Periodicity in a Logistic Type System with Several Delays

FENGDE CHEN* AND JINLIN SHI
Department of Mathematics, Fuzhou University
Fuzhou, Fujian, 350002, P.R. China
fdehen@263.net fdchen@fzu.edu.cn

(Received October 2002; accepted February 2004)

Abstract—With the help of a continuation theorem based on Gaines and Mawhin’s coincidence degree, easily verifiable criteria are established for the global existence of positive periodic solutions of the following several delays logistic type system:

\[ \frac{du(t)}{dt} = u(t) \left[ a(t) - \sum_{j=1}^{m} b_j(t) (u(t - \tau_j(t)))^{\beta_j} \right], \]

where \( a(t), b_j(t) \) are positive periodic continuous functions with periodic \( \omega > 0 \), \( \tau_j(t) \) are nonnegative continuous periodic functions with periodic \( \omega > 0 \). After that, by constructing a suitable Lyapunov functional, some sufficient conditions which guarantee the stability of the positive periodic solutions are obtained. © 2004 Elsevier Ltd. All rights reserved.

Keywords—Periodic solutions, Delay equation, Logistic type system, Coincidence degree.

1. INTRODUCTION

The qualitative behavior of logistic equation with several delays

\[ \frac{du(t)}{dt} = u(t) \left[ a - \sum_{j=1}^{n} b_j u(t - \tau_j) \right], \tag{1} \]

and some of its generalizations have been studied extensively, where \( a, b_j, \tau_j(j = 1, 2, \ldots, m) \) are positive constants. For a recent contribution to the study of (1), we refer to the works of Gopalsamy [1], Gyori and Ladas [2], Lenhart and Travis [3], Kuang [4], Yu [5], Cao [6], Li [7,8] and Chen [9]. In [10], Zhang and Gopalsamy have considered the following periodic delay logistic equation:

\[ \frac{dx(t)}{dt} = r(t)x(t) \left[ 1 - \frac{x(t - n\tau)}{k(t)} \right], \tag{2} \]

This work was supported by the start-up fund of Fuzhou University under Grant 0030824228 and the Foundation of Developing Science and Technology of Fuzhou University under Grant 2003-QX-21.

*Author to whom all correspondence should be addressed.
where \( r \) and \( k \) are positive periodic functions of periodic \( \tau \) and \( n \) is a positive integer. In [1], Gopalsamy has considered the following delay logistic equation:
\[
\frac{du(t)}{dt} = ru(t) \left[ 1 - \left( \frac{u(t - \tau)}{k} \right)^n \right],
\]
where \( r, k, \tau, n \) are positive constants. Recently, in [11], Yan et al. have considered the following delay logistic equation:
\[
\frac{du(t)}{dt} = u(t) \left[ a(t) + b(t)x^p(t - m\omega) - c(t)x^q(t - m\omega) \right],
\]
where \( a, b, c \) are positive periodic functions of periodic \( \omega \) and \( p, q \) are positive constants with \( q > p \). Therefore, the author has been motivated to consider the periodic type logistic equation with several delays
\[
\frac{du(t)}{dt} = u(t) \left[ a(t) - \sum_{j=1}^{m} b_j(t) (u(t - \tau_j(t)))^{\theta_j} \right].
\]
Throughout this paper, we assume system (5) satisfies the following.

\((H_1)\). \( a(t), b_j(t) \) are positive periodic continuous functions with periodic \( \omega > 0 \), \( \tau_j(t) \) are non-negative periodic continuous functions with periodic \( \omega > 0 \), \( \theta_j \) are positive constants.

We will consider solutions of equation (5) with the initial condition
\[
\varphi(s) \geq 0, \quad \varphi(0) > 0, \quad \varphi \in C \left([-\tau^*, 0], R^+ \right),
\]
where \( \tau^* = \max_{1 \leq j \leq m} \{ \max_{0 \leq i \leq \omega} \tau_j(i) \} \).

The main purpose of this paper is to derive a set of easily verifiable sufficient conditions for the existence of unique globally attractive positive periodic solutions of (5). The outline of the paper is as follows. In Section 2, we introduce the continuation theorem based on the coincidence degree theory of [12]. This is then used to prove the results which guarantee the existence of positive periodic solution of system (5). Also, as we can see, the results obtained about system (5) (Theorem 2.1 of the next section) can be immediately generalized to a more general case (Theorem 2.2 of the next section). In Section 3, by constructing a suitable Lyapunov functional, some sufficient conditions which guarantee the global attractivity of the positive periodic solution are obtained.

2. EXISTENCE OF POSITIVE PERIODIC SOLUTIONS

In order to obtain the existence of positive periodic solutions of (5), for the reader’s convenience, in the following, we shall summarize a few concepts and results from [12] that will be basic for this section.

Let \( X, Z \) be normed vector spaces, \( L : \text{Dom} \ L \subset X \to Z \) be a linear mapping, \( N : X \to Z \) be a continuous mapping. The mapping \( L \) will be called a Fredholm mapping of index zero if \( \dim \ker L = \text{Codim} \text{Im} L < +\infty \) and \( \text{Im} L \) is closed in \( Z \). If \( L \) is a Fredholm mapping of index zero, there exist continuous projectors \( P : X \to X \) and \( Q : Z \to Z \) such that \( \text{Im} P = \ker L, \text{Im} L = \ker Q = \text{Im} (I - Q) \). It follows that \( L | \text{Dom} L \cap \ker P : (I - P)X \to \text{Im} L \) is invertible. We denote the inverse of that map by \( K_P \). If \( \Omega \) is an open bounded subset of \( X \), the mapping \( N \) will be called \( L \)-compact on \( \Omega \) if \( QN(\Omega) \) is bounded and \( K_P(I - Q)N : \Omega \to X \) is compact. Since \( \text{Im} Q \) is isomorphic to \( \ker L \), there exists an isomorphism \( J : \text{Im} Q \to \ker L \).

In the proof of our existence theorem below, we will use the continuation theorem of [12, p. 40].
LEMMA 2.1. (CONTINUATION THEOREM). Let $L$ be a Fredholm mapping of index zero and let $N$ be $L$-compact on $\Omega$. Suppose the following.

(a) For each $\lambda \in (0, 1)$, every solution $x$ of $Lx = \lambda Nx$ is such that $x \notin \partial \Omega$.

(b) $QN \neq 0$ for each $x \in \partial \Omega \cap \text{Ker } L$ and
\[
\deg\{JQN, \Omega \cap \text{Ker } L, 0\} \neq 0.
\]
Then, the equation $Lx = Nx$ has at least one solution lying in $\text{Dom } L \cap \Omega$.

LEMMA 2.2. The domain $R^+ = \{x \mid x > 0\}$ is invariant with respect to (5).

PROOF. Since
\[
u(t) = \frac{1}{\omega} \int_0^\omega g(t) dt, \quad g^\prime = \min_{t \in [0, \omega]} g(t), \quad g^\ast = \max_{t \in [0, \omega]} g(t),
\]
where $g$ is a continuous $\omega$-periodic function.

Our main result on the global existence of a positive periodic solution of (5) is stated in the following theorem.

THEOREM 2.1. Under Assumption (H), system (5) with initial condition (6) has at least one positive $\omega$-periodic solution $u^*(t)$ and there exists a positive constant $B$ such that $\|u^*\| \leq B$, where $\|u^*\| = \max_{t \in [0, \omega]} |u^*(t)|$.

PROOF. From Lemma 2.2, we can make the change of variables
\[
u(t) = \exp\{x(t)\},
\]
then (5) can be reformulated as
\[
od x(t) \over dt = a(t) - \sum_{j=1}^m b_j(t) \exp \{\theta_j x(t - \tau_j(t))\}.
\]
In order to apply Lemma 2.1 (continuation theorem) to (8), we first define
\[
X = Z = \{x(t) \in C(R, R), x(t + \omega) = x(t)\}
\]
and
\[
\|x\| = \max_{t \in [0, \omega]} |x(t)|,
\]
for any $x \in X$ (or $Z$). Then, $X$ and $Z$ are Banach spaces with the norm $\|\cdot\|$. Let
\[
Nx = a(t) - \sum_{j=1}^m b_j(t) \exp \{\theta_j x(t - \tau_j(t))\}, \quad x \in X,
\]
\[
Lx = \dot{x} = \frac{dx(t)}{dt}, \quad Px = \frac{1}{\omega} \int_0^\omega x(t) dt, \quad x \in X; \quad Qz = \frac{1}{\omega} \int_0^\omega z(t) dt, \quad z \in Z.
\]
Then, it follows that
\[ \text{Ker } L = \mathbb{R}, \quad \text{Im } L = \left\{ z \in Z : \int_0^\omega z(t) \, dt = 0 \right\} \text{ is closed in } Z, \]
\[ \dim \text{Ker } L = 1 = \text{codim Im } L, \]
and \( P, Q \) are continuous projectors such that
\[ \text{Im } P = \text{Ker } L, \quad \text{Ker } Q = \text{Im } L = \text{Im}(I - Q). \]

Therefore, \( L \) is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to \( L \)) \( K_P : \text{Im } L \to \text{Ker } P \cap \text{Dom } L \) reads
\[ K_P(z) = \int_0^t z(s) \, ds - \frac{1}{\omega} \int_0^\omega \int_0^t z(s) \, ds \, dt. \]

Thus,
\[ QN x = \frac{1}{\omega} \int_0^\omega \left[ a(s) - \sum_{j=1}^m b_j(s) \exp \{ \theta_j x(t - \tau_j(s)) \} \right] \, ds \]
\[ K_P(I - Q)N x = \int_0^t \left[ a(s) - \sum_{j=1}^m b_j(s) \exp \{ \theta_j x(s - \tau_j(s)) \} \right] \, ds \]
\[ - \frac{1}{\omega} \int_0^\omega \int_0^t \left[ a(s) - \sum_{j=1}^m b_j(s) \exp \{ \theta_j x(s - \tau_j(s)) \} \right] \, ds \, dt \]
\[ - \left( \frac{t}{\omega} - \frac{1}{2} \right) \int_0^\omega \left[ a(s) - \sum_{j=1}^m b_j(s) \exp \{ \theta_j x(s - \tau_j(s)) \} \right] \, ds. \]

Obviously, \( QN \) and \( K_P(I - Q)N \) are continuous. It is not difficult to show that \( K_P(I - Q)N(\bar{\Omega}) \) is compact for any open bounded \( \Omega \subset X \) by using the Arzela-Ascoli theorem. Moreover, \( QN(\bar{\Omega}) \) is clearly bounded. Thus, \( N \) is \( L \)-compact on \( \bar{\Omega} \) with any open bounded set \( \Omega \subset X \).

Now we reach the position to search for an appropriate open bounded subset \( \Omega \) for the application of the continuation theorem (Lemma 2.1). Corresponding to the operator equation \( Lx = \lambda Nx, \lambda \in (0, 1) \), we have
\[ \frac{dx(t)}{dt} = \lambda \left[ a(t) - \sum_{j=1}^m b_j(t) \exp \{ \theta_j x(t - \tau_j(t)) \} \right]. \quad (9) \]

Assume that \( x = x(t) \in X \) is a solution of (9) for a certain \( \lambda \in (0, 1) \). Integrating (9) over the interval \([0, \omega]\), we obtain
\[ \int_0^\omega \left[ a(t) - \sum_{j=1}^m b_j(t) \exp \{ \theta_j x(t - \tau_j(t)) \} \right] \, dt = 0. \]

That is,
\[ \int_0^\omega \left[ \sum_{j=1}^m b_j(t) \exp \{ \theta_j x(t - \tau_j(t)) \} \right] \, dt = \bar{a}\omega, \quad (10) \]
From (9), (10), it follows that
\[
\int_0^\omega |\dot{x}(t)| \, dt = \lambda \int_0^\omega \left[ a(t) - \sum_{j=1}^m b_j(t) \exp \{ \theta_j x(t - \tau_j(t)) \} \right] \, dt
\]
\[
\leq \int_0^\omega |a(t)| \, dt + \int_0^\omega \left[ \sum_{j=1}^m b_j(t) \exp \{ \theta_j x(t - \tau_j(t)) \} \right] \, dt
\]
\[
= (\bar{a} + \bar{A}) \omega,
\]
where \(\bar{A} = 1/\omega \int_0^\omega |a(t)| \, dt\). Note that \(x(t) \in X\), then there exist \(\xi, \tau \in [0, \omega]\) such that
\[
x(\xi) = \min_{t \in [0, \omega]} x(t), \quad x(\tau) = \max_{t \in [0, \omega]} x(t).
\] (12)

Then, by (10) and (12), we have
\[
\bar{a} \omega = \int_0^\omega \left[ \sum_{j=1}^m b_j(t) \exp \{ \theta_j x(t - \tau_j(t)) \} \right] \, dt
\]
\[
\geq \begin{cases}
\exp \{ \theta^l x(\xi) \} \sum_{j=1}^m \bar{b}_j \omega, & x(\xi) \geq 0, \\
\exp \{ \theta^u x(\xi) \} \sum_{j=1}^m \bar{b}_j \omega, & x(\xi) \leq 0,
\end{cases}
\]
where \(\theta^l = \min \{\theta_j, j = 1, 2, \ldots, m\}\), \(\theta^u = \max \{\theta_j, j = 1, 2, \ldots, m\}\). That is,
\[
x(\xi) \leq \begin{cases}
\frac{1}{\theta^l} \ln \left( \frac{\bar{a}}{\sum_{j=1}^m \bar{b}_j} \right), & x(\xi) \geq 0,
\end{cases}
\]
\[
x(\xi) \leq \begin{cases}
\frac{1}{\theta^u} \ln \left( \frac{\bar{a}}{\sum_{j=1}^m \bar{b}_j} \right), & x(\xi) \leq 0,
\end{cases}
\]
or we can say there exists a constant \(M_0\) such that \(x(\xi) \leq M_0\), then
\[
x(t) \leq x(\xi) + \int_0^\omega |\dot{x}(t)| \, dt \leq M_0 + (\bar{A} + \bar{a}) \omega.
\] (13)

On the other hand, by (10) and (12), we also have
\[
\bar{a} \omega = \int_0^\omega \left[ \sum_{j=1}^m b_j(t) \exp \{ \theta_j x(t - \tau_j(t)) \} \right] \, dt
\]
\[
\leq \begin{cases}
\exp \{ \theta^l x(\tau) \} \sum_{j=1}^m \bar{b}_j \omega, & x(\tau) \leq 0, \\
\exp \{ \theta^u x(\tau) \} \sum_{j=1}^m \bar{b}_j \omega, & x(\tau) \geq 0,
\end{cases}
\]
where \(\theta^l = \min \{\theta_j, j = 1, 2, \ldots, m\}\), \(\theta^u = \max \{\theta_j, j = 1, 2, \ldots, m\}\). That is,
\[
x(\tau) \geq \begin{cases}
\frac{1}{\theta^l} \ln \left( \frac{\bar{a}}{\sum_{j=1}^m \bar{b}_j} \right), & x(\tau) \leq 0,
\end{cases}
\]
\[
x(\tau) \geq \begin{cases}
\frac{1}{\theta^u} \ln \left( \frac{\bar{a}}{\sum_{j=1}^m \bar{b}_j} \right), & x(\tau) \geq 0,
\end{cases}
\]
or we can say there exists a constant $M_1$ such that $x(t) \geq M_1$, then

$$x(t) \geq x(\tau) - \int_{0}^{\omega} |\dot{x}(t)| \, dt \geq M_1 - (\bar{A} + \bar{a}) \omega. \tag{14}$$

It follows from (13) and (14) that

$$\max_{t \in [0, \omega]} |x(t)| \leq \max \{|M_1 - (\bar{A} + \bar{a}) \omega|, |M_0 + (\bar{A} + \bar{a}) \omega|\} := M. \tag{15}$$

Clearly, $M$ is independent on the choice of $\lambda$. Noticing the fact that $a(t)$ and $b_j(t)$ are all positive functions, so, one can easily show that equation

$$f(x) := \bar{a} - \sum_{j=1}^{n} b_j x^{\theta_j} = 0 \tag{16}$$

has a unique positive solution $x_0$ and $f(x) > 0$, for all $x \in (0, x_0)$, $f(x) < 0$, for all $x \in (x_0, +\infty)$. Let $H = M + C$, where $C$ is chosen sufficiently large such that the unique solution $x_0$ of (16) satisfies $\|\ln x_0\| < C$. Let $\Omega = \{x \in X \mid \|x\| < H\}$. It is clear that $\Omega$ verifies Requirement (a) in Lemma 2.1. When $x \in \partial \Omega \cap R$, $x$ is constant with $x = \pm H$. Then,

$$QN \varphi = \left( \bar{a} - \sum_{j=1}^{n} b_j \exp\{\theta_j x\} \right) \neq 0.$$ 

Furthermore, in view of the assumptions in Theorem 2.1, direct calculation produces

$$\text{deg}\{JQN, \Omega \cap \text{Ker} \, L, 0\} \neq 0.$$ 

Here, $J$ can be the identity mapping since $\text{Im} \, P = \text{Ker} \, L$. By now, we have proved that $\Omega$ verifies all the requirements in Lemma 2.1. Hence, (8) has at least one solution $x^*(t)$ in $\text{Dom} \, L \cap \Omega$. Set $u^*(t) = \exp\{x^*(t)\}$, then by the method of (7), we know that $u^*(t)$ is a positive $\omega$-periodic solution of (5). The existence of positive constant $B$ is clear, since $x^*(t)$ lies in $\Omega$. This completes the proof of the claim. \(\Box\)

From the proof of Theorem 2.1, one can observe that Theorem 2.1 remains valid if some or all of the terms with discrete delay in (5) are replaced by distributed delays (finite or infinite).

In fact, recent research shows that for a realistic ecosystem, it is better to consider the system with both discrete delay (time varying) and continuous delay synchronously, so let us consider a logistic type system with both discrete delays and continuous delays, that is, system

$$\frac{du(t)}{dt} = u(t) \left\{ r(t) - b(t)u^p(t) - \sum_{j=1}^{n} b_j(t)u(t - \tau_j(t))^\theta_j - \sum_{j=1}^{n} e_j(t) \int_{-\sigma_j}^{0} u^\gamma(t + s) \, d\mu_j(s) \right\}, \tag{17}$$

where $r(t)$, $b_j(t)$, $e_j(t)$, $\tau_j(t)$, $j = 1, 2, \ldots, n$ are all continuous, real-valued positive periodic functions with periodic $\omega > 0$, $\tau_j(t)$, $j = 1, 2, \ldots, n$ are nonnegative continuous $\omega$-periodic functions. $\int_{-\sigma_j}^{0} d\mu_j(s) = 1$, $p$, $\theta_j$, $\gamma_j$, $j = 1, 2, \ldots, n$ are all positive constants.

We will consider solutions of equation (17) with initial condition

$$u(s) = \varphi(s) \geq 0, \quad \varphi(0) > 0, \quad \varphi \in C \left([-\tau^*, 0], R^+\right), \tag{18}$$

where $\tau^* = \max_{1 \leq j \leq n} \{\max_{0 \leq \tau \leq \omega} \tau_j(t), \sigma_j\}$. 


Theorem 2.2. System (17) with initial condition (18) has at least one positive $\omega$-periodic solution.

**Proof.** The proof of Theorem 2.2 is similar to that of Theorem 2.1, so we omit the details here.

**Remark 1.** Theorem 4.1 of [13] is the special case of Theorem 2.2 if we take $p = 1$, $\theta_j = 1$, $\gamma_j = 1$, $j = 1, 2, \ldots, n$. Theorem 2.1 of [8] and Theorem 1 of [7] are all special cases of Theorem 2.2 if we take $a(t) \equiv 0$, $e_j(t) \equiv 0$, $\theta_j \equiv 1$, $j = 1, 2, \ldots, n$.

As a direct corollary of Theorem 2.2, one has the following.

**Theorem 2.3.** The logistic type modal with continuous delays

$$\frac{du(t)}{dt} = a(t) - \sum_{j=1}^{n} e_j(t) \int_{-\sigma_j}^{0} u(t + s) d\mu_j(s),$$

where all the coefficients of system (19) satisfy the same restrictions as that of system (17). Then, system (19) has at least one positive $\omega$-periodic solution.

### 3. Global Stability of Positive Periodic Solution

The aim of this section is to derive sufficient conditions which guarantee the global attractivity of a positive periodic solution of system (17). From Theorem 2.2, we know that system (17) with initial condition (18) has at least one positive periodic solution $u^*(t)$. Let $c = \min_{0 < t < \infty} \{u^*(t)\}$ and $y(t) = u(t)/c$. Then, system (17) is transformed into

$$\frac{dy(t)}{dt} = y(t) \left\{ \tau(t) - a(t) - \sum_{j=1}^{n} \beta_j(t) y(t - \tau_j(t)) \gamma_j - \sum_{j=1}^{n} \gamma_j(t) \int_{-\sigma_j}^{0} y(t + s) d\mu_j(s) \right\}.$$  

We also consider the solutions of equation (20) with initial condition

$$u(s) = \varphi(s) \geq 0, \quad \varphi(0) > 0, \quad \varphi \in C([-\tau^*, 0], \mathbb{R}^+),$$

where $\tau^* = \max_{1 \leq j \leq n} \{\max_{0 < t < \infty} \{\tau_j(t), \sigma_j\}\}$. Obviously, $y^*(t) = u^*(t)/c$ is the periodic solution of system (20), if the periodic solution of system (20) is globally attractive, so is the periodic solution of system (5).

**Theorem 3.1.** Assume

(i) $\tau_j(t), j = 1, 2, \ldots, n$ are continuous differentiable periodic functions such that

$$c^0 b(t) > \sum_{j=1}^{n} \frac{\beta_j(t) \xi_j^{-1}(t)}{1 - \tau_j'(\xi_j^{-1}(t))} + \sum_{j=1}^{n} \gamma_j(t) \int_{-\sigma_j}^{0} e_j(t - s) d\mu_j(s),$$

where $\xi_j^{-1}(t)$ is the inverse function of $\xi_j(t) = t - \tau_j(t)$;

(ii) $p \geq \max\{\theta_j, \gamma_j\}$.

Then, system (17) has a unique positive periodic solution $u^*(t)$ which is globally attractive.

**Proof.** When $a \geq b$, $a \geq 1, y = a^2 - b^2$ is an increasing function. From condition (22) and the periodicity of coefficients of system (20), there exists a positive constant $A$ such that

$$c^0 b(t) - \sum_{j=1}^{n} \frac{\beta_j(t) \xi_j^{-1}(t)}{1 - \tau_j'(\xi_j^{-1}(t))} - \sum_{j=1}^{n} \gamma_j(t) \int_{-\sigma_j}^{0} e_j(t - s) d\mu_j(s) > A, \quad t \geq 0.$$
Let $y(t)$ be any positive solution of system (20) with initial condition (21). Now constructing $V(t)$ as

$$V(t) = \left[ \ln y(t) - \ln y^*(t) \right] + \sum_{j=1}^{n} \int_{t-	au_j(t)}^{t} \frac{c^\theta_j b_j (\xi_j^{-1}(s))}{1 - \tau_j' (\xi_j^{-1}(s))} \left[ (y(s))^{\theta_j} - (y^*(s))^{\theta_j} \right] ds$$

$$+ \sum_{j=1}^{n} c^{\gamma_j} \int_{t-s_j}^{t} e_j(l-s) \left[ (y(l))^{\gamma_j} - (y^*(l))^{\gamma_j} \right] dl \, d\mu_j(s).$$

(23)

By direct computation, we have

$$D^+ V(t) \leq -c^p b(t) \left[ (y(t))^p - (y^*(t))^p \right] + \sum_{j=1}^{n} c^\theta_j b_j (\xi_j^{-1}(t)) \left[ (y(t))^{\theta_j} - (y^*(t))^{\theta_j} \right]$$

$$+ \sum_{j=1}^{n} c^{\gamma_j} e_j(t-s) d\mu_j(s) \left[ (y(t+s))^\gamma_j - (y^*(t+s))^\gamma_j \right]$$

$$+ \sum_{j=1}^{n} c^\theta_j b_j (\xi_j^{-1}(t)) \left[ (y(t))^\theta_j - (y^*(t))^\theta_j \right]$$

$$- \sum_{j=1}^{n} c^{\gamma_j} \int_{t-s_j}^{t} e_j(l-s) d\mu_j(s) \left[ (y(l))^\gamma_j - (y^*(l))^\gamma_j \right]$$

$$+ \sum_{j=1}^{n} c^{\gamma_j} \int_{t-s_j}^{t} e_j(l-s) d\mu_j(s) \left[ (y(l+s))^\gamma_j - (y^*(l+s))^\gamma_j \right]$$

$$- \sum_{j=1}^{n} c^{\gamma_j} \int_{t-s_j}^{t} e_j(l-s) d\mu_j(s) \left[ (y(l))^\gamma_j - (y^*(l))^\gamma_j \right]$$

From $y^*(t) = u^*(t)/c$, we know $y^*(t) \geq 1$. Since when $a \geq 1$, and $x > 0$, $y = a^x - b^x$ is an increasing function, for $p \geq \max_j \{\theta_j, \gamma_j\}$, we get

$$\left| (y(t))^{\theta_j} - (y^*(t))^{\theta_j} \right| \leq \left| (y(t))^p - (y^*(t))^p \right|,$$

$$\left| (y(t))^{\gamma_j} - (y^*(t))^{\gamma_j} \right| \leq \left| (y(t))^p - (y^*(t))^p \right|, \quad j = 1, 2, \ldots, n.$$ 

(24)

Therefore,

$$D^+ V(t) \leq -c^p b(t) \left[ (y(t))^p - (y^*(t))^p \right] + \sum_{j=1}^{n} c^\theta_j b_j (\xi_j^{-1}(t)) \left[ (y(t))^{\theta_j} - (y^*(t))^{\theta_j} \right]$$

$$+ \sum_{j=1}^{n} c^{\gamma_j} \int_{t-s_j}^{t} e_j(l-s) d\mu_j(s) \left[ (y(l))^\gamma_j - (y^*(l))^\gamma_j \right]$$

$$\leq - \left( c^p b(t) - \sum_{j=1}^{n} \frac{c^\theta_j b_j (\xi_j^{-1}(t))}{1 - \tau_j' (\xi_j^{-1}(t))} - \sum_{j=1}^{n} c^{\gamma_j} \int_{t-s_j}^{t} e_j(l-s) d\mu_j(s) \right) \left| (y(t))^p - (y^*(t))^p \right|$$

$$\leq -A \left| (y(t))^p - (y^*(t))^p \right| < 0.$$ 

From above, we know that $V(t)$ is a decreasing function on $[0, +\infty)$, thus, $0 \leq V(t) \leq V(0)$, and $\lim_{t \to +\infty} V(t) = V^* \geq 0$.

Now we claim that $V^* = 0$. Since $y^*(t)$ is a positive periodic solution of system (20), then $\ln y^*(t)$ is bounded. From $\left| \ln y(t) \right| \leq \left| \ln y(t) - \ln y^*(t) \right| + \left| \ln y^*(t) \right| \leq V(t) + \left| \ln y^*(t) \right|$, it follows $\ln y(t)$ is bounded also. So, we assume that $\left| \ln y(t) \right| \leq M_0, \left| \ln y^*(t) \right| \leq M_0$. From this, it follows

$$\left| (y(t))^p - (y^*(t))^p \right| = \exp \left\{ p \ln y(t) \right\} - \exp \left\{ p \ln y^*(t) \right\} = \exp \left\{ p \ln y(t) \right\} \left| \ln y(t) - \ln y^*(t) \right|,$$
where \( \tilde{y}(t) \) lies between \( y(t) \) and \( y^*(t) \). So,

\[
m |\ln y(t) - \ln y^*(t)| \leq |(y(t))^p - (y^*(t))^p| \leq M |\ln y(t) - \ln y^*(t)|,
\]

where \( m = \exp\{-M_0 p\}, M = \exp\{M_0 p\} \), so,

\[
D^+ V(t) \leq -A |(y(t))^p - (y^*(t))^p| \leq -Am |\ln y(t) - \ln y^*(t)| \leq -Am V(t).
\]

We claim \( V^* = 0 \). Otherwise, \( V^* > 0 \), and we have \( V(t) \geq V^* > 0 \), it follows \( D^+ V(t) \leq -Av V^* \), which implies

\[
V(t) \leq V(0) - Av V^* \to -\infty \quad (t \to \infty).
\]

This contradicts with positivity of \( V(t) \), so \( V^* = 0 \). Now, from \( |(y(t))^p - (y^*(t))^p| \leq M |\ln y(t) - \ln y^*(t)| \leq MV(t) \), it follows

\[
\lim_{t \to +\infty} |(y(t))^p - (y^*(t))^p| = 0.
\]

Therefore,

\[
\lim_{t \to +\infty} |y(t) - y^*(t)| = 0.
\]

This ends the proof of the theorem.

**Remark 2.** The aim of change variable \( y(t) = u(t)/c \) is to ensure \( y^*(t) = u^*(t)/c \geq 1 \) such that inequality (24) holds. By using system (17), one could obtain

\[
u^*(t) \geq \min \left\{ \left[ \left( \frac{r}{\sum_{j=1}^{n} (b_j + e_j) + b} \right)^{1/h'} \right], \left[ \left( \frac{r}{\sum_{j=1}^{n} (b_j + e_j) + b} \right)^{1/h''} \right] \right\} := L,
\]

where \( h' = \max_j \{\theta_j, \gamma_j\}, h'' = \min_j \{\theta_j, \gamma_j\} \). So, we could choose arbitrary constant \( c \) such that \( 0 < c \leq L \), and \( c \) satisfies inequality (22), then Theorem 3.1 holds.

**Remark 3.** (Condition (ii)). \( p \geq \max_j \{\theta_j, \gamma_j\} \) is to ensure that inequality (24) holds. And so, it is interesting to consider the case \( p = \theta_j = \gamma_j \). In this case, we need not make the change of variable \( y(t) = u(t)/c \), as a direct corollary of Theorem 3.1, one has the following.

**Theorem 3.2.** Assume

(i) \( \tau_j(t), j = 1, 2, \ldots, n \) are continuous differentiable periodic functions such that

\[
b(t) > \sum_{j=1}^{n} \frac{b_j}{1 - \tau_j^{-1}(t)} + \sum_{j=1}^{n} \int_{-\sigma_j}^{0} e_j(t-s) d\mu_j(s),
\]

where \( \xi_j^{-1}(t) \) is the inverse function of \( \xi_j(t) = t - \tau_j(t) \); (ii) \( p = \theta_j = \gamma_j \).

Then, system (17) has a unique positive periodic solution \( u^*(t) \) which is globally attractive.

**References**