Note

Minimally circular-imperfect graphs with a major vertex

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Abstract

In this note, we characterize the minimally circular-imperfect graphs that have a major vertex.

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All graphs considered in this paper are finite and simple, i.e., finite graphs without multiedges and loops. Undefined concepts and terminologies can be found in [3].

Let \( G \) be a graph. A proper subgraph of \( G \) is a subgraph which is not the graph \( G \) itself. A subgraph \( H \) of \( G \) is called an induced subgraph if \( E(H) = \{uv|u \in V(H), v \in V(H), \text{ and } uv \in E(G)\} \). A set \( S \) of vertices is deleted from a graph \( G \) if \( S \) together with all the edges with at least one end in \( S \) is removed from \( G \), and the resulting graph is denoted by \( G \setminus S \).

The chromatic number of \( G \), denoted by \( \chi(G) \), is the least integer \( n \) such that \( G \) can be proper colored with \( n \) colors. The clique number of \( G \), denoted by \( \omega(G) \), is the maximum number of vertices contained in a complete subgraph of \( G \).

A graph \( G \) is called a perfect graph if \( \chi(H) = \omega(H) \) for each induced subgraph \( H \) of \( G \), and a graph is called imperfect if it is not perfect. An odd hole in a graph is an induced subgraph that is isomorphic to an odd circuit of length at least five (an odd circuit is a circuit with odd length).

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Theorem 1 (Chudnovsky et al. [4]). (Strong Perfect Graph Theorem) The only minimally imperfect graphs are the odd holes and their complements.

Suppose that $k$ and $d$ are integers with $k \geq 2d$. A $(k, d)$-circular coloring of a graph $G$ is a mapping $\psi : V(G) \mapsto \{0, 1, \ldots, k - 1\}$ such that $d \leq |\psi(u) - \psi(v)| \leq k - d$ whenever $uv \in E(G)$. A graph $G$ is called $(k, d)$-circular colorable if it admits a $(k, d)$-circular coloring. The circular chromatic number of $G$, denoted by $\chi_c(G)$, is the minimum $\frac{k}{d}$ such that $G$ is $(k, d)$-circular colorable. The concept of circular coloring was first introduced in 1988 by Vince who first named it as star coloring, and it got the current name from Zhu [9]. It was proved elsewhere [2,7] that

$$\chi(G) - 1 < \chi_c(G) \leq \chi(G) \quad \text{for any graph } G. \quad (1)$$

A graph called $G^k_d$ makes important roles in the study of circular coloring which is analogous to what $K_l$ does in the coloring. Given two positive integers $k$ and $d$ such that $k \geq 2d$, the graph $G^k_d$ is defined as follows:

$$V(G^k_d) = \{v_0, v_1, v_2, \ldots, v_{k-1}\},$$

$$E(G^k_d) = \{v_i v_j | d \leq |j - i| \leq k - d \mod k\}.$$ 

It is proved elsewhere that $\chi_c(G^k_d) = \frac{k}{d}$ [2,7]. So, if a graph $G$ contains a subgraph which is isomorphic to $G^k_d$ for some $k$ and $d$, then $\chi_c(G) \geq \frac{k}{d}$.

In [10], Zhu introduced the concept of circular clique number, which, denoted by $\omega_c(G)$, is the maximum fractional $\frac{k}{d}$ such that $G^k_d$ admits a homomorphism to $G$. Given integers $k$ and $d$, we use $gcd(k, d)$ to denote the greatest common divisor of $k$ and $d$. Zhu proved in [10] that

Theorem 2 (Zhu [10]). For any graph $G$,

$$\omega(G) \leq \omega_c(G) < \omega(G) + 1, \quad (2)$$

and $\omega_c(G) = \frac{k}{d}$ for some $k$ and $d$ with $gcd(k, d) = 1$ indicates that $G$ contains an induced subgraph isomorphic to $G^k_d$.

It is clear that $\omega_c(G) \leq \chi_c(G)$ for any graph $G$. Therefore, by (1) and (2),

$$\omega(G) \leq \omega_c(G) \leq \chi_c(G) \leq \chi(G) \quad \text{for any graph } G. \quad (3)$$

Analogous to the concept of perfect graph, Zhu also introduced in [10] the concept of circular-perfect graph, and gave some sufficient conditions and necessary conditions for a graph to be circular-perfect. A graph $G$ is called circular-perfect if $\omega_c(H) = \chi_c(H)$ for each induced subgraph $H$ of $G$.

Theorem 3 (Bang-Jensen and Huang [1], Zhu [10]). For any integers $k \geq 2d$, $G^k_d$ is circular-perfect.
In [1], Bang-Jensen and Huang present a family of graphs, they call them \textit{convex-round graphs}, and show that each of these graphs is a circular-perfect graph. In fact, $G_k^d$ for $k \geq 2d$ is a convex-round graph.

Compared with the family of perfect graphs, the family of circular-perfect graphs has many different properties. Take as an example, we know that a graph is perfect iff its complement is perfect. But this is not true again while considering the family of circular-perfect graphs. There are numerous circular-perfect graphs with noncircular perfect complements.

A graph is called \textit{circular-imperfect} if it is not circular perfect. A \textit{minimally circular-imperfect graph} is a circular-imperfect graph of which each proper induced subgraph is circular-perfect.

By the definition of $G_k^d$, one can see easily that $G_{2n}^n + 1$ is an odd circuit of length $2n + 1$ and $G_{2n}^n + 1$ is the complement of $G_{2n}^n$. Let $\mathcal{C} = \{G_{2n}^n + x, G_{2n}^n + x | n \geq 2\}$, where $G + x$ means that $x \notin V(G)$ is adjacent to every vertex of $G$. A vertex that is adjacent to each of the other vertices in a graph is usually called a \textit{major vertex}.

In this note, we will prove that

\textbf{Theorem 4.} Every graph $G \in \mathcal{C}$ is minimally circular-imperfect, and every minimally circular-imperfect graph with a major vertex is in $\mathcal{C}$.

To prove this result, we need the following lemmas.

\textbf{Lemma 1} (Steffen and Zhu [6], Zhu [8]). Let $G$ be a graph with $\chi(G) = n$. If there is a subset $A$ of vertices such that for each $n$-coloring of $G$, the set of vertices which receive the same color is either disjoint from $A$ or contained in $A$, then $\chi(G) = n$.

As a direct consequence of Lemma 1, if a graph $G$ has a major vertex $u$, then

\[ \chi_c(G) = \chi(G) = \chi(G - u) + 1. \]  \hspace{1cm} (4)

\textbf{Lemma 2.} If a graph $G$ has a major vertex, then $\omega(G) = \chi_c(G)$.

\textbf{Proof.} If it is not the case, assume that $\omega_c(G) = \frac{k}{d} \neq \lfloor \frac{k}{d} \rfloor$ for some $k$ and $d$ with $gcd(k, d) = 1$. By Theorem 2, $\omega(G) = \lfloor \frac{k}{d} \rfloor$, and $G$ contains a subgraph $G'$ isomorphic to $G_{\frac{k}{d}}^n$. It is certainly that $u \notin V(G')$. Now, we look at the graph $G' + u$, $G' + u$ contains a complete subgraph on $\lfloor \frac{k}{d} \rfloor + 1$ vertices, and this indicates that $\omega(G) \geq \lfloor \frac{k}{d} \rfloor + 1$. This ends the proof. \hfill \Box

\textbf{Lemma 3.} The graph obtained from a perfect graph by adding a major vertex is perfect.

This is a direct consequence of the fact that, when adding a major vertex to a graph, both the chromatic number and the clique number increase by one.

\textbf{Proof of Theorem 4.} First we show that every graph in $\mathcal{C}$ is minimally circular-imperfect. For $G = G_{2n+1}^n + u$ ($r = 2$ or $r = n$), we have $\chi_c(G) = r + 2$ by (4), and $\omega_c(G) = r + 1$ by Lemma 2. So, $G$ is circular-imperfect. Since $G - u$ is minimally imperfect, by Lemma 3, the only proper induced subgraph of $G$ that is not perfect is the graph $G - u$ that is known to be

B. Xu / Discrete Mathematics 301 (2005) 239–242

241
circular-perfect. By (3), every perfect graph is circular-perfect. Therefore, \( G \) is minimally circular-imperfect.

Conversely, let \( G \) be a minimally circular-imperfect graph with a major vertex \( u \). By (3) and Lemma 3, \( G - u \) is not perfect and hence by the Strong Perfect Graph Theorem, \( G - u \) contains an induced subgraph \( G' \) that is an odd hole or the complement of an odd hole. By the first part of the proof, \( G' + u \) is then minimally circular-imperfect and so \( G = G' + u \). The theorem is then proved. \( \square \)

Let \( G \) be a graph with a major vertex \( u \). If neither \( G - u \) nor the complement of \( G - u \) contains an odd hole, then by the Strong Perfect Graph Theorem, \( G - u \) is perfect. By Lemma 3, \( G \) is perfect and hence circular-perfect. As a corollary of Theorem 4, we have

**Corollary 1.** Let \( G \) be a graph with a major vertex \( u \). Then, \( G \) is circular-perfect iff neither \( G - u \) nor the complement of \( G - u \) contains an odd hole.

**Comments:** It seems that the family of forbidden subgraphs for circular-perfect graphs is much bigger than that of perfect graphs. One can find a minimally circular-imperfect graph from either the Petersen graph or its complement, one is the graph obtained from the Petersen graph by removing two adjacent vertices, another is the graph obtained from the complement of the Petersen graph by removing three vertices that induce a path in the Petersen graph. One can also find numerous minimally circular-imperfect graphs in the family of serial-parallel graphs. These minimally circular-imperfect graphs look so different, it seems difficult to give a simple structural characterization to them. Can we find other families of minimally circular-imperfect graphs which have a simple structural property?

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