'(ta+(1-t)b)|^q\,dt \le \int_0^1 \left[t|f'(a)|^q+(1-t)|f'(b)|^q\right]dt = \frac{|f'(a)|^q+|f'(b)|^q}{2}. \tag{2.6}$$

Further, since

$$\int_0^1 |1-2t|^p\,dt = \int_0^{1/2}(1-2t)^p\,dt+\int_{1/2}^1 (2t-1)^p\,dt = 2\int_0^{1/2}(1-2t)^p\,dt=\frac{1}{p+1}, \tag{2.7}$$

a combination of (2.5)–(2.7) immediately gives the required inequality (2.4).

3. APPLICATIONS TO SPECIAL MEANS

In the literature, the following means for *positive* real numbers α, β, $\alpha\ne\beta$ are well known:

$$\begin{aligned}
A(\alpha,\beta) &= \frac{\alpha+\beta}{2}, && \text{arithmetic mean},\\
G(\alpha,\beta) &= \sqrt{\alpha\beta}, && \text{geometric mean},\\
L(\alpha,\beta) &= \frac{\beta-\alpha}{\ln\beta-\ln\alpha}, && \text{logarithmic mean},\\
I(\alpha,\beta) &= \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right)^{1/(\beta-\alpha)}, && \text{identric mean},\\
L_p(\alpha,\beta) &= \left[\frac{\beta^{p+1}-\alpha^{p+1}}{(p+1)(\beta-\alpha)}\right]^{1/p}, && \text{generalized log-mean}, \qquad p\ne -1,0.
\end{aligned}$$

There are several results connecting these means, e.g., see [8] for some new relations; however, very few results are known for *arbitrary* real numbers. For this, it is clear that we can extend some of the above means as follows:

$$\begin{aligned}
A(\alpha,\beta) &= \frac{\alpha+\beta}{2}, && \alpha,\beta\in\mathbb{R},\\
\overline{L}(\alpha,\beta) &= \frac{\beta-\alpha}{\ln|\beta|-\ln|\alpha|}, && \alpha,\beta\in\mathbb{R}\backslash\{0\},\\
L_n(\alpha,\beta) &= \left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{1/n}, && n\in\mathbb{N},\quad n\ge 1,\quad \alpha,\beta\in\mathbb{R},\quad \alpha<\beta.
\end{aligned}$$

Now we shall use the results of Section 2 to prove the following new inequalities connecting the above means for *arbitrary* real numbers.

PROPOSITION 3.1. *Let $a,b\in\mathbb{R}$, $a<b$ and $n\in\mathbb{N}$, $n\ge 2$. Then, the following inequality holds:*

$$|A(a^n,b^n)-L_n(a,b)| \le \frac{n(b-a)}{4}A\left(|a|^{n-1},|b|^{n-1}\right).$$

PROOF. The proof is immediate from Theorem 2.2 applied for $f(x) = x^n$, $x \in \mathbb{R}$.

PROPOSITION 3.2. *Let $a, b \in \mathbb{R}$, $a < b$, and $n \in \mathbb{N}$, $n \geq 2$. Then, for all $p > 1$, the following inequality holds:*

$$|A(a^n, b^n) - L_n(a,b)| \leq \frac{n(b-a)}{2(p+1)^{1/p}} \left[A\left(|a|^{(n-1)p/(p-1)},\ |b|^{(n-1)p/(p-1)} \right) \right]^{(p-1)/p}.$$

PROOF. The proof is immediate from Theorem 2.3 applied for $f(x) = x^n$, $x \in \mathbb{R}$.

PROPOSITION 3.3. *Let $a, b \in \mathbb{R}$, $a < b$, and $0 \notin [a, b]$. Then, the following inequality holds:*

$$\left| A\left(a^{-1}, b^{-1}\right) - \overline{L}^{-1}(a,b) \right| \leq \frac{(b-a)}{4} A\left(|a|^{-2}, |b|^{-2}\right).$$

PROOF. The proof is obvious from Theorem 2.2 applied for $f(x) = 1/x$, $x \in [a, b]$.

PROPOSITION 3.4. *Let $a, b \in \mathbb{R}$, $a < b$, and $0 \notin [a, b]$. Then, for $p > 1$, the following inequality holds:*

$$\left| A\left(a^{-1}, b^{-1}\right) - \overline{L}^{-1}(a,b) \right| \leq \frac{(b-a)}{2(p+1)^{1/p}} \left[A\left(|a|^{-2p/(p-1)}, |b|^{-2p/(p-1)} \right) \right]^{(p-1)/p}.$$

PROOF. The proof is obvious from Theorem 2.3 applied for $f(x) = 1/x$, $x \in [a, b]$.

4. APPLICATIONS TO TRAPEZOIDAL FORMULA

Let d be a division of the interval $[a, b]$, i.e., $d : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$, and consider the trapezoidal formula

$$T(f, d) = \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} (x_{i+1} - x_i).$$

It is well known that if the mapping $f : [a, b] \to \mathbb{R}$ is twice differentiable on (a, b) and $M = \max_{t \in (a,b)} |f''(x)| < \infty$, then

$$\int_a^b f(x)\, dx = T(f, d) + E(f, d), \tag{4.1}$$

where the approximation error $E(f, d)$ of the integral $\int_a^b f(x)\, dx$ by the trapezoidal formula $T(f, d)$ satisfies

$$|E(f, d)| \leq \frac{M}{12} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^3. \tag{4.2}$$

It is clear that if the mapping f is not twice differentiable or the second derivative is not bounded on (a, b), then (4.2) cannot be applied. In recent papers [9–11], Dragomir and Wang have shown that the remainder term $E(f, d)$ can be estimated in terms of the first derivative only. These estimates have a wider range of applications. Here, we shall propose some new estimates of the remainder term $E(f, d)$ which supplement, in a sense, those established in [9–11].

PROPOSITION 4.1. *Let f be a differentiable mapping on I^o, $a, b \in I^o$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then in (4.1), for every division d of $[a, b]$,the following holds:*

$$\begin{aligned} |E(f, d)| &\leq \frac{1}{8} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \left(|f'(x_i)| + |f'(x_{i+1})| \right) \\ &\leq \frac{\max\{|f'(a)|, |f'(b)|\}}{4} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2. \end{aligned}$$

PROOF. Applying Theorem 2.2 on the subinterval $[x_i, x_{i+1}]$ ($i = 0, \dots, n-1$) of the division d, we get

$$\left| \frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i) - \int_{x_i}^{x_{i+1}} f(x)\,dx \right| \le \frac{(x_{i+1} - x_i)^2 \left(|f'(x_i)| + |f'(x_{i+1})|\right)}{8}.$$

Summing over i from 0 to $n-1$ and taking into account that $|f'|$ is convex, we deduce, by the triangle inequality, that

$$\begin{aligned} \left| T(f,d) - \int_a^b f(x)\,dx \right| &\le \frac{1}{8} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \left(|f'(x_i)| + |f'(x_{i+1})|\right) \\ &\le \frac{\max\{|f'(a)|, |f'(b)|\}}{4} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2. \end{aligned}$$

PROPOSITION 4.2. *Let f be a differentiable mapping on I^o, $a, b \in I^o$ with $a < b$, and let $p > 1$. If $|f'|^{p/(p-1)}$ is convex on $[a,b]$, then in (4.1) for every division d of $[a,b]$, the following holds:*

$$\begin{aligned} |E(f,d)| &\le \frac{1}{2(p+1)^{1/p}} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \left[\frac{|f'(x_i)|^{p/(p-1)} + |f'(x_{i+1})|^{p/(p-1)}}{2} \right]^{(p-1)/p} \\ &\le \frac{\max\{|f'(a)|, |f'(b)|\}}{2(p+1)^{1/p}} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2. \end{aligned}$$

PROOF. The proof uses Theorem 2.3 and is similar to that of Proposition 4.1.

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