Computers and Mathematics with Applications 64 (2012) 1612–1615

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

A stronger version of the second mean value theorem for integrals

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ARTICLE INFO

Article history:
Received 8 September 2011
Received in revised form 30 December 2011
Accepted 3 January 2012

Keywords:
Second mean value theorem for integral

ABSTRACT

We prove a stronger version of the classic second mean value theorem for integrals.

1. Introduction

The purpose of this paper is to prove a stronger version of the classic second mean value theorem for integrals. Direct impulse for preparing this note appeared in the course of discussion on the following theorem, our attention to which was called by professor Z. Kominek, the referee of our book [1].

Theorem 1. *Let $f : [a, b] \to [0, \infty)$ be a monotonic function, whereas $g : [a, b] \to \mathbb{R}$ be a Lebesgue integrable function. If the function f is non-decreasing, then there exists $\xi \in [a, b]$ such that*

$$\int_a^b f(x)\, g(x)\, dx = f(b-) \int_\xi^b g(x)\, dx, \tag{1}$$

where $f(b-) := \lim_{x\to b^-} f(x)$.

If the function f is non-increasing, then there exists $\eta \in [a, b]$ such that

$$\int_a^b f(x)\, g(x)\, dx = f(a+) \int_a^\eta g(x)\, dx, \tag{2}$$

where $f(a+) := \lim_{x\to a^+} f(x)$.

The above theorem can be found in the handbook [2] (see Lemma 5 on page 493) where it is used for proving the Jordan test for convergence of the trigonometric Fourier series (see also [3] where only the classic version of the second mean value theorem for integral is applied to this aim). Directly from this theorem some other classic theorems can be deduced (consider Theorem 2), discovered by P.O. Bonnet for the Riemann integrable functions (more precisely, for the continuous functions), called the mean value theorem of the second kind for integrals.

2. The second mean value theorem for integrals

We begin with presenting a version of this theorem for the Lebesgue integrable functions. Let us note that many authors give this theorem only for the case of the Riemann integrable functions (see for example [4,5]). However the proofs in both

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0898-1221/$ – see front matter
doi:10.1016/j.camwa.2012.01.008

cases proceed in the same way. Below, for the order, we present the proof of this theorem when f is a non-decreasing function (proof for a non-increasing function proceeds similarly).

Theorem 2. *Let $f : [a, b] \to [0, \infty)$ be a monotonic function and $g : [a, b] \to \mathbb{R}$ be a Lebesgue integrable function. Then there exists $\xi \in [a, b]$ such that*

$$\int_a^b f(x)\,g(x)\,dx = f(a+)\int_a^{\xi} g(x)\,dx + f(b-)\int_{\xi}^b g(x)\,dx.$$

Proof. Let f be a non-decreasing function. We take $h(x) := f(x) - f(a+)$. Certainly, h is non-decreasing and according to formula (1) there exists $\xi \in [a, b]$ such that

$$\int_a^b \big(f(x) - f(a+)\big)g(x)\,dx = \big(f(b-) - f(a+)\big)\int_{\xi}^b g(x)\,dx,$$

from where, after applying some simple algebra, we get

$$\begin{aligned}\int_a^b f(x)\,g(x)\,dx &= f(a+)\left(\int_a^b g(x)\,dx - \int_{\xi}^b g(x)\,dx\right) + f(b-)\int_{\xi}^b g(x)\,dx \\ &= f(a+)\int_a^{\xi} g(x)\,dx + f(b-)\int_{\xi}^b g(x)\,dx\end{aligned}$$

which gives the expected equality. □

One can indicate, without any difficulties, the examples of functions f and g satisfying the thesis of Theorem 2, but belonging to the definitely wider class of functions than in the assumptions of this theorem.

Example 1. Let $\xi \in (0, \pi)$, $a, b \in \mathbb{R}$, $g(x) = \sin x$ and

$$f(x) = \begin{cases} b, & x \in [-\xi, \xi], \\ a, & x \in [-\pi, -\xi) \cup (\xi, \pi]. \end{cases}$$

Then we have

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\,g(x)\,dx &= f(-\pi)\int_{-\pi}^{\xi} g(x)\,dx + f(\pi)\int_{\xi}^{\pi} g(x)\,dx \\ &= f(-\pi)\int_{-\pi}^{-\xi} g(x)\,dx + f(\pi)\int_{-\xi}^{\pi} g(x)\,dx.\end{aligned}$$

The above example implies additionally that the value of ξ, from the thesis of Theorem 2, does not have to be uniquely determined (if we take $a = b$, then ξ can be any number from the interval $(0, \pi)$). Moreover, any odd function can be taken here as the respective function $g : (-\pi, \pi) \to \mathbb{R}$.

Also the next example is curiously connected with Theorem 2.

Example 2. Let $a, b, c, d \in \mathbb{R}$. We set

$$f(x) = \begin{cases} a, & x \in \left[-\pi, -\frac{\pi}{2}\right), \\ c, & x \in \left[-\frac{\pi}{2}, 0\right), \\ d, & x \in \left[0, \frac{\pi}{2}\right), \\ b, & x \in \left[\frac{\pi}{2}, \pi\right]. \end{cases}$$

Thus we have

$$\int_{-\pi}^{\pi} f(x)\sin x\,dx = -a - c + d + b.$$

Hence, there exists $\xi \in [-\pi, \pi]$ such that

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\sin x\,dx &= f(-\pi)\int_{-\pi}^{\xi}\sin x\,dx + f(\pi)\int_{\xi}^{\pi}\sin x\,dx \\ &= -a + b + (b - a)\cos\xi,\end{aligned}$$

if and only if $-|b - a| \le d - c \le |b - a|$ (see also Example 3).

3. Main result

The above examples highlighting Theorem 2 suggest a possibility of some generalization which follows and is the main result of this paper.

Theorem 3. *Let* $[a,b]\subset\mathbb{R}$, $f:[a,b]\to[0,\infty)$ *be a monotonic function and* $g:[a,b]\to\mathbb{R}$ *be a Lebesgue integrable function. Let* $f(a+):=\lim_{x\to a^+}f(x)$ *and* $f(b-):=\lim_{x\to b^-}f(x)$. *We assume that* $f(a+)\neq f(b-)$.

(i) *(standard version) If for every* $\eta\in(a,b)$ *we have*

$$0\neq\int_a^\eta g(x)\,dx\neq\int_a^b g(x)\,dx, \tag{3}$$

then for any

$$A\in\big(-\infty,\min\{f(a+),f(b-)\}\big]\quad\text{and}\quad B\in\big[\max\{f(a+),f(b-)\},\infty\big)$$

there exists $\xi=\xi(A,B)\in[a,b]$ *such that*

$$f(a+)<f(b-)\Rightarrow\int_a^b f(x)\,g(x)\,dx=A\int_a^\xi g(x)\,dx+B\int_\xi^b g(x)\,dx; \tag{4}$$

$$f(a+)>f(b-)\Rightarrow\int_a^b f(x)\,g(x)\,dx=B\int_a^\xi g(x)\,dx+A\int_\xi^b g(x)\,dx. \tag{5}$$

(ii) *(generalization) Let* $A_0,B_0\in\mathbb{R}$, $\xi_0\in(a,b]$ *and*

$$\int_a^b f(x)\,g(x)\,dx=A_0\int_a^{\xi_0} g(x)\,dx+B_0\int_{\xi_0}^b g(x)\,dx.$$

If $A_0<B_0$ *and* $\int_a^{\xi_0}g(x)\,dx\neq0$, *then for any* $A<A_0$ *there exists* $\xi=\xi(A)\in(a,\xi_0)$ *such that*

$$\int_a^b f(x)\,g(x)\,dx=A\int_a^\xi g(x)\,dx+B_0\int_\xi^b g(x)\,dx. \tag{6}$$

If $A_0>B_0$ *and* $\int_{\xi_0}^b g(x)\,dx\neq0$, *then for any* $B<B_0$ *there exists* $\eta=\eta(B)\in(\xi_0,b)$ *such that*

$$\int_a^b f(x)\,g(x)\,dx=A_0\int_a^\eta g(x)\,dx+B\int_\eta^b g(x)\,dx. \tag{7}$$

Proof. First we prove (ii). Let us assume that $A<A_0<B_0$ and $\int_a^{\xi_0}g(x)\,dx\neq0$. We consider the function

$$F(t):=A\int_a^t g(x)\,dx+B_0\int_t^b g(x)\,dx-A_0\int_a^{\xi_0} g(x)\,dx-B_0\int_{\xi_0}^b g(x)\,dx,$$

where $t\in[a,\xi_0]$. We have

$$F(t)=(A-A_0)\int_a^{\xi_0} g(x)\,dx+(A-B_0)\int_{\xi_0}^t g(x)\,dx.$$

Certainly, F is the continuous function (see Theorem 25 and proof of this theorem at the end of paragraph 46 in handbook [2]) and

$$F(a)\,F(\xi_0)=(A-A_0)\,(B_0-A_0)\left(\int_a^{\xi_0} g(x)\,dx\right)^2<0.$$

In view of the Darboux property it means that there exists $\xi=\xi(A)\in(a,\xi_0)$ such that $F(\xi)=0$ which easily implies equality (6).

Now let $A_0>B_0>B$ and $\int_{\xi_0}^b g(x)\,dx\neq0$. We define the following auxiliary function

$$\begin{aligned}G(t)&:=A_0\int_a^t g(x)\,dx+B\int_t^b g(x)\,dx-A_0\int_a^{\xi_0} g(x)\,dx-B_0\int_{\xi_0}^b g(x)\,dx\\&=(A_0-B)\int_{\xi_0}^t g(x)\,dx+(B-B_0)\int_{\xi_0}^b g(x)\,dx,\end{aligned}$$

for $t \in [\xi_0, b]$. Obviously, G is the continuous function and

$$G(\xi_0)\,G(b) = (A_0 - B_0)\,(B - B_0)\left(\int_{\xi_0}^{b} g(x)\,dx\right)^2 < 0.$$

Hence, by the Darboux property, there exists $\eta = \eta(B) \in (\xi_0, b)$ such that $G(\eta) = 0$ which implies equality (7).

Now we prove (i). From assumption (3) it results that

$$\int_a^{\eta} g(x)\,dx \neq 0 \quad \text{and} \quad \int_{\eta}^{b} g(x)\,dx \neq 0,$$

for any $\eta \in (a, b)$. Hence, we easily obtain (4), as well as (5), from (ii). □

Next example indicates that the above theorem assumption, concerning the integral $\int_a^{\xi} g(x)\,dx$ which should be not equal to zero, is essential not only in regard to the proof of this theorem but also seems to be natural in applications.

Example 3. Let $A \in \mathbb{R}$, $a, b \in \mathbb{R}$, $a \neq b$. If $A \in \big[\min\{0, 2(b-a)\}, \max\{0, 2(b-a)\}\big]$, then there exists $\xi \in [-\pi, \pi]$ such that

$$A = a\int_{-\pi}^{\xi} \sin x\,dx + b\int_{\xi}^{\pi} \sin x\,dx. \tag{8}$$

Let us notice that the choice of number a (number b, respectively) is here completely arbitrary.

We will show that equality (8) is true. For this purpose we consider the function

$$F(t) := a\int_{-\pi}^{t} \sin x\,dx + b\int_{t}^{\pi} \sin x\,dx, \quad t \in [-\pi, \pi].$$

We have

$$F(0) = 2(b-a), \qquad F(-\pi) = F(\pi) = 0, \qquad F\left(-\frac{\pi}{2}\right) = F\left(\frac{\pi}{2}\right) = b - a.$$

Since F is the continuous function, therefore the Darboux property implies that there exists $\xi \in [0, \pi]$ for which equality (8) holds true.

The above thesis can be strengthened in the following way: if $\pm A \in \big[\min\{b-a, 2(b-a)\}, \max\{b-a, 2(b-a)\}\big]$, then there exists such $\xi \in [0, \frac{\pi}{2}]$ that equality (8) holds true, whereas, if $\pm A \in \big[\min\{0, b-a\}, \max\{0, b-a\}\big]$, then there exists such $\xi \in [\frac{\pi}{2}, \pi]$ that equality (8) holds true.

Acknowledgments

The authors wish to express their gratitude to the Reviewers for several helpful comments concerning first version of the paper.

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