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Cardinal invariants above the continuum

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Abstract

We prove some consistency results about $\mathfrak{b}(\lambda)$ and $\mathfrak{d}(\lambda)$, which are natural generalisations of the cardinal invariants of the continuum \mathfrak{b} and \mathfrak{d} . We also define invariants $\mathfrak{b}_{\text{cl}}(\lambda)$ and $\mathfrak{d}_{\text{cl}}(\lambda)$, and prove that almost always $\mathfrak{b}(\lambda) = \mathfrak{b}_{\text{cl}}(\lambda)$ and $\mathfrak{d}(\lambda) = \mathfrak{d}_{\text{cl}}(\lambda)$.

1. Introduction

The cardinal invariants of the continuum have been extensively studied. They are cardinals, typically between ω_1 and 2^ω , whose values give structural information about ${}^\omega\omega$. The survey paper [2] contains a wealth of information about these cardinals.

In this paper we study some natural generalisations to higher cardinals. Specifically, for λ regular, we define cardinals $\mathfrak{b}(\lambda)$ and $\mathfrak{d}(\lambda)$ which generalise the well-known invariants of the continuum \mathfrak{b} and \mathfrak{d} .

For a fixed value of λ , we will prove that there are some simple constraints on the triple of cardinals $(\mathfrak{b}(\lambda), \mathfrak{d}(\lambda), 2^\lambda)$. We will also prove that any triple of cardinals obeying these constraints can be realised.

We will then prove that there is essentially no correlation between the values of the triple $(\mathfrak{b}(\lambda), \mathfrak{d}(\lambda), 2^\lambda)$ for different values of λ , except the obvious one that $\lambda \mapsto 2^\lambda$ is non-decreasing. This generalises Easton's celebrated theorem (see [3]) on the possible behaviours of $\lambda \mapsto 2^\lambda$; since his model was built using Cohen forcing, one can show that in that model $\mathfrak{b}(\lambda) = \lambda^+$ and $\mathfrak{d}(\lambda) = 2^\lambda$ for every λ .

$\mathfrak{b}(\lambda)$ and $\mathfrak{d}(\lambda)$ are defined using the co-bounded filter on λ , and for $\lambda > \omega$ we can replace the co-bounded filter by the club filter to get invariants $\mathfrak{b}_{\text{cl}}(\lambda)$ and $\mathfrak{d}_{\text{cl}}(\lambda)$.

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We finish the paper by proving that these invariants are essentially the same as those defined using the co-bounded filter.

Some investigations have been made into generalising the other cardinal invariants of the continuum, for example in [7] Zapletal considers $\mathfrak{s}(\lambda)$ which is a generalised version of the splitting number \mathfrak{s} . His work has a different flavour to ours, since getting $\mathfrak{s}(\lambda) > \lambda^+$ needs large cardinals.

2. Definitions and elementary facts

It will be convenient to define the notions of “bounding number” and “dominating number” in quite a general setting. To avoid some trivialities, all partial orderings \mathbb{P} mentioned in this paper (with the exception of the notions of forcing) will be assumed to have the property that $\forall p \in \mathbb{P} \exists q \in \mathbb{P} p <_{\mathbb{P}} q$.

Definition 1. Let \mathbb{P} be a partial ordering. Then

- $U \subseteq \mathbb{P}$ is *unbounded* if and only if $\forall p \in \mathbb{P} \exists q \in U q \not\leq_{\mathbb{P}} p$.
- $D \subseteq \mathbb{P}$ is *dominating* if and only if $\forall p \in \mathbb{P} \exists q \in D p \leq_{\mathbb{P}} q$.
- $\mathfrak{b}(\mathbb{P})$ is the least cardinality of an unbounded subset of \mathbb{P} .
- $\mathfrak{d}(\mathbb{P})$ is the least cardinality of a dominating subset of \mathbb{P} .

The next lemma collects a few elementary facts about the cardinals $\mathfrak{b}(\mathbb{P})$ and $\mathfrak{d}(\mathbb{P})$.

Lemma 1. *Let \mathbb{P} be a partial ordering, and suppose that $\beta = \mathfrak{b}(\mathbb{P})$ and $\delta = \mathfrak{d}(\mathbb{P})$ are infinite. Then*

$$\beta = \text{cf}(\beta) \leq \text{cf}(\delta) \leq \delta \leq |\mathbb{P}|.$$

Proof. To show that β is regular, suppose for a contradiction that $\text{cf}(\beta) < \beta$. Let B be an unbounded family of cardinality β , and write $B = \bigcup_{\alpha < \text{cf}(\beta)} B_{\alpha}$ with $|B_{\alpha}| < \beta$. For each α find p_{α} such that $\forall p \in B_{\alpha} p \leq p_{\alpha}$, then find q such that $\forall \alpha < \text{cf}(\beta) p_{\alpha} \leq q$. Then $\forall p \in B p \leq q$, contradicting the assumption that B was unbounded.

Similarly, suppose that $\text{cf}(\delta) < \delta$. Let D be dominating with cardinality δ and write $D = \bigcup_{\alpha < \text{cf}(\delta)} D_{\alpha}$ where $|D_{\alpha}| < \delta$. For each α find p_{α} such that $\forall p \in D_{\alpha} p_{\alpha} \not\leq p$, and then find q such that $\forall \alpha < \text{cf}(\delta) p_{\alpha} \leq q$. Then $\forall p \in D q \not\leq p$, contradicting the assumption that D was dominating. \square

The next result shows that we cannot hope to say much more.

Lemma 2. *Let β and δ be infinite cardinals with $\beta = \text{cf}(\beta)$ and $\delta^{<\beta} = \delta$. Define a partial ordering $\mathbb{P} = \mathbb{P}(\beta, \delta)$ in the following way; the underlying set is $\beta \times [\delta]^{<\beta}$, and $(\rho, x) \leq (\sigma, y)$ if and only if $\rho \leq \sigma$ and $x \subseteq y$.*

Then $\mathfrak{b}(\mathbb{P}) = \beta$ and $\mathfrak{d}(\mathbb{P}) = \delta$.

Proof. Let $B \subseteq \mathbb{P}$ be unbounded. If $|B| < \beta$ then we can define

$$\rho = \sup \{ \sigma \mid \exists y (\sigma, y) \in B \} < \beta,$$

$$x = \bigcup \{ y \mid \exists \sigma (\sigma, y) \in B \} \in [\delta]^{<\beta}.$$

But then (ρ, x) is a bound for B , so $|B| \geq \beta$ and hence $\mathfrak{b}(\mathbb{P}) \geq \beta$. On the other hand the set $\{(\alpha, \emptyset) \mid \alpha < \beta\}$ is clearly unbounded, so that $\mathfrak{b}(\mathbb{P}) = \beta$.

Let $D \subseteq \mathbb{P}$ be dominating. If $|D| < \delta$ then $\bigcup \{ y \mid \exists \sigma (\sigma, y) \in D \} \neq \delta$, and this is impossible, so that $\mathfrak{d}(\mathbb{P}) \geq \delta$. On the other hand GCH holds and $\text{cf}(\delta) \geq \beta$, so that $|\mathbb{P}| = \beta \times \delta^{<\beta} = \delta$. Hence $\mathfrak{d}(\mathbb{P}) = \delta$. \square

Definition 2. Let \mathbb{P}, \mathbb{Q} be posets and $f : \mathbb{P} \rightarrow \mathbb{Q}$ a function. f embeds \mathbb{P} cofinally into \mathbb{Q} if and only if

- $\forall p, p' \in \mathbb{P} \ p \leq_{\mathbb{P}} p' \Leftrightarrow f(p) \leq_{\mathbb{Q}} f(p')$.
- $\forall q \in \mathbb{Q} \ \exists p \in \mathbb{P} \ q \leq_{\mathbb{Q}} f(p)$. That is, $\text{rge}(f)$ is dominating.

Lemma 3. If $f : \mathbb{P} \rightarrow \mathbb{Q}$ embeds \mathbb{P} cofinally into \mathbb{Q} then

$$\mathfrak{b}(\mathbb{P}) = \mathfrak{b}(\mathbb{Q}) \quad \text{and} \quad \mathfrak{d}(\mathbb{P}) = \mathfrak{d}(\mathbb{Q}).$$

Proof. Easy. \square

Lemma 4. Let \mathbb{P} be any partial ordering. Then there is $\mathbb{P}^* \subseteq \mathbb{P}$ such that \mathbb{P}^* is a dominating subset of \mathbb{P} and \mathbb{P}^* is well-founded.

Proof. We enumerate \mathbb{P}^* recursively. Suppose that we have already enumerated elements $\langle b_\alpha : \alpha < \beta \rangle$ into \mathbb{P}^* . If $\{ b_\alpha \mid \alpha < \beta \}$ is dominating then we stop, otherwise we choose b_β so that $b_\beta \not\leq b_\alpha$ for all $\alpha < \beta$.

Clearly the construction stops and enumerates a dominating subset \mathbb{P}^* of \mathbb{P} . To see that \mathbb{P}^* is well-founded observe that $b_\beta < b_\alpha \Rightarrow \beta < \alpha$. \square

Notice that the identity embeds \mathbb{P}^* cofinally in \mathbb{P} , so $\mathfrak{b}(\mathbb{P}^*) = \mathfrak{b}(\mathbb{P})$ and $\mathfrak{d}(\mathbb{P}^*) = \mathfrak{d}(\mathbb{P})$.

We also need some information about the preservation of $\mathfrak{b}(\mathbb{P})$ and $\mathfrak{d}(\mathbb{P})$ by forcing.

Lemma 5. Let \mathbb{P} be a partial ordering with $\mathfrak{b}(\mathbb{P}) = \beta$, $\mathfrak{d}(\mathbb{P}) = \delta$.

- Let $V[G]$ be a generic extension of V such that every set of ordinals of size less than β in $V[G]$ is covered by a set of size less than β in V . Then $V[G] \models \mathfrak{b}(\mathbb{P}) = \beta$.
- Let $V[G]$ be a generic extension of V such that every set of ordinals of size less than δ in $V[G]$ is covered by a set of size less than δ in V . Then $V[G] \models \mathfrak{d}(\mathbb{P}) = \delta$.

Proof. We do the first part, the second is very similar. The hypothesis implies that β is a cardinal in $V[G]$, and since “ B is unbounded” is upwards absolute from V to $V[G]$ it is clear that $V[G] \models \mathfrak{b}(\mathbb{P}) \leq \beta$. Suppose for a contradiction that we have C

in $V[G]$ unbounded with $V[G] \models |C| < \beta$. By our hypothesis there is $D \in V$ such that $C \subseteq D$ and $V \models |D| < \beta$, but now D is unbounded, contradicting the definition of β . \square

With these preliminaries out of the way, we can define the cardinals which will concern us in this paper.

Definition 3. Let λ be a regular cardinal.

1. If $f, g \in {}^\lambda \lambda$ then $f <^* g$ iff $\exists \alpha < \lambda \forall \beta > \alpha f(\beta) < g(\beta)$.
2. $\mathfrak{b}(\lambda) =_{\text{def}} \mathfrak{b}({}^\lambda \lambda, <^*)$.
3. $\mathfrak{d}(\lambda) =_{\text{def}} \mathfrak{d}({}^\lambda \lambda, <^*)$.

These are defined by analogy with some “cardinal invariants of the continuum” (for a reference on cardinal invariants see [2]) known as \mathfrak{b} and \mathfrak{d} . In our notation $\mathfrak{b} = \mathfrak{b}(\omega)$ and $\mathfrak{d} = \mathfrak{d}(\omega)$.

Lemma 6. *If λ is regular then*

- $\lambda^+ \leq \mathfrak{b}(\lambda)$.
- $\mathfrak{b}(\lambda) = \text{cf}(\mathfrak{b}(\lambda))$.
- $\mathfrak{b}(\lambda) \leq \text{cf}(\mathfrak{d}(\lambda))$.
- $\mathfrak{d}(\lambda) \leq 2^\lambda$.
- $\text{cf}(2^\lambda) > \lambda$.

Proof. The first claim follows from the following trivial fact.

Fact 1. *Let $\{f_\alpha \mid \alpha < \lambda\} \subseteq {}^\lambda \lambda$. Then there is a function $f \in {}^\lambda \lambda$ such that $\forall \alpha < \lambda f_\alpha <^* f$.*

Proof. Define $f(\beta) = \sup \{f_\gamma(\beta) + 1 \mid \gamma < \beta\}$. Then $f(\beta) > f_\gamma(\beta)$ for $\gamma < \beta < \lambda$. \square

The next three claims follow easily from our general results on $\mathfrak{b}(\mathbb{P})$ and $\mathfrak{d}(\mathbb{P})$, and the last is just König’s well-known theorem on cardinal exponentiation. \square

We will prove that these are essentially the only restrictions provable in ZFC. One could view this as a refinement of Easton’s classical result (see [3]) on $\lambda \mapsto 2^\lambda$.

3. Hechler forcing

In this section we show how to force that certain posets can be cofinally embedded in $({}^\lambda \lambda, <^*)$. This is a straightforward generalisation of Hechler’s work in [4], where he treats the case $\lambda = \omega$.

We start with a brief review of our forcing notation. $p \leq q$ means that p is stronger than q , a κ -closed forcing notion is one in which every decreasing chain of length less

than κ has a lower bound, and a κ -dense forcing notion is one in which every sequence of dense open sets of length less than κ has non-empty intersection.

If x is an ordered pair then x_0 will denote the first component of x and x_1 the second component.

Definition 4. Let λ be regular. $\mathbb{D}(\lambda)$ is the notion of forcing whose conditions are pairs (s, F) with $s \in {}^{<\lambda}\lambda$ and $F \in {}^\lambda\lambda$, ordered as follows; $(s, F) \leq (t, F')$ if and only if

1. $\text{dom}(t) \leq \text{dom}(s)$ and $t = s \upharpoonright \text{dom}(t)$.
2. $s(\alpha) \geq F'(\alpha)$ for $\text{dom}(t) \leq \alpha < \text{dom}(s)$.
3. $F(\alpha) \geq F'(\alpha)$ for all α .

We will think of a generic filter G as adding a function $f_G : \lambda \rightarrow \lambda$ given by $f_G = \bigcup \{ s \mid \exists F (s, F) \in G \}$. It is easy to see that

$$G = \{ (t, F) \mid t = f_G \upharpoonright \text{dom}(t), \text{dom}(t) \leq \alpha \Rightarrow F(\alpha) \leq f_G(\alpha) \},$$

so that $V[f_G] = V[G]$ and we can talk about functions from λ to λ being $\mathbb{D}(\lambda)$ -generic.

Lemma 7. Let $\lambda^{<\lambda} = \lambda$, and set $\mathbb{P} = \mathbb{D}(\lambda)$. Then

1. \mathbb{P} is λ -closed.
2. \mathbb{P} is λ^+ -c.c.
3. If $g : \lambda \rightarrow \lambda$ is \mathbb{P} -generic over V then $\forall f \in {}^\lambda\lambda \cap V \ f <^* g$.

Proof. 1. Let $\gamma < \lambda$ and suppose that $\langle (t_\alpha, F_\alpha) : \alpha < \gamma \rangle$ is a descending γ -sequence of conditions from \mathbb{P} . Defining $t = \bigcup \{ t_\alpha \mid \alpha < \gamma \}$ and $F : \beta \mapsto \sup \{ F_\alpha(\beta) \mid \alpha < \gamma \}$, it is easy to see that (t, F) is a lower bound for the sequence.

2. Observe that if (s, F) and (s, F') are two conditions with the same first component then they are compatible, because if $H : \beta \mapsto F(\beta) \cup F'(\beta)$ the condition (s, H) is a common lower bound. There are only $\lambda^{<\lambda} = \lambda$ possible first components, so that \mathbb{P} clearly has the λ^+ -c.c.

3. Let $f \in {}^\lambda\lambda \cap V$, and let (t, F) be an arbitrary condition. Let us define $F' : \beta \mapsto (F(\beta) \cup f(\beta)) + 1$, then (t, F') refines (t, F) and forces that $f(\alpha) < f_G(\alpha)$ for all $\alpha \geq \text{dom}(t)$. \square

If $\mu = \text{cf}(\mu) > \lambda = \lambda^{<\lambda}$ then it is straightforward to iterate $\mathbb{D}(\lambda)$ with $< \lambda$ -support for μ steps and get a model where $\text{b}(\lambda) = \text{d}(\lambda) = \mu$. Getting a model where $\text{b}(\lambda) < \text{d}(\lambda)$ is a little harder, but we can do it by a “nonlinear iteration” which will embed a well-chosen poset cofinally into $({}^\lambda\lambda, <^*)$.

Theorem 1. Let $\lambda = \lambda^{<\lambda}$, and suppose that \mathbb{Q} is any well-founded poset with $\text{b}(\mathbb{Q}) \geq \lambda^+$. Then there is a forcing $\mathbb{D}(\lambda, \mathbb{Q})$ such that

1. $\mathbb{D}(\lambda, \mathbb{Q})$ is λ -closed and λ^+ -c.c.
2. $V^{\mathbb{D}(\lambda, \mathbb{Q})} \models \mathbb{Q}$ can be cofinally embedded into $({}^\lambda\lambda, <^*)$

- 3. If $V \models \mathfrak{b}(\mathbb{Q}) = \beta$ then $V^{\mathbb{D}(\lambda, \mathbb{Q})} \models \mathfrak{b}(\lambda) = \beta$.
- 4. If $V \models \mathfrak{d}(\mathbb{Q}) = \delta$ then $V^{\mathbb{D}(\lambda, \mathbb{Q})} \models \mathfrak{d}(\lambda) = \delta$.

Proof. We will define the conditions and ordering for $\mathbb{D}(\lambda, \mathbb{Q})$ by induction on \mathbb{Q} . The idea is to iterate $\mathbb{D}(\lambda)$ “along \mathbb{Q} ” so as to get a cofinal embedding of \mathbb{Q} into ${}^\lambda\lambda$. It will be convenient to define a new poset \mathbb{Q}^+ which consists of \mathbb{Q} together with a new element *top* which is greater than all the elements of \mathbb{Q} .

We will define for each $a \in \mathbb{Q}^+$ a notion of forcing \mathbb{P}_a . If $a \in \mathbb{Q}^+$ then we will denote $\{c \in \mathbb{Q} \mid c < a\}$ by \mathbb{Q}/a . It will follow from the definition that if $c < a$ then \mathbb{P}_c is a complete subordering of \mathbb{P}_a , and that the map $p \in \mathbb{P}_a \mapsto p \upharpoonright \mathbb{Q}/c$ is a projection from \mathbb{P}_a to \mathbb{P}_c .

Suppose that for all $b <_{\mathbb{Q}} a$ we have already defined \mathbb{P}_b .

- 1. p is a condition in \mathbb{P}_a if and only if
 - (a) p is a function, $\text{dom}(p) \subseteq \mathbb{Q}/a$ and $|\text{dom}(p)| < \lambda$.
 - (b) For all $b \in \text{dom}(p)$, $p(b) = (t, \dot{F})$ where $t \in {}^{<\lambda}\lambda$ and \dot{F} is a \mathbb{P}_b -name for a member of ${}^\lambda\lambda$.
- 2. If $p, q \in \mathbb{P}_a$ then $p \leq q$ if and only if
 - (a) $\text{dom}(q) \subseteq \text{dom}(p)$.
 - (b) For all $b \in \text{dom}(q)$, if $p(b) = (s, \dot{H})$ and $q(b) = (t, \dot{I})$ then
 - (i) $t = s \upharpoonright \text{dom}(t)$.
 - (ii) $p \upharpoonright (\mathbb{Q}/b) \Vdash_{\mathbb{P}_b} \text{dom}(t) \leq \alpha < \text{dom}(s) \Rightarrow s(\alpha) > \dot{I}(\alpha)$.
 - (iii) $p \upharpoonright (\mathbb{Q}/b) \Vdash_{\mathbb{P}_b} \forall \alpha \dot{H}(\alpha) \geq \dot{I}(\alpha)$.

We define $\mathbb{D}(\lambda, \mathbb{Q}) = \mathbb{P}_{\text{top}}$, and verify that this forcing does what we claimed. The verification is broken up into a series of claims.

Claim 1. $\mathbb{D}(\lambda, \mathbb{Q})$ is λ -closed.

Proof. Let $\gamma < \lambda$ and let $\langle p_\alpha : \alpha < \gamma \rangle$ be a descending γ -sequence of conditions. We will define a new condition p with $\text{dom}(p) = \bigcup_\alpha \text{dom}(p_\alpha)$. For each $b \in \text{dom}(p_\alpha)$ let $p_\alpha(b) = (t_\alpha(b), \dot{F}_\alpha(b))$.

Let $p(b) = (t, \dot{F})$ where $t = \bigcup \{t_\alpha(b) \mid b \in \text{dom}(p_\alpha)\}$ and $\dot{F}(b)$ is a \mathbb{P}_b -name for the pointwise supremum of $\{\dot{F}_\alpha(b) \mid b \in \text{dom}(p_\alpha)\}$. Then it is easy to check that p is a condition and is a lower bound for $\langle p_\alpha : \alpha < \gamma \rangle$. \square

Claim 2. $\mathbb{D}(\lambda, \mathbb{Q})$ is λ^+ -c.c.

Proof. Let $\langle p_\alpha : \alpha < \lambda^+ \rangle$ be a family of conditions. Since $\lambda = \lambda^{<\lambda}$ we may assume that the domains form a Δ -system with root r . We may also assume that for $b \in r$, $p_\alpha(b) = (t_b, \dot{F}_\alpha(b))$ where t_b is independent of α . It is now easy to see that any two conditions in the family are compatible. \square

Claim 3. If $c < a$ then \mathbb{P}_c is a complete subordering of \mathbb{P}_a , and the map $p \mapsto p \upharpoonright \mathbb{Q}/c$ is a projection from \mathbb{P}_a to \mathbb{P}_c .

Proof. This is routine. \square

If G is $\mathbb{D}(\lambda, \mathbb{Q})$ -generic, then for each $a \in \mathbb{Q}$ we can define $f_G^a \in {}^\lambda \lambda \cap V[G]$ by $f_G^a = \bigcup \{ t(a)_0 \mid t \in G \}$. It is these functions that will give us a cofinal embedding of \mathbb{Q} into $({}^\lambda \lambda \cap V[G], <^*)$, via the map $a \mapsto f_G^a$.

Claim 4. *If $a <_{\mathbb{Q}} b$ then $f_G^a <^* f_G^b$.*

Proof. Let p be a condition and let \dot{F} be the canonical \mathbb{P}_b -name for f_G^a . Refine p to q in the following way; $q(c) = p(c)$ for $c \neq b$, and if $p(b) = (t, \dot{H})$ then $q(b) = (t, \dot{I})$ where \dot{I} names the pointwise maximum of \dot{F} and \dot{H} .

Then q forces that $f_G^b(\alpha)$ is greater than $f_G^a(\alpha)$ for $\alpha \geq \text{dom}(t)$. \square

Notice that by the same proof f_G^b dominates every function in $V^{\mathbb{P}_a}$.

Claim 5. *If $a \not<_{\mathbb{Q}} b$ then $f_G^a \not<^* f_G^b$.*

Proof. If $b <_{\mathbb{Q}} a$ then we showed in the last claim that $f_G^b <^* f_G^a$, so we may assume without loss of generality that $b \not<_{\mathbb{Q}} a$.

Let p be a condition and let $\alpha < \lambda$. Choose β large enough that $\{\text{dom}(p(a)_0), \text{dom}(p(b)_0), \alpha\} \subseteq \beta$. Let $p(b) = (t, \dot{F})$, and find $q \in \mathbb{P}_b$ such that $q \leq p \upharpoonright (\mathbb{Q}/b)$ and q decides $\dot{F} \upharpoonright (\beta + 1)$.

Let p_1 be the condition such that $p_1(c) = p(c)$ if $c \not<_{\mathbb{Q}} b$ and $p_1(c) = q(c)$ if $c <_{\mathbb{Q}} b$. Then p_1 refines p and $p_1(a) = p(a)$, $p_1(b) = p(b)$.

Let $p(a) = (s, \dot{H})$. Find $r \in \mathbb{P}_a$ such that $r \leq p_1 \upharpoonright (\mathbb{Q}/a)$ and r decides $\dot{H} \upharpoonright (\beta + 1)$.

Let p_2 be the condition such that $p_2(c) = p_1(c)$ if $c \not<_{\mathbb{Q}} a$ and $p_2(c) = r(c)$ if $c <_{\mathbb{Q}} a$. Then p_2 refines p_1 and $p_2(a) = p(a)$, $p_2(b) = p(b)$.

Now it is easy to extend p_2 to a condition which forces $f_G^a(\beta) > f_G^b(\beta)$. \square

Claim 6. *The map $a \mapsto f_G^a$ embeds \mathbb{Q} cofinally into $({}^\lambda \lambda, <^*)$ in the generic extension by $\mathbb{D}(\lambda, \mathbb{Q})$.*

Proof. We have already checked that the map is order-preserving. It remains to be seen that its range is dominating.

Let G be $\mathbb{D}(\lambda, \mathbb{Q})$ -generic and let $f \in {}^\lambda \lambda \cap V[G]$. Then $f = (\dot{f})^G$ for some canonical name \dot{f} , and by the λ^+ -c.c. we may assume that there is $X \subseteq \mathbb{Q}$ such that $|X| = \lambda$ and \dot{f} only involves conditions p with $\text{dom}(p) \subseteq X$. Now $\text{b}(\mathbb{Q}) \geq \lambda^+$ so that we can find $a \in \mathbb{Q}$ with $X \subseteq \mathbb{Q}/a$.

This implies that \dot{f} is a \mathbb{P}_a -name for a function in ${}^\lambda \lambda$, so that $f <^* f_G^a$ and we are done. \square

Claim 7. *If $V \models \text{b}(\mathbb{Q}) = \beta$ then $V^{\mathbb{D}(\lambda, \mathbb{Q})} \models \text{b}(\lambda) = \beta$.*

Proof. Let G be $\mathbb{D}(\lambda, \mathbb{Q})$ -generic. By Lemma 3 it will suffice to show that $V[G] \models \text{b}(\mathbb{Q}) = \beta$. This follows from Lemma 5, the fact that $\mathbb{D}(\lambda, \mathbb{Q})$ is λ^+ -c.c. and the assumption that $\text{b}(\mathbb{Q}) \geq \lambda^+$. \square

Claim 8. *If $V \models \mathfrak{d}(\mathbb{Q}) = \delta$ then $V^{\mathbb{D}(\lambda, \mathbb{Q})} \models \mathfrak{d}(\lambda) = \delta$.*

Proof. Exactly like the last claim. \square

This finishes the proof of Theorem 1. \square

4. Controlling the invariants at a fixed cardinal

In this section we show how to force that the triple $(\mathfrak{b}(\lambda), \mathfrak{d}(\lambda), 2^\lambda)$ can be anything “reasonable” for a fixed value of λ .

Theorem 2. *Let $\lambda = \lambda^{<\lambda}$ and let GCH hold at all cardinals $\rho \geq \lambda$. Let β, δ, μ be cardinals such that $\lambda^+ \leq \beta = \text{cf}(\beta) \leq \text{cf}(\delta)$, $\delta \leq \mu$ and $\text{cf}(\mu) > \lambda$.*

Then there is a forcing $\mathbb{M}(\lambda, \beta, \delta, \mu)$ such that in the generic extension $\mathfrak{b}(\lambda) = \beta$, $\mathfrak{d}(\lambda) = \delta$ and $2^\lambda = \mu$.

Proof. In V define $\mathbb{Q} = \mathbb{P}(\beta, \delta)$, as in Lemma 2. We know that $V \models \mathfrak{b}(\mathbb{Q}) = \beta$ and $V \models \mathfrak{d}(\mathbb{Q}) = \delta$. Fix \mathbb{Q}^* a cofinal well-founded subset of \mathbb{Q} , and then define a new well-founded poset \mathbb{R} as follows.

Definition 5. The elements of \mathbb{R} are pairs (p, i) where either $i = 0$ and $p \in \mu$ or $i = 1$ and $p \in \mathbb{Q}^*$. $(p, i) \leq (q, j)$ iff $i = j = 0$ and $p \leq q$ in μ , $i = j = 1$ and $p \leq q$ in \mathbb{Q}^* , or $i = 0$ and $j = 1$.

Now we set $\mathbb{M}(\lambda, \beta, \delta, \mu) = \mathbb{D}(\lambda, \mathbb{R})$. It is routine to use the closure and chain condition to argue that \mathbb{M} makes $2^\lambda = \mu$. Since \mathbb{R} contains a cofinal copy of \mathbb{Q}^* , it is also easy to see that \mathbb{M} forces $\mathfrak{b}(\lambda) = \beta$ and $\mathfrak{d}(\lambda) = \delta$. \square

5. A first attempt at the main theorem

We now aim to put together the basic modules as described in the previous section, so as to control the function $\lambda \mapsto (\mathfrak{b}(\lambda), \mathfrak{d}(\lambda), 2^\lambda)$ for all regular λ . A naive first attempt would be to imitate Easton’s construction from [3]; this almost works, and will lead us towards the right construction.

Let us briefly recall the statement and proof of Easton’s theorem on the behaviour of $\lambda \mapsto 2^\lambda$.

Lemma 8 (Easton’s lemma). *If \mathbb{P} is κ -c.c. and \mathbb{Q} is κ -closed then \mathbb{P} is κ -c.c. in $V^{\mathbb{Q}}$ and \mathbb{Q} is κ -dense in $V^{\mathbb{P}}$. In particular ${}^{<\kappa}\text{ON} \cap V^{\mathbb{P} \times \mathbb{Q}} = {}^{<\kappa}\text{ON} \cap V^{\mathbb{P}}$.*

Theorem 3 (Easton’s theorem). *Let $F : \text{REG} \rightarrow \text{CARD}$ be a class function such that $\text{cf}(F(\lambda)) > \lambda$ and $\lambda < \mu \Rightarrow F(\lambda) \leq F(\mu)$. Let GCH hold. Then there is a class*

forcing \mathbb{P} which preserves cardinals and cofinalities, such that in the extension $2^\lambda = F(\lambda)$ for all regular λ .

Proof (sketch). The “basic module” is $\mathbb{P}(\lambda) = Add(\lambda, F(\lambda))$. \mathbb{P} is the “Easton product” of the $\mathbb{P}(\lambda)$, to be more precise $p \in \mathbb{P}$ iff

- (1) p is a function with $dom(p) \subseteq REG$ and $p(\beta) \in \mathbb{P}(\beta)$ for all $\beta \in dom(p)$.
- (2) For all inaccessible γ , $dom(p) \cap \gamma$ is bounded in γ .

\mathbb{P} is ordered by pointwise refinement. There are certain complications arising from the fact that we are doing class forcing; we ignore them in this sketch.

If β is regular then we may factor \mathbb{P} as $\mathbb{P}_{<\beta} \times \mathbb{P}(\beta) \times \mathbb{P}_{>\beta}$ in the obvious way. $\mathbb{P}_{\geq\beta}$ is always β -closed.

It follows from GCH and the Δ -system lemma that if γ is Mahlo or the successor of a regular cardinal then $\mathbb{P}_{<\gamma}$ is γ -c.c. On the other hand, if γ is a non-Mahlo inaccessible or the successor of a singular cardinal, then $\mathbb{P}_{<\gamma}$ is in general only γ^+ -c.c.

In particular for γ regular $\mathbb{P}_{\leq\gamma} = \mathbb{P}_{<\gamma^+}$ is always γ^+ -c.c. so that by Easton’s lemma ${}^\gamma ON \cap V^{\mathbb{P}} = {}^\gamma ON \cap V^{\mathbb{P} \leq \gamma}$. This implies that in the end we have only added $F(\gamma)$ many subsets of γ .

It remains to be seen that cardinals and cofinalities are preserved. It will suffice to show that regular cardinals remain regular. If γ is Mahlo or the successor of a regular cardinal, then Easton’s lemma implies that ${}^{<\gamma} ON \cap V^{\mathbb{P}} = {}^{<\gamma} ON \cap V^{\mathbb{P} < \gamma}$, and since γ is regular in $V^{\mathbb{P} < \gamma}$ (by γ -c.c.) γ is clearly regular in $V^{\mathbb{P}}$.

Now suppose that $\gamma = \mu^+$ for μ singular. If γ becomes singular in $V^{\mathbb{P}}$ let its new cofinality be β , where we see that $\beta < \mu$ and β is regular in V . ${}^\beta ON \cap V^{\mathbb{P}} = {}^\beta ON \cap V^{\mathbb{P} \leq \beta}$, so that γ will have cofinality β in $V^{\mathbb{P} \leq \beta}$. This is absurd as $\mathbb{P}_{\leq\beta}$ is β^+ -c.c. and $\beta^+ < \mu < \gamma$. A very similar argument will work in case γ is a non-Mahlo inaccessible. \square

Suppose that we replace $Add(\lambda, F(\lambda))$ by $\mathbb{P}(\lambda) = \mathbb{M}(\lambda, \beta(\lambda), \delta(\lambda), \mu(\lambda))$, where $\lambda \mapsto (\beta(\lambda), \delta(\lambda), \mu(\lambda))$ is a function obeying the constraints given by Lemma 6. Let \mathbb{P} be the Easton product of the $\mathbb{P}(\lambda)$. Then exactly as in the proof of Easton’s theorem it will follow that \mathbb{P} preserves cardinals and cofinalities, and that $2^\lambda = \mu(\lambda)$ in $V^{\mathbb{P}}$.

Lemma 9. *If λ is inaccessible or the successor of a regular cardinal then $b(\lambda) = \beta(\lambda)$, $d(\lambda) = \delta(\lambda)$ and $2^\lambda = \mu(\lambda)$ in $V^{\mathbb{P}}$.*

Proof. For any λ , ${}^\lambda \lambda \cap V^{\mathbb{P}} = {}^\lambda \lambda \cap V^{\mathbb{P} \leq \lambda}$. $b(\lambda)$ and $d(\lambda)$ have the right values in $V^{\mathbb{P}(\lambda)}$ by design, and these values are not changed by λ -c.c. forcing. So assuming $\mathbb{P}_{<\lambda}$ is λ -c.c. those invariants have the right values in $V^{\mathbb{P} \leq \lambda}$, and hence in $V^{\mathbb{P}}$. \square

We need some way of coping with the successors of singular cardinals and the non-Mahlo inaccessibles. Zapletal pointed out that at the first inaccessible in an Easton iteration we are certain to add many Cohen subsets, so that there really is a need to modify the construction.

6. Tail forcing

Easton’s forcing to control $\lambda \mapsto 2^\lambda$ can be seen as a kind of iterated forcing in which we choose each iterand from the ground model, or equivalently as a kind of product forcing. Silver’s “reverse Easton forcing” is an iteration in which the iterand at λ is defined in $V^{\mathbb{P}^\lambda}$. The “tail forcing” which we describe here is a sort of hybrid.

We follow the conventions of Baumgartner’s paper [1] in our treatment of iterated forcing, except that when we have $\dot{\mathbb{Q}} \in V^{\mathbb{P}}$ and form $\mathbb{P} * \dot{\mathbb{Q}}$ we reserve the right not to take all \mathbb{P} -names for members of \mathbb{Q} (as long as we take enough names that the set of their denotations is forced to be dense). For example if $\mathbb{Q} \in V$ we will only take names \hat{q} for $q \in \mathbb{Q}$, so $\mathbb{P} * \dot{\mathbb{Q}}$ will just be $\mathbb{P} \times \mathbb{Q}$.

We will describe a kind of iteration which we call “Easton tail iteration” in which at successor stages we choose iterands from V , but at limit stage λ we choose $\dot{\mathbb{Q}}_\lambda$ in a different way; possibly $\mathbb{Q}_\lambda \notin V$, but we will arrange things so that the generic G_λ factors at many places below λ and any final segment of G_λ essentially determines \mathbb{Q}_λ . This idea comes from Magidor and Shelah’s paper [5].

We assume for simplicity that in the ground model all limit cardinals are singular or inaccessible. In the application that we intend this is no restriction, as the ground model will obey GCH.

Definition 6. A forcing iteration \mathbb{P}_γ with iterands $\langle \dot{\mathbb{Q}}_\beta : \beta + 1 < \gamma \rangle$ is an *Easton tail iteration* iff

1. The iteration has Easton support, that is to say a direct limit is taken at inaccessible limit stages and an inverse limit elsewhere.
2. $\dot{\mathbb{Q}}_\beta = 0$ unless β is a regular cardinal.
3. If β is the successor of a regular cardinal then $\mathbb{Q}_\beta \in V$.
4. For all regular β , $\mathbb{P}_{\beta+1}$ is β^+ -c.c.
5. For λ a limit cardinal \mathbb{P}_λ is λ^{++} -c.c. if λ is singular, and λ^+ -c.c. if λ is inaccessible.
6. For all regular α with $\alpha + 1 < \gamma$ there exists an iteration \mathbb{P}_γ^α dense in \mathbb{P}_γ such that $\mathbb{P}_{\alpha+1}^\alpha = \mathbb{P}_{\alpha+1}$, and for β with $\alpha + 1 < \beta \leq \gamma$
 - (a) \mathbb{P}_β^α factors as $\mathbb{P}_{\alpha+1} \times \mathbb{P}^\alpha \upharpoonright (\alpha + 1, \beta)$.
 - (b) If β is inaccessible or the successor of a singular, and $p \in \mathbb{P}_\gamma^\alpha$, then $p(\beta)$ is a name depending only on $\mathbb{P}^\alpha \upharpoonright (\alpha + 1, \beta)$.
 - (c) $\mathbb{P}^\alpha \upharpoonright (\alpha + 1, \beta)$ is α^+ -closed.

Clause 6 is of course the interesting one. It holds in a trivial way if \mathbb{P}_γ is just a product with Easton supports. Clauses (6a) and (6b) should really be read together, as the factorisation in (6a) only makes sense because (6b) already applies to $\bar{\beta} < \beta$, and conversely (6b) only makes sense once we have the factorisation from (6a).

The following result shows that Easton tail iterations do not disturb the universe too much.

Lemma 10. *Let \mathbb{P}_γ be an Easton tail iteration. Then*

1. *For all regular $\alpha < \gamma$, ${}^\alpha ON \cap V^{\mathbb{P}_\gamma} = {}^\alpha ON \cap V^{\mathbb{P}_{\alpha+1}}$.*
2. *\mathbb{P}_γ preserves all cardinals and cofinalities.*

Proof. Exactly like Theorem 3. \square

In the next section we will see how to define a non-trivial Easton tail iteration. If γ^+ is the successor of a regular then it will suffice to choose $\mathbb{Q}_{\gamma^+} \in V$ as any γ^+ -closed and γ^{++} -c.c. forcing. The interesting (difficult) stages are the ones where we have to cope with the other sorts of regular cardinal, here we will have to maintain the hypotheses on the chain condition and factorisation properties of the iteration. It turns out that slightly different strategies are appropriate for inaccessible and successors of singulars.

7. The main theorem

Theorem 4. *Let GCH hold. Let $\lambda \mapsto (\beta(\lambda), \delta(\lambda), \mu(\lambda))$ be a class function from REG to $CARD^3$, with $\lambda^+ \leq \beta(\lambda) = \text{cf}(\beta(\lambda)) \leq \text{cf}(\delta(\lambda)) \leq \delta(\lambda) \leq \mu(\lambda)$ and $\text{cf}(\mu(\lambda)) > \lambda$ for all λ .*

Then there exists a class forcing \mathbb{P}_∞ , preserving all cardinals and cofinalities, such that in the generic extension $\mathfrak{b}(\lambda) = \beta(\lambda)$, $\mathfrak{d}(\lambda) = \delta(\lambda)$ and $2^\lambda = \mu(\lambda)$ for all λ .

Proof. We will define by induction on γ a sequence of Easton tail iterations \mathbb{P}_γ , and then take a direct limit to get a class forcing \mathbb{P}_∞ . The proof that \mathbb{P}_∞ has the desired properties is exactly as in [3], so we will concentrate on defining the \mathbb{P}_γ . As we define the \mathbb{P}_γ we will also define dense subsets \mathbb{P}_γ^α intended to witness clause 6 in the definition of an Easton tail iteration.

Much of the combinatorics in this section is very similar to that in Section 3. Accordingly we have only sketched the proofs of some of the technical assertions about closure and chain conditions.

The easiest case to cope with is that where we are looking at the successor of a regular cardinal. So let γ be regular and assume that we have defined \mathbb{P}_{γ^+} (which is equivalent to $\mathbb{P}_{\gamma+1}$ since we do trivial forcing at all points between γ and γ^+), and $\mathbb{P}_{\gamma^+}^\alpha$ (which is equivalent to $\mathbb{P}_{\gamma+1}^\alpha$) for all $\alpha \leq \gamma$.

Definition 7. $\mathbb{Q}_{\gamma^+} = \mathbb{M}(\gamma^+, \beta(\gamma^+), \delta(\gamma^+), \mu(\gamma^+))$ as defined in Section 4. $\mathbb{P}_{\gamma^+} = \mathbb{P}_{\gamma^+} \times \mathbb{Q}_{\gamma^+}$, and $\mathbb{P}_{\gamma^+}^\alpha = \mathbb{P}_{\gamma^+}^\alpha \times \mathbb{Q}_{\gamma^+}$ for $\alpha \leq \gamma$.

It is now easy to check that this definition maintains the conditions for being an Easton tail iteration. Since \mathbb{P}_{γ^+} is γ^+ -c.c. and we know that $\gamma^+ \cap V^{\mathbb{P}_\infty} = \gamma^+ \cap V^{\mathbb{P}_{\gamma^+}}$, we will get the desired behaviour at γ^+ in $V^{\mathbb{P}_\infty}$.

Next we consider the case of a cardinal λ^+ , where λ is singular. Suppose we have defined \mathbb{P}_{λ^+} (that is \mathbb{P}_λ) and $\mathbb{P}_{\lambda^+}^\alpha$ appropriately. Let \mathbb{R} be a well-founded poset of cardinality $\mu(\lambda)$ with $\mathfrak{b}(\mathbb{R}) = \beta(\lambda)$ and $\mathfrak{d}(\mathbb{R}) = \delta(\lambda)$, as defined in Section 4. Let \mathbb{R}^+ be \mathbb{R} with the addition of a maximal element *top*. We will define $\mathbb{P}_{\lambda^+} * \dot{\mathbb{Q}}_a$ by induction on $a \in \mathbb{R}^+$, and then set $\mathbb{P}_{\lambda^{++}} = \mathbb{P}_{\lambda^+} * \dot{\mathbb{Q}}_{top}$. In the induction we will maintain the hypothesis that, for each $\alpha < \lambda$, $\mathbb{P}_{\lambda^+}^\alpha * \dot{\mathbb{Q}}_a$ can be factored as $\mathbb{P}_{\alpha+1} \times (\mathbb{P}^\alpha \upharpoonright (\alpha + 1, \lambda^+) * \dot{\mathbb{Q}}_a)$. Let us now fix a , and suppose that we have defined $\mathbb{P}_{\lambda^+} * \dot{\mathbb{Q}}_b$ for all b below a in \mathbb{R}^+ .

Definition 8. Let $b < a$, and let τ be a function from λ^+ to λ^+ . Then τ is *symmetric* iff for all $\alpha < \lambda$, whenever $G_0 \times G_1$ and $G'_0 \times G_1$ are two generics for $\mathbb{P}_{\alpha+1} \times (\mathbb{P}^\alpha \upharpoonright (\alpha + 1, \lambda^+) * \dot{\mathbb{Q}}_b)$, then $\tau^{G_0 \times G_1} = \tau^{G'_0 \times G_1}$.

Of course the (technically illegal) quantification over generic objects in this definition can be removed using the truth lemma, to see that the collection of symmetric names really is a set in V .

Definition 9. (p, q) is a condition in $\mathbb{P}_{\lambda^+} * \dot{\mathbb{Q}}_a$ iff

1. $p \in \mathbb{P}_{\lambda^+}$.
2. q is a function, $\text{dom}(q) \subseteq \mathbb{R}^+ / a$ and $|\text{dom}(q)| \leq \lambda$.
3. For each $b \in \text{dom}(q)$, $q(b)$ is a pair (s, \dot{F}) where $s \in {}^{<\lambda^+} \lambda^+$ and \dot{F} is a symmetric $\mathbb{P}_{\lambda^+} * \dot{\mathbb{Q}}_b$ -name for a function from λ^+ to λ^+ .

Definition 10. Let (p, q) and (p', q') be conditions in $\mathbb{P}_{\lambda^+} * \dot{\mathbb{Q}}_a$. (p', q') refines (p, q) iff

1. p' refines p in \mathbb{P}_{λ^+} .
2. $\text{dom}(q) \subseteq \text{dom}(q')$.
3. For each $b \in \text{dom}(q)$, if we let $q(b) = (s, F)$ and $q'(b') = (s', F')$, then
 - (a) s' extends s .
 - (b) If $\alpha \in \text{lh}(s') - \text{lh}(s)$ then $(p', q' \upharpoonright (\mathbb{R}^+ / b)) \Vdash s'(\alpha) \geq F(\alpha)$.
 - (c) For all α , $(p', q' \upharpoonright (\mathbb{R}^+ / b)) \Vdash F'(\alpha) \geq F(\alpha)$.

Definition 11. We define $\mathbb{P}_{\lambda^{++}}$ as $\mathbb{P}_{\lambda^+} * \dot{\mathbb{Q}}_{top}$. If $\alpha < \lambda$ then we define $\mathbb{P}_{\lambda^{++}}^\alpha$ as $\mathbb{P}_{\lambda^+}^\alpha * \dot{\mathbb{Q}}_{top}$.

It is now routine to check that this definition satisfies the chain condition and factorisation demands from Definition 6. The chain condition argument works because $2^\lambda = \lambda^+$, and any incompatibility in $\dot{\mathbb{Q}}_{top}$ is caused by a disagreement in the first coordinate at some $b \in \mathbb{R}$. The factorisation condition (clause 6(a)) holds because symmetric names can be computed using any final segment of the \mathbb{P}_λ -generic, and the closure condition (clause (6c)) follows from the fact that the canonical name for the pointwise sup of a series of functions with symmetric names is itself a symmetric name.

Now we check that we have achieved the desired effect on the values of $\mathfrak{b}(\lambda^+)$ and $\mathfrak{d}(\lambda^+)$.

Lemma 11. *The function added by $\mathbb{P}_{\lambda^+} * \dot{\mathbb{Q}}_{top}$ at $b \in \mathbb{R}^+$ eventually dominates all functions in $\lambda^+ \lambda^+ \cap V^{\mathbb{P}_{\lambda^+} * \dot{\mathbb{Q}}_b}$.*

Proof. It suffices to show that if \dot{F} is a $\mathbb{P}_{\lambda^+} * \dot{\mathbb{Q}}_b$ -name for a function from λ^+ to λ^+ then there is a symmetric name \dot{F}' such that $\Vdash \forall \beta F(\beta) \leq F'(\beta)$.

For each regular $\alpha < \lambda$ we factor $\mathbb{P}_{\lambda^+} * \dot{\mathbb{Q}}_b$ as $\mathbb{P}_{\alpha+1} \times (\mathbb{P}^\alpha \upharpoonright (\alpha + 1, \lambda^+) * \dot{\mathbb{Q}}_b)$. In $V^{\mathbb{P}^\alpha \upharpoonright (\alpha+1, \lambda^+) * \dot{\mathbb{Q}}_b}$ we may treat \dot{F} as a $\mathbb{P}_{\alpha+1}$ -name and define

$$G_\alpha(\beta) = \sup(\{ \gamma \mid \exists p \in \mathbb{P}_{\alpha+1} p \Vdash \dot{F}(\beta) = \gamma \}).$$

Notice that we may also treat G_α as a $\mathbb{P}_{\lambda^+} * \dot{\mathbb{Q}}_b$ -name, and that if $\alpha < \bar{\alpha}$ then $\Vdash G_\alpha \leq G_{\bar{\alpha}}$. Now let F' be the canonical name for a function such that $\forall \beta F'(\beta) = \sup_{\alpha < \lambda} G_\alpha(\beta)$; then it is easy to see that F' is a symmetric name for a function from λ^+ to λ^+ and that $\Vdash \forall \beta F(\beta) \leq F'(\beta)$. \square

Lemma 12. *In $V^{\mathbb{P}_{\lambda^+}}$ there is a copy of \mathbb{R} embedded cofinally into $\lambda^+ \lambda^+$.*

Proof. Let \dot{f} name a function from λ^+ to λ^+ in $V^{\mathbb{P}_{\lambda^+}}$. As \mathbb{P}_{λ^+} has the λ^{++} -c.c. we may assume that \dot{f} only depends on λ^+ many coordinates in \mathbb{R} , and hence (since $\mathfrak{b}(\mathbb{R}) = \beta(\lambda^+) \geq \lambda^{++}$) that \dot{f} is a $\mathbb{P}_{\lambda^+} * \dot{\mathbb{Q}}_b$ -name for some $b \in \mathbb{R}$. By the preceding lemma the function which is added at coordinate b will dominate \dot{f} . \square

It remains to be seen what we should do for λ inaccessible. The construction is very similar to that for successors of singulars, with the important difference that we need to work with a larger class of names for functions in order to guarantee that we dominate everything that we ought to. This in turn leads to a slight complication in the definition of \mathbb{P}^α .

Suppose that we have defined \mathbb{P}_λ and $\mathbb{P}_\lambda^\alpha$ appropriately. Let \mathbb{R} be a poset with the appropriate properties ($|\mathbb{R}| = \mu(\lambda)$, $\mathfrak{b}(\mathbb{R}) = \beta(\lambda)$, $\mathfrak{d}(\mathbb{R}) = \delta(\lambda)$) and let \mathbb{R}^+ be \mathbb{R} with a maximal element called *top* adjoined. As before we define $\mathbb{P}_\lambda * \dot{\mathbb{Q}}_a$ by induction on $a \in \mathbb{R}^+$. In the induction we will maintain the hypothesis that for each $\alpha < \lambda$ there is a dense subset of $\mathbb{P}_\lambda^\alpha * \dot{\mathbb{Q}}_a$ which factorises as $\mathbb{P}_{\alpha+1} \times (\mathbb{P}^\alpha \upharpoonright (\alpha + 1, \lambda) * \dot{\mathbb{Q}}_a^\alpha)$. Let us fix a , and suppose that we have defined everything for all b below a .

Definition 12. $(p, (\mu, q))$ is a condition in $\mathbb{P}_\lambda * \dot{\mathbb{Q}}_a$ iff

1. $p \in \mathbb{P}_\lambda$.
2. $\mu < \lambda$, μ is regular.
3. q is a function, $\text{dom}(q) \subseteq \mathbb{R}^+ / a$ and $|\text{dom}(q)| < \lambda$.
4. For each $b \in \text{dom}(q)$, $q(b)$ is a pair (s, \dot{F}) where $s \in {}^{<\lambda^+} \lambda^+$ and \dot{F} is a $\mathbb{P}^\mu \upharpoonright (\mu + 1, \lambda) * \dot{\mathbb{Q}}_b^\mu$ -name for a function from λ^+ to λ^+ .

Definition 13. Let $(p, (\mu, q))$ and $(p', (\mu', q'))$ be conditions in $\mathbb{P}_\lambda * \dot{\mathbb{Q}}_a$. $(p', (\mu', q'))$ refines $(p, (\mu, q))$ iff

1. p' refines p in \mathbb{P}_{λ^+} .
2. $\text{dom}(q) \subseteq \text{dom}(q')$.
3. $\mu' \geq \mu$.
4. For each $b \in \text{dom}(q)$, if we let $q(b) = (s, F)$ and $q(b') = (s', F')$, then
 - (a) s' extends s .
 - (b) If $\alpha \in \text{lh}(s') - \text{lh}(s)$ then $(p', q' \upharpoonright (\mathbb{R}^+/b)) \Vdash s'(\alpha) \geq F(\alpha)$.
 - (c) For all α , $(p', q' \upharpoonright (\mathbb{R}^+/b)) \Vdash F'(\alpha) \geq F(\alpha)$.

Notice that since any $\mathbb{P}_\lambda * \dot{\mathbb{Q}}_b$ -generic induces a $\mathbb{P}^\mu \upharpoonright (\mu + 1, \lambda) * \dot{\mathbb{Q}}_b^\mu$ -generic, there is a natural interpretation of any $\mathbb{P}^\mu \upharpoonright (\mu + 1, \lambda) * \dot{\mathbb{Q}}_b^\mu$ -name as a $\mathbb{P}_\lambda * \dot{\mathbb{Q}}_b$ -name. We are using this fact implicitly when we define the ordering on the conditions. We need to maintain the hypothesis on factorising the forcing, so we make the following definition.

Definition 14. Let μ be regular with $\mu < \lambda$. $\mathbb{P}_{\mu+1} \times (\mathbb{P}^\mu \upharpoonright (\mu + 1, \lambda) * \dot{\mathbb{Q}}_a^\mu)$ is defined as the set of $(p_0, (p_1, (v, q)))$ such that $p_0 \in \mathbb{P}_{\mu+1}$, $p_1 \in \mathbb{P}^\mu \upharpoonright (\mu + 1, \lambda)$ and $(p_0 \smallfrown p_1, (v, q)) \in \mathbb{P}_\lambda * \dot{\mathbb{Q}}_a$ with $v \geq \mu$.

The key point here is that the factorisation makes sense, because for such a condition $q(b)$ depends only on $\mathbb{P}^\mu \upharpoonright (\mu + 1, \lambda) * \dot{\mathbb{Q}}_b^\mu$.

Definition 15. We define $\mathbb{P}_{\lambda+1}$ as $\mathbb{P}_\lambda * \dot{\mathbb{Q}}_{\text{top}}$, and $\mathbb{P}_{\lambda+1}^\mu$ as $\mathbb{P}_\lambda * \dot{\mathbb{Q}}_{\text{top}}^\mu$.

As in the case of a successor of a singular, it is straightforward to see that we have satisfied the chain condition and factorisation conditions. To finish the proof we need to check that the forcing at λ has achieved the right effect, which will be clear exactly as in the singular case when we have proved the following lemma.

Lemma 13. *The function added by $\mathbb{P}_\lambda * \dot{\mathbb{Q}}_{\text{top}}$ at $b \in \mathbb{R}$ eventually dominates all functions in ${}^\lambda \lambda \cap V^{\mathbb{P}_\lambda * \dot{\mathbb{Q}}_b}$.*

Proof. We do a density argument. Suppose that \dot{f} names a function in ${}^\lambda \lambda \cap V^{\mathbb{P}_\lambda * \dot{\mathbb{Q}}_b}$, and let $(p, (\mu, q))$ be a condition in $\mathbb{P}_\lambda * \dot{\mathbb{Q}}_{\text{top}}$. We factor $\mathbb{P}_\lambda * \dot{\mathbb{Q}}_b$ as $\mathbb{P}_{\mu+1} \times (\mathbb{P}^\mu \upharpoonright (\mu + 1, \lambda) * \dot{\mathbb{Q}}_b^\mu)$, and use the fact that $\mathbb{P}_{\mu+1}$ has μ^+ -c.c. in $V^{\mathbb{P}^\mu \upharpoonright (\mu+1, \lambda) * \dot{\mathbb{Q}}_b^\mu}$ to find a $\mathbb{P}^\mu \upharpoonright (\mu + 1, \lambda) * \dot{\mathbb{Q}}_b^\mu$ -name \dot{g} such that $\Vdash \forall \beta \dot{f}(\beta) \leq \dot{g}(\beta)$.

Now we can refine $(p, (\mu, q))$ in the natural way by strengthening the second component of $q(b)$ to dominate \dot{g} . This gives a condition which forces that the function added at b will eventually dominate \dot{f} . \square

This concludes the proof of Theorem 4. \square

8. Variations

In this section we discuss the invariants that arise if we work with the club filter in place of the co-bounded filter. It turns out that this does not make too much difference. All the results here are due to Shelah.

Definition 16. Let λ be regular.

1. Let $f, g \in {}^\lambda\lambda$. $f <_{cl} g$ iff there is $C \subseteq \lambda$ closed and unbounded in λ such that $\alpha \in C \Rightarrow f(\alpha) < g(\alpha)$.
2. $b_{cl}(\lambda) =_{def} b({}^\lambda\lambda, <_{cl})$.
3. $d_{cl}(\lambda) =_{def} d({}^\lambda\lambda, <_{cl})$.

Theorem 5. $d_{cl}(\lambda) \leq d(\lambda) \leq d_{cl}(\lambda)^\omega$.

Proof. If a family of functions is dominating with respect to $<^*$ it is dominating with respect to $<_{cl}$, so that $d_{cl}(\lambda) \leq d(\lambda)$.

For the converse, let us fix $D \subseteq {}^\lambda\lambda$ such that D is dominating with respect to $<_{cl}$ and $|D| = d_{cl}(\lambda)$. We may assume that every function in D is increasing (replace each $f \in D$ by $f^* : \gamma \mapsto \bigcup_{\alpha \leq \gamma} f(\alpha)$).

Let $g_0 \in {}^\lambda\lambda$. Define by induction f_n, g_n and C_n such that

1. $f_n \in D$.
2. C_n is club in λ , and $\alpha \in C_n \Rightarrow g_n(\alpha) < f_n(\alpha)$.
3. $C_{n+1} \subseteq C_n$.
4. g_n is increasing.
5. $g_{n+1}(\beta) > g_n(\beta)$ for all β .
6. $g_{n+1}(\beta) > f_n(\min(C_n - (\beta + 1)))$ for all β .

Now let $\alpha = \min(\bigcap_{n < \omega} C_n)$. We will prove that $g_0(\gamma) \leq \bigcup_n f_n(\gamma)$ for $\gamma > \alpha$.

Fix some $\gamma > \alpha$. For each n we know that $C_n \cap \gamma \neq \emptyset$, so that if we define $\gamma_n = \sup(C_n \cap (\gamma + 1))$ then γ_n is the largest point of C_n less than or equal to γ . Notice that $\min(C_n - (\gamma + 1)) = \min(C_n - (\gamma_n + 1))$.

Since $C_{n+1} \subseteq C_n$, $\gamma_{n+1} \leq \gamma_n$, so that for all sufficiently large n (say $n \geq N$) we have $\gamma_n = \bar{\gamma}$ for some fixed $\bar{\gamma}$. We claim that $g_0(\gamma) \leq f_{N+1}(\bar{\gamma})$, which we will prove by building a chain of inequalities. Let us define

$$\delta = \min(C_N - (\gamma + 1)) = \min(C_N - (\bar{\gamma} + 1)).$$

Then

$$g_0(\gamma) \leq g_N(\gamma) \leq g_N(\delta) < f_N(\delta) < g_{N+1}(\bar{\gamma}) < f_{N+1}(\bar{\gamma}) \leq f_{N+1}(\gamma),$$

where the key point is that $\bar{\gamma} \in C_{N+1}$ and hence $g_{N+1}(\bar{\gamma}) < f_{N+1}(\bar{\gamma})$.

Now it is easy to manufacture a family of size $d_{cl}(\lambda)^\omega$ which is dominating with respect to $<^*$, so that $d(\lambda) \leq d_{cl}(\lambda)^\omega$. \square

Theorem 6. $b_{cl}(\lambda) = b(\lambda)$.

Proof. If a family of functions is unbounded with respect to $<_{cl}$ it is unbounded with respect to $<^*$, so that $b(\lambda) \leq b_{cl}(\lambda)$.

Suppose for a contradiction that $b(\lambda) < b_{cl}(\lambda)$, and fix $U_0 \subseteq \lambda$ such that $|U_0| = b(\lambda)$ and U_0 is unbounded with respect to $<^*$. We may assume without loss of generality that every function in U_0 is increasing. We perform an inductive construction in ω steps, whose aim is to produce a bound for U_0 with respect to $<^*$.

By assumption U_0 is bounded with respect to $<_{cl}$, so choose g_0 which bounds it modulo the club filter. Choose also club sets $\{C_f^0 \mid f \in U_0\}$ such that $\alpha \in C_f^0 \Rightarrow f(\alpha) < g_0(\alpha)$. For each $f \in U_0$ define another function $f^{[0]}$ by $f^{[0]} : \beta \mapsto f(\min(C_f^0 - (\beta + 1)))$.

For $n \geq 1$ define $U_n = U_{n-1} \cup \{f^{[n-1]} \mid f \in U_{n-1}\}$. By induction it will follow that $|U_n| = b(\lambda)$, so that we may choose g_n such that $g_n(\beta) > g_{n-1}(\beta)$ for all β , and g_n bounds U_n modulo clubs. We choose clubs $\{C_f^n \mid f \in U_n\}$ such that

1. $\alpha \in C_f^n \Rightarrow f(\alpha) < g_n(\alpha)$.
2. If $f \in U_{n-1}$, then $C_f^n \subseteq C_f^{n-1}$.
3. If $n \geq 2$ and $f \in U_{n-2}$ then $C_f^n \subseteq C_{f^{[n-2]}}^{n-1}$, where this makes sense because in this case $f^{[n-2]} \in U_{n-1}$.

For each $f \in U_n$ we define $f^{[n]} : \beta \mapsto f(\min(C_f^n - (\beta + 1)))$ to finish round n of the inductive construction.

Now we claim that the pointwise sup of the sequence $\langle g_n : n < \omega \rangle$ is an upper bound for U_0 with respect to $<^*$. Let us fix $f \in U_0$, and then let $\alpha = \min(\bigcap C_f^n)$.

We will now give a very similar argument to that of Theorem 5. Fix $\gamma > \alpha$. We define $\gamma_n = \sup(C_f^n \cap (\gamma + 1))$ and observe that $\gamma_{n+1} \leq \gamma_n$, so we may find N and $\bar{\gamma}$ such that $n \geq N \Rightarrow \gamma_n = \bar{\gamma}$.

Let $\delta = \min(C_F^N - (\gamma + 1)) = \min(C_F^N - (\bar{\gamma} + 1))$. We now get a chain of inequalities

$$f(\gamma) \leq f(\delta) = f^{[N]}(\bar{\gamma}) < g_{N+1}(\bar{\gamma}) \leq g_{N+1}(\gamma).$$

This time the key point is that $\bar{\gamma} \in C_f^{N+2} \subseteq C_{f^{[N]}}^{N+1}$, so that $f^{[N]}(\bar{\gamma}) < g_{N+1}(\bar{\gamma})$.

We have proved that $\gamma > \alpha \Rightarrow f(\gamma) \leq \bigcup_n g_n(\gamma)$, so that every function $f \in U_0$ is bounded on a final segment of λ by $\gamma \mapsto \bigcup_n g_n(\gamma)$. This contradicts the choice of U_0 as unbounded with respect to $<^*$, so we are done. \square

It is natural to ask whether the first result can be improved to show that $b_{cl}(\lambda) = b(\lambda)$. This can be done for λ sufficiently large, at the cost of using a powerful result from Shelah’s paper [6].

Definition 17. Let $\theta = cf(\theta) < \mu$.

- (1) $\mathcal{P}^\theta(\mu) = \{X \subseteq \mu \mid |X| = \theta\}$.
- (2) $\mu^{[\theta]}$ is the least cardinality of a family $P \subseteq \mathcal{P}^\theta(\mu)$ such that

$$\forall A \in \mathcal{P}^\theta(\mu) \exists B \subseteq P (|B| < \theta \wedge A \subseteq \bigcup B).$$

One of the main results of [6] is that ZFC proves a weak form of the GCH.

Theorem 7. *Let $\mu > i_\omega$. Then $\mu^{[\theta]} = \mu$ for all sufficiently large $\theta < i_\omega$.*

It is easy to see that if $P \subseteq \mathcal{P}^\theta \mu$ is such that

$$\forall A \in \mathcal{P}^\theta(\mu) \exists B \subseteq P (|B| < \theta \wedge A \subseteq \bigcup B),$$

then $\forall A \in \mathcal{P}^\theta(\mu) \exists C \in P |A \cap C| = \theta$. This is all we use in what follows, and in fact we could get away with $\forall A \in \mathcal{P}^\theta(\mu) \exists C \in P |A \cap C| = \aleph_0$.

Theorem 8. *Let $\lambda = \text{cf}(\lambda) > i_\omega$. Then $\mathfrak{d}(\lambda) = \mathfrak{d}_{\text{cl}}(\lambda)$.*

Proof. Let $\mu = \mathfrak{d}_{\text{cl}}(\lambda)$. Then $\mu > i_\omega$, so that we may apply Theorem 7 to find a regular $\theta < i_\omega$ such that $\mu^{[\theta]} = \mu$. Let us fix $P \subseteq \mathcal{P}^\theta(\mu)$ such that $|P| = \mu$ and $\forall A \in \mathcal{P}^\theta(\mu) \exists C \in P |A \cap C| = \theta$.

Now let $D \subseteq {}^\lambda \lambda$ be such that $|D| = \mu$ and D is dominating in $({}^\lambda \lambda, <_{\text{cl}})$. We may suppose that D consists of increasing functions. Enumerate D as $\langle h_\alpha : \alpha < \theta \rangle$, and then define $h_A : \gamma \mapsto \bigcup_{\alpha \in A} h_\alpha(\gamma)$ for each $A \in P$. Since $\theta < i_\omega \leq \lambda$, $h_A \in {}^\lambda \lambda$. We will prove that $\{h_A \mid A \in P\}$ is dominating in $({}^\lambda \lambda, <^*)$.

We will do a version of the construction from Theorem 5. Let $g_0 \in {}^\lambda \lambda$. Define by induction f_α, g_α and C_α for $\alpha < \theta$, with the following properties.

1. $f_\alpha \in D$.
2. C_α is club in λ , and $\beta \in C_\alpha \Rightarrow g_\alpha(\beta) < f_\alpha(\beta)$.
3. $\alpha < \bar{\alpha} \Rightarrow C_{\bar{\alpha}} \subseteq C_\alpha$.
4. g_α is increasing.
5. If $\alpha < \bar{\alpha}$, then $g_{\bar{\alpha}}(\beta) > g_\alpha(\beta)$ for all β .
6. If $\alpha < \bar{\alpha}$, then $g_{\bar{\alpha}}(\beta) > f_\alpha(\min(C_\alpha - (\beta + 1)))$ for all β .

This is easy, because $\theta < \lambda$. By the choice of P we may find a set $A \in P$ such that $|\{h_\beta \mid \beta \in A\} \cap \{f_\alpha \mid \alpha < \theta\}| = \theta$. Enumerate the first ω many α such that $f_\alpha \in \{h_\beta \mid \beta \in A\}$ as $\langle \alpha_n : n < \omega \rangle$.

We may now repeat the proof of Theorem 5 with $f_{\alpha_n}, g_{\alpha_n}$ and C_{α_n} in place of f_n, g_n and C_n . We find that for all sufficiently large γ we have $g_0(\gamma) \leq g_{\alpha_0}(\gamma) \leq \bigcup_n f_{\alpha_n}(\gamma)$. By the definition of h_A and the fact that $\{f_{\alpha_n} \mid n < \omega\} \subseteq \{h_\beta \mid \beta \in A\}$, $\bigcup_n f_{\alpha_n}(\gamma) \leq h_A(\gamma)$ for all γ , so that $g_0 <^* h_A$. This shows $\{h_A \mid A \in P\}$ to be dominating, so we are done. \square

We do not know whether it can ever be the case that $\mathfrak{d}_{\text{cl}}(\lambda) < \mathfrak{d}(\lambda)$. This is connected with some open and apparently difficult questions in pcf theory.

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