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On some properties of doughnut graphs

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Abstract

The class of doughnut graphs is a subclass of 5-connected planar graphs. It is known that a doughnut graph admits a straight-line grid drawing with linear area, the outerplanarity of a doughnut graph is 3, and a doughnut graph is k-partitionable. In this paper we show that a doughnut graph exhibits a recursive structure. We also give an efficient algorithm for finding a shortest path between any pair of vertices in a doughnut graph. We also propose a nice application of a doughnut graph based on its properties.

Keywords: Planar graphs; Recursive structure; Hamiltonian; Fault tolerance; Connectivity

1. Introduction

A five-connected planar graph G is called a doughnut graph if G has an embedding Γ such that (a) Γ has two vertex-disjoint faces each of which has exactly p vertices, $p > 3$, and all the other faces of Γ has exactly three vertices; and (b) G has the minimum number of vertices satisfying condition (a). Fig. 1(a) illustrates a doughnut graph where F_1 and F_2 are two vertex disjoint faces. Faces F_1 and F_2 are depicted by thick lines. The name of doughnut graph was chosen in [1] for such a graph since the graph has a doughnut like embedding, as illustrated in Fig. 1(b). The class of doughnut graphs is an interesting class of graphs which was recently introduced in graph drawing literature for it's beautiful area-efficient drawing properties [1–3]. A doughnut graph admits a straight-line grid drawing with linear area [1,3]. Any spanning subgraph of a doughnut graph also admits straight-line grid drawing with linear area [2,3]. The outerplanarity of this class is 3 [3].

Given a graph $G = (V, E)$, k natural numbers $n_1, n_2, \ldots, n_k$ such that $\sum_{i=1}^{k} n_i = |V|$, we wish to find a k-partition $V_1, V_2, \ldots, V_k$ of the vertex set V such that $|V_i| = n_i$ and V_i induces a connected subgraph of G for each i, $1 \le i \le k$. The problem of finding a k-partition of a given graph often appears in the load distribution among different power plants and the fault-tolerant routing of communication networks [4,5]. A doughnut graph is k-partitionable [6].

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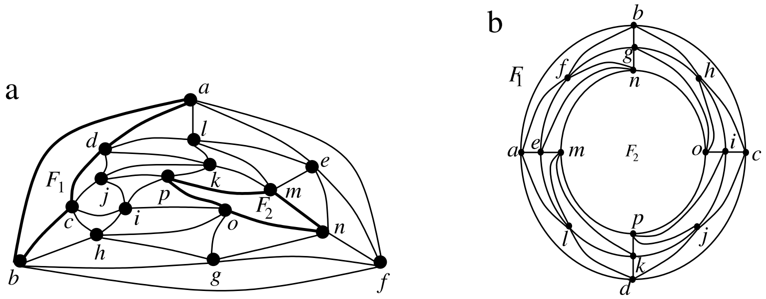

Fig. 1. (a) A doughnut graph G, and (b) doughnut embedding of G.

A class of graph has recursive structure if every instance of it can be created by connecting the smaller instances of the same class of graphs. In this paper, we show that any instance of a doughnut graph can be constructed by connecting smaller instances of doughnut graphs. We show that one can find a shortest path between any pair of vertices u and v of a doughnut graph G in $O(l_s)$ time where l_s is the length of shortest path between u and v by exploiting its beautiful structure. We study the other topological properties like degree, diameter, connectivity and fault tolerance. We show that it's diameter is $\lfloor p/2 \rfloor + 2$. It has maximal fault tolerance, and has ring embedding since it is Hamilton-connected. One may explore the suitability of a doughnut graph as an interconnection network since some of its properties are similar to that of the graph classes usually used for interconnection networks.

The remainder of the paper is organized as follows. In Section 2, we give some definitions and preliminary results. Section 3 provides recursive structure of a doughnut graph. Finding a shortest path between any pair of vertices of doughnut graphs is presented in Section 4. Section 5 summarizes the topological properties of doughnut graphs. Finally Section 6 concludes the paper. An early version of this paper is presented at [7].

2. Preliminaries

In this section we give some definitions.

Let $G = (V, E)$ be a connected simple graph with the vertex set V and the edge set E. Throughout the paper, we denote by n the number of vertices in G, that is, $n = |V|$, and denote by m the number of edges in G, that is, $m = |E|$. An *edge* joining the vertices u and v is denoted by (u, v). The *degree* of a vertex v, denoted by $d(v)$, is the number of edges incident to v in G. We denote by $\Delta(G)$ the maximum of the degrees of all vertices in G. G is called *r-regular* if every vertex of G has degree r. We call a vertex v a *neighbor* of a vertex u in G if G has an edge (u, v). The *connectivity* $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected graph or a single-vertex graph K_1. G is called *k-connected* if $\kappa(G) \geq k$. A *path* in G is an ordered list of distinct vertices $v_1, v_2, \ldots, v_q \in V$ such that $(v_{i-1}, v_i) \in E$ for all $2 \leq i \leq q$. The vertices v_1 and v_q are the end-vertices of the path $v_1, v_2, \ldots, v_q$. The *length* of a path is one less than the number of vertices on the path. A path is called a *u,v-path* if its two end-vertices are u and v, respectively. The shortest path between two vertices u and v of G is a u, v-path of G with the least length. The *distance* from u to v, denoted by $d(u, v)$, is the length of a shortest u, v-path. The *diameter* of G is $\max_{u,v \in V(G)} d(u, v)$.

A graph is *planar* if it can be embedded in the plane so that no two edges intersect geometrically except at a vertex to which they are both incident. A *plane graph* is a planar graph with a fixed embedding. A plane graph G divides the plane into connected regions called *faces*. Each of the bounded regions is called an *inner face* and the unbounded region is called the *outer face*. Let $v_1, v_2, \ldots, v_l$ be all the vertices in the clockwise order on the contour of a face f in G. We often denote f by $f(v_1, v_2, \ldots, v_l)$. For a face f in G we denote by $V(f)$ the set of vertices of G on the boundary of face f. We call two faces F_1 and F_2 *vertex-disjoint* if $V(F_1) \bigcap V(F_2) = \emptyset$.

A *maximal planar* graph is one to which no edge can be added without losing planarity. Thus in any embedding of a maximal planar graph G with $n \geq 3$, each faces of G is triangulated, and hence an embedding of a maximal planar graph is a triangulated plane graph. It can be derived from the Euler's formula for planar graphs that if G is a maximal planar graph with n vertices and m edges then $m = 3n - 6$, for more details see [8].

Let G be a 5-connected planar graph, let Γ be any planar embedding of G and let p be an integer such that $p > 3$. We call G a *p-doughnut* graph if the following Conditions (d_1) and (d_2) hold: (d_1) Γ has two vertex-disjoint faces

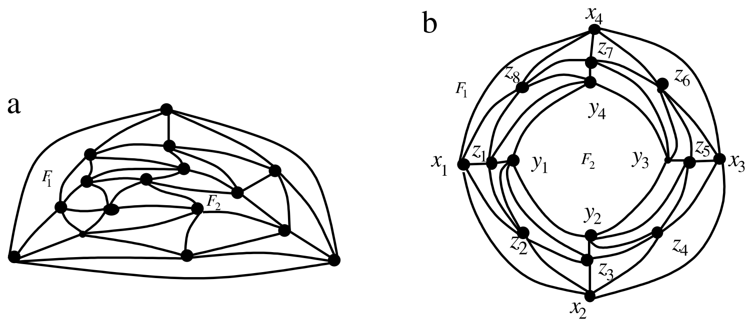

Fig. 2. (a) A p-doughnut graph G where $p = 4$, and (b) doughnut embedding of G.

each of which has exactly p vertices, and all the other faces of Γ have exactly three vertices; and (d_2) G has the minimum number of vertices satisfying Condition (d_1). In general, we call a p-doughnut graph for $p > 3$ a *doughnut graph*. The following result is known for doughnut graphs [1].

Lemma 1. *Let G be a p-doughnut graph. Then G is 5-regular and has exactly $4p$ vertices. Furthermore, G has three vertex-disjoint cycles C_1, C_2 and C_3 with p, $2p$ and p vertices, respectively, such that $V(C_1) \cup V(C_2) \cup V(C_3) = V(G)$.*

For a cycle C in a plane graph G, we denote by $G(C)$ the plane subgraph of G inside C excluding C. Let C_1, C_2 and C_3 be the three vertex-disjoint cycles of a p-doughnut graph G with p, $2p$ and p vertices, respectively, such that $V(C_1) \cup V(C_2) \cup V(C_3) = V(G)$. Then we call a planar embedding Γ of G a *doughnut embedding* of G if C_1 is the outer face and C_3 is an inner face of Γ, $G(C_1)$ contains C_2 and $G(C_2)$ contains C_3. We call C_1 the *outer cycle*, C_2 the *middle cycle* and C_3 the *inner cycle* of Γ. Fig. 2(b) illustrates the doughnut embedding of the doughnut graph in Fig. 2(a).

The following results on doughnut embeddings are known for doughnut graphs [1].

Lemma 2. *A p-doughnut graph always has a doughnut embedding.*

Lemma 3. *Let Γ be a doughnut embedding of a p-doughnut graph G and let C_1, C_2 and C_3 be the outer cycle, the middle cycle and the inner cycle of Γ, respectively. Then either condition* (a) *or condition* (b) *holds for any vertex u of C_2.*

(a) *The vertex u has exactly two consecutive neighbors on C_1 and exactly one neighbor on C_3.*
(b) *The vertex u has exactly two consecutive neighbors on C_3 and exactly one neighbor on C_1.*

Furthermore, for any two consecutive vertices u and v on C_2, if u holds condition (a) *then v holds condition* (b) *or vice-versa.*

Before going further we need some definitions. Let Γ be a doughnut embedding of G and let C_1, C_2 and C_3 be the outer cycle, middle cycle and the inner cycle of Γ, respectively. Let z_i be a vertex on C_2. Without loss of generality, by Lemma 3 we assume that z_i has exactly two consecutive neighbors on C_1. Let x and x' be the two neighbors of z_i on C_1 such that x' is the counter clockwise next vertex to x on C_1. We call x the *left neighbor* of z_i on C_1 and x' the *right neighbor* of z_i on C_1. Similarly we define the left neighbor and the right neighbor of z_i on C_3 if a vertex z_i on C_2 has two neighbors on C_3. Let $z_1, z_2, \ldots, z_{2p}$ be the vertices of C_2 in counter clockwise order such that z_1 has exactly one neighbor on C_1. Let x_1 be the neighbor of z_1 on C_1, and let $x_1, x_2, \ldots, x_p$ be the vertices of C_1 in the counter clockwise order. Let $y_1, y_2, \ldots, y_p$ be the vertices on C_3 in counter clockwise order such that y_1 and y_p are the right neighbor and the left neighbor of z_1, respectively. Fig. 2(b) illustrates the labeling of vertices of a doughnut embedding Γ of G in Fig. 2(a) as mentioned above. In the rest of the paper, we consider a doughnut embedding Γ of a doughnut graph G such that the vertices of cycles C_1, C_2 and C_3 are labeled as mentioned above. We now have the following lemmas from [1].

Lemma 4. *Let G be a p-doughnut graph and let Γ be a doughnut embedding of G. Let z_i be a vertex of C_2. Then the following conditions hold.*

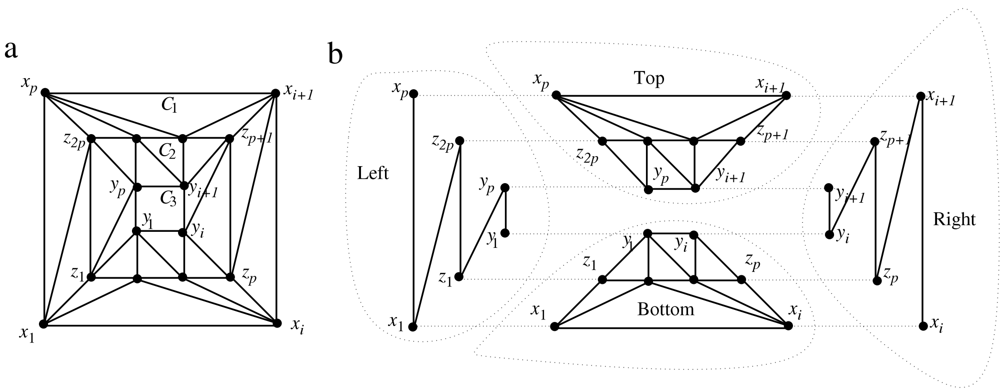

Fig. 3. (a) A doughnut embedding of a p-doughnut graph G where $p = 4$, and (b) illustration for four partition of edges of G.

(a) *The vertex z_i has exactly two neighbors on C_1 and exactly one neighbor on C_3 if i is even. The neighbors of z_i on C_1 are x_p and x_1 in a counter clockwise order if $i = 2p$, otherwise the neighbors of z_i on C_1 are $x_{i/2}$ and $x_{i/2+1}$ in a counter clockwise order. The neighbor of z_i on C_3 is $y_{i/2}$.*

(b) *The vertex z_i has exactly two neighbors on C_3 and exactly one neighbor on C_1 if i is odd. The neighbors of z_i on C_3 are y_1 and y_p in a counter clockwise order if $i = 1$, otherwise the neighbors of z_i on C_3 are $y_{\lceil i/2\rceil-1}$ and $y_{\lceil i/2\rceil}$ in a counter clockwise order. The neighbor of z_i on C_1 is $x_{\lceil i/2\rceil}$.*

Lemma 5. *Let G be a p-doughnut graph and let Γ be a doughnut embedding of G. Let x_i be a vertex of C_1. Then x_i has exactly three neighbors z_{2p}, z_1, z_2 on C_2 in a counter clockwise order if $i = 1$, otherwise x_i has exactly three neighbors $z_{2i-2}, z_{2i-1}, z_{2i}$ on C_2 in a counter clockwise order.*

Lemma 6. *Let G be a p-doughnut graph and let Γ be a doughnut embedding of G. Let y_i be a vertex of C_3. Then y_i has exactly three neighbors z_{2p-1}, z_{2p}, z_1 in a counter clockwise order if $i = p$, otherwise y_i has exactly three neighbors $z_{2i-1}, z_{2i}, z_{2i+1}$ on C_2 in a counter clockwise order.*

3. Recursive structure of doughnut graphs

A class of graphs has a recursive structure if every instance of it can be created by connecting the smaller instances of the same class of graphs. We now show that the doughnut graphs have a recursive structure. We now need some definitions. Let D be a straight-line grid drawing of a p-doughnut graph G with linear area as illustrated in Fig. 3(a). We partition the edges of D as follows. The *left partition* consists of the edges—(i) (x_1, x_p), (ii) (z_1, z_{2p}), (iii) (y_1, y_p), (iv)(x_1, z_{2p}) and (v) (z_1, y_p); and the *right partition* consists of the edges—(i) (z_p, z_{p+1}), (ii) the edge between the two neighbors of z_p on C_1 if z_p has two neighbors on C_1 otherwise the edge between the two neighbors of z_{p+1} on C_1, (iii) the edge between the two neighbors of z_p on C_3 if z_p has two neighbors on C_3 otherwise the edge between the two neighbors of z_{p+1} on C_3, (iv) the edge between z_p and its right neighbor on C_1 if z_p has two neighbors on C_1 otherwise the edge between z_{p+1} and its left neighbor on C_1, and (v) the edge between z_p and its right neighbor on C_3 if z_p has two neighbors on C_3 otherwise the edge between z_{p+1} and its left neighbor on C_3. The graph G is divided into two connected components if we delete the edges of the left and the right partitions from G. We call the connected component that contains vertex x_p the *top partition* of edges and we call the connected component that contains vertex x_1 the *bottom partition* of edges. Fig. 3(b) illustrates four partitions of edges (indicated by dotted lines) of a p-doughnut graph G in Fig. 3(a) where $p = 4$.

We now construct a $(p_1 + p_2)$-doughnut graph G from a p_1-doughnut graph G_1 and a p_2-doughnut graph G_2. We first construct two graphs G'_1 and G'_2 from G_1 and G_2, respectively, as follows. We partition the edges of G_1 into left, right, top and bottom partitions. Then we identify the vertex x_{i+1} of the top partition to the vertex y_i of the right partition, the vertex z_{p_1+1} of the top partition to the vertex z_{p_1} of the right partition, and the vertex y_{i+1} of the top partition to the vertex x_i of the right partition. Thus we construct G'_1 from G_1. Fig. 4(c) illustrates G'_1 which is constructed from G_1 in Fig. 4(a) where $p_1 = 4$. In case of construction of G'_2, after partitioning (left, right, top, bottom) the edges of G_2 we identify the vertex y'_{p_2} of left partition to the vertex x'_1 of the bottom partition, vertex

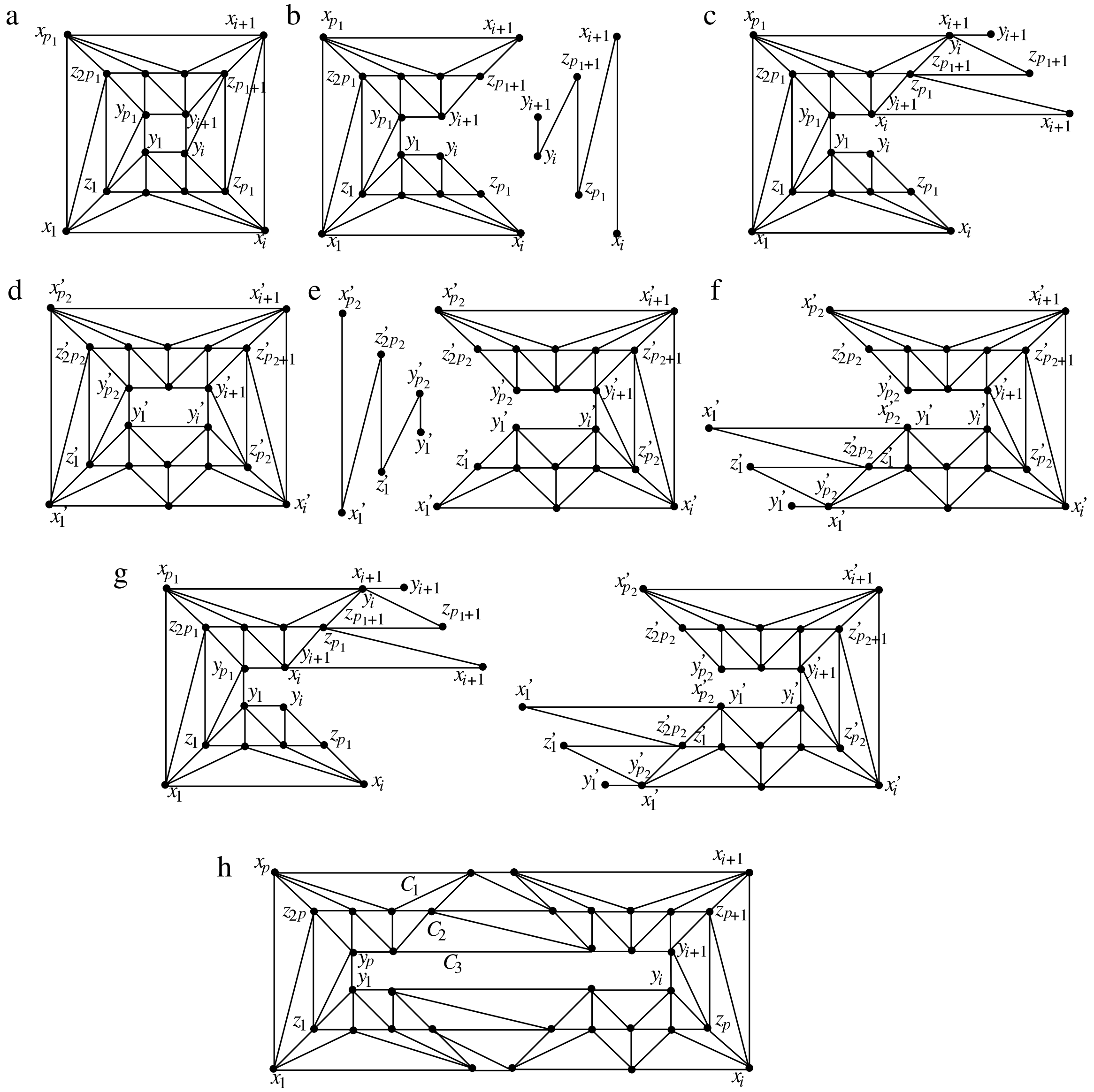

Fig. 4. Illustration for construction of a $(p_1 + p_2)$-doughnut graph G from a p_1-doughnut graph G_1 and a p_2-doughnut graph G_2 where $p_1 = 4$ and $p_2 = 5$.

z'_{2p_2} of the left partition to the vertex z'_1 of the bottom partition, and the vertex x'_{p_2} of left partition to the vertex y'_1. Fig. 4(f) illustrates G'_2 which is constructed from G_2 in Fig. 4(d) where $p_2 = 5$. We finally construct a $(p_1 + p_2)$-doughnut graph G as follows. We identify the vertices $y_{i+1}, z_{p_1+1}, x_{i+1}$ of G'_1 to the vertices of $x'_{p_2}, z'_{2p_2}, y'_{p_2}$ of G'_2, respectively; and identify the vertices of y_i, z_{p_1}, x_i of G'_1 to the vertices of x'_1, z'_1, y'_1 of G'_2, respectively. Clearly the resulting graph G is a $(p_1 + p_2)$-doughnut graph as illustrated in Fig. 4(h).

We thus have the following theorem.

Theorem 1. *Let G_1 be a p_1-doughnut graph and let G_2 be a p_2-doughnut graph. Then one can construct $(p_1 + p_2)$-doughnut graph G from G_1 and G_2.*

4. Finding a shortest path

In this section, we present a simple efficient algorithm to find a shortest path between any pair of vertices. We have the following theorem.

Theorem 2. *Let G be a p-doughnut graph and let Γ be a doughnut embedding of G. Let C_1, C_2 and C_3 be the three vertex disjoint cycles of Γ such that C_1 is the outer cycle, C_2 is the middle cycle and C_3 is the inner cycle. Then the shortest path between any pair of vertices u and v of G can be found in $O(l_s)$ time where l_s is the length of the shortest path between u and v.*

To prove the theorem, we need the following lemmas.

Lemma 7. *Let G be a p-doughnut graph and let Γ be a doughnut embedding of G. Let C_1, C_2 and C_3 be the three vertex disjoint cycles of Γ such that C_1 is the outer cycle, C_2 is the middle cycle and C_3 is the inner cycle. Then the shortest path between any two vertices on C_1 (C_3) contains only the vertices of C_1 (C_3), respectively.*

Proof. We only prove for the case where both of the vertices are on C_1 since the proof is similar if both of the vertices are on C_3. Let x_i and x_j be two vertices of C_1. For contradiction, we assume that P is a shortest path between x_i and x_j which contains vertices other than the vertices of cycle C_1. Then (i) G would have a non-triangulated face other than F_1 and F_2 or (ii) a vertex of C_2 would have degree more than five or (iii) the graph G would be non-planar, a contradiction to the properties of a doughnut graph. Therefore the shortest path between any two vertices of C_1 contains only the vertices of C_1. ■

Lemma 8. *Let G be a p-doughnut graph and let Γ be a doughnut embedding of G. Let C_1, C_2 and C_3 be the outer, the middle and the inner cycle of Γ, respectively. Let z_i and z_j be two non-adjacent vertices on C_2 and the length of the shorter (between clockwise and counter clockwise) path between them along C_2 is l. Then the length of any path between z_i and z_j is at least $\lceil l/2\rceil + 1$.*

Proof. Without loss of generality we assume that $i < j$ and the shortest path between z_i and z_j along C_2 is in the counter clockwise direction. We prove the claim by induction on l. Since z_i and z_j are non-adjacent, then $l \geq 2$. The claim is true for $l = 2$ where $j = i + 2$, and the shortest path between these two vertices has length $\lceil 2/2\rceil + 1 = 2$.

Assume that $l > 2$ and the claim is true for all pairs of vertices of C_2 with the shorter distance $l' < l$ between them along C_2. In this case $j = i + l$. Let P be any path between z_i and z_j. We now show that the length of P is at least $\lceil l/2\rceil + 1$.

We first consider the case where P contains some vertex z_k of cycle C_2 such that $i < k < j$. If z_k is adjacent to z_i, then by induction hypothesis, the length of any path between z_k and z_j has length $\lceil (l-1)/2\rceil + 1$ and therefore the length of P is at least $1 + \lceil (l-1)/2\rceil + 1 \geq \lceil l/2\rceil + 1$. From the same line of reasoning, we can show that if z_k is adjacent to z_j, then the length of P is at least $\lceil l/2\rceil + 1$. Thus we assume that z_k is adjacent to neither z_i nor z_j. Then from induction hypothesis, the length of any path between z_i and z_k is at least $\lceil (k-i)/2\rceil + 1$ and the length of any path between z_k and z_j is at least $\lceil (j-k)/2\rceil + 1$. Therefore the length of P is at least $\lceil l/2\rceil + 1$. Hence, no path containing vertices of the cycle C_2 other than z_i and z_j has length less than $\lceil l/2\rceil + 1$.

Thus we assume that P does not contain any vertices of C_2 other than z_i and z_j. Therefore there are only two different paths to consider for each pair of vertices z_i and z_j, one containing only vertices of C_1 and the other containing only vertices of C_3 other than z_i and z_j. If P contains only the vertices of C_1 other than z_i and z_j, then by Lemma 4, the rightmost (or only) neighbor of z_i and the leftmost (or only) neighbor of z_j on C_1 are $x_{\lfloor i/2\rfloor+1}$ and $x_{\lceil j/2\rceil}$, respectively. Therefore the length of P is at least $1 + \lceil j/2\rceil - \lfloor i/2\rfloor - 1 + 1 \geq \lceil l/2\rceil + 1$ as illustrated in Fig. 5(a) and (b). On the other hand, if P contains only the vertices of C_3 other than z_i and z_j, then by Lemma 4, the rightmost (or only) neighbor of z_i and the leftmost (or only) neighbor of z_j on C_3 are $y_{\lfloor i/2\rfloor}$ and $y_{\lceil j/2\rceil}$, respectively. Therefore the length of P is at least $1 + \lceil j/2\rceil - \lfloor i/2\rfloor + 1 \geq \lceil l/2\rceil + 1$ as illustrated in Fig. 5(c) and (d). ■

We are now ready to prove Theorem 2.

Proof. The vertices of G lie on three vertex disjoint cycles C_1, C_2 and C_3 where C_1 is the outer cycle, C_2 is the middle cycle and C_3 is the inner cycle. We have four cases to consider.

Case 1: Both u and v are either on C_1 or on C_3.

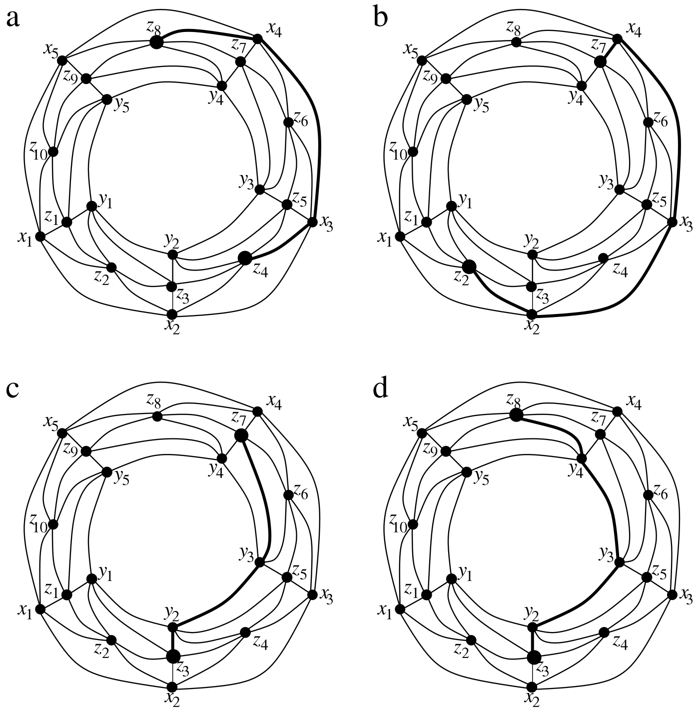

Fig. 5. Illustration of shortest path between two vertices on C_2 of a doughnut graph.

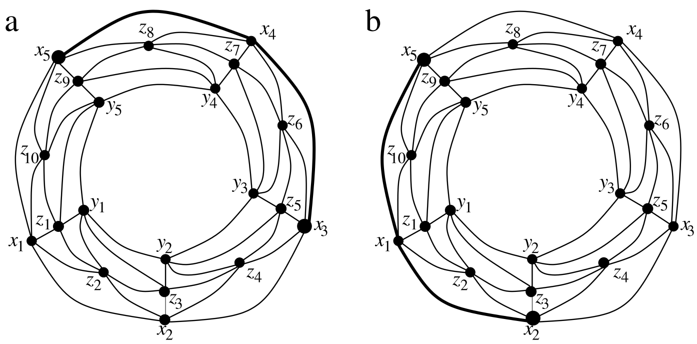

Fig. 6. Illustration for Case 1.

Without loss of generality, we assume that both the u and v are on C_1, since the case is similar where both of u and v are on C_3. Let $x_i = u$ and $x_j = v$. Without loss of generality, we may assume that $i < j$. Let us take the path $P_1 = x_i, x_{i+1}, \ldots, x_j$ if $(j - i) < \lceil p/2 \rceil$ otherwise $P_1 = x_i, x_{i-1}, \ldots, x_j$. By Lemma 7, P_1 is the shortest path between x_i and x_j. Fig. 6 illustrates the case where (i) $u = x_3$ and $v = x_5$, and (ii) $u = x_2$ and $v = x_5$.

Case 2: Both u and v are on C_2.

We assume that $z_i = u$ and $z_j = v$, respectively. The shortest path between z_i and z_j consists of edge (z_i, z_j) if z_i and z_j are adjacent. We thus assume that z_i and z_j are not adjacent. Without loss of generality, we also assume that $i < j$. We now define a path between z_i and z_j. We have the following four types of paths to consider — (i) we take path $P_2 = z_i, x_{i/2+1}, \ldots, x_{j/2}, z_j$ if both i and j are even; (ii) we take path $P_2 = z_i, y_{\lceil i/2 \rceil}, \ldots, y_{\lceil j/2 \rceil - 1}, z_j$ if both i and j are odd; (iii) we take path $P_2 = z_i, x_{i/2+1}, \ldots, x_{\lceil j/2 \rceil}, z_j$ if i is even and j is odd; (iv) we take path $P_2 = z_i, x_{\lceil i/2 \rceil}, \ldots, x_{j/2}, z_j$ if i is odd and j is even. The paths of Types (i), (iii) and (iv) contain vertices of C_1 and C_2. By Lemma 4, $x_{i/2+1}$ and $x_{\lceil i/2 \rceil}$ are neighbors of z_i and by Lemma 5, z_j is a neighbor of $x_{j/2}$ and $x_{\lceil j/2 \rceil}$. The path of Type (ii) contains vertices of C_2 and C_3. By Lemma 4, $y_{\lceil i/2 \rceil}$ is a neighbor of z_i and by Lemma 6, z_j is a neighbor

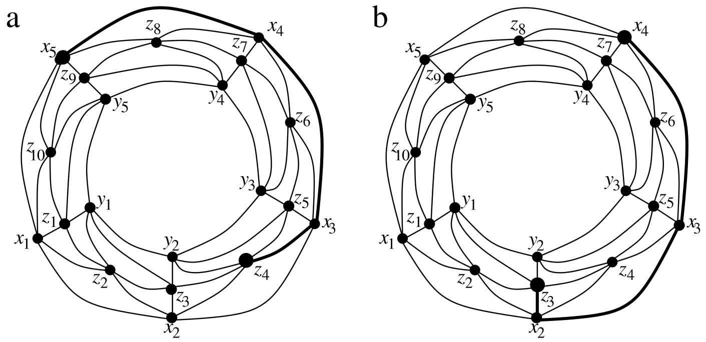

Fig. 7. Illustration for case 3.

of $y_{\lceil j/2 \rceil}$. It is easy to verify that each of the paths P_2 as mentioned above has length $\lceil l/2 \rceil + 1$ and by Lemma 8, these paths are the shortest paths between z_i and z_j.

Case 3: One of u and v is on C_2, and the other one is on C_1 or C_3.

We assume that u is on C_2 and the v is on C_1. Let $z_i = u$ and $x_j = v$. We also assume that $\lceil i/2 \rceil < j$. For odd value of i, we take $P_3 = z_i, x_{\lceil i/2 \rceil}, x_{\lceil i/2 \rceil+1}, \ldots, x_j$ if $j - \lceil i/2 \rceil < \lceil p/2 \rceil$ otherwise $P_3 = z_i, x_{\lceil i/2 \rceil}, x_{\lceil i/2 \rceil-1}, \ldots, x_j$. For even value of i, we take $P_3 = z_i, x_{i/2}, x_{i/2+1}, \ldots, x_j$ if $j - \lceil i/2 \rceil < \lceil p/2 \rceil$ otherwise $P_3 = z_i, x_{i/2}, x_{i/2-1}, \ldots, x_j$. Each of the paths P_3 contain vertices of C_2 and C_1. By Lemma 4, $x_{i/2}$ and $x_{\lceil i/2 \rceil}$ are neighbors of z_i. We can prove that both of the paths are the shortest path since each of them are the subpaths of the shortest path of Subcase 2(b) and the length of the shortest path between z_i and y_j is $j - \lceil i/2 \rceil + 1$. Fig. 7(a) illustrates an example where $z_4 = u$ and $x_5 = v$. The shortest path $P_3 = z_4, x_3, x_4, x_5$. Fig. 7(b) illustrates an example where $z_3 = u$ and $x_4 = v$. The shortest path $P_3 = z_3, x_2, x_3, x_4$.

Case 4: One of u and v is on C_1, and the other one is on C_3.

We assume that u is on C_1 and v is on C_3. Let $x_i = u$ and $y_j = v$. Without loss of generality, we assume that $i < j$. Let us take the path $P_4 = x_i, z_{2i}, y_i, y_{i+1}, \ldots, y_j$ if $j - i < \lceil p/2 \rceil$ otherwise let us take path $P_4 = x_i, z_{2i-2}, y_{i-1}, y_{i-2}, \ldots, y_j$. Each of the paths P_4 contain vertices of C_1, C_2 and C_3. By Lemma 5, z_{2i} and z_{2i-2} are neighbors of x_i, and by Lemma 4, y_i is a neighbor of z_{2i} and y_{i-1} is a neighbor of z_{2i-2}. We now prove that P_4 is the shortest path between x_i and y_j. We prove only for the case where y_j is to the counter clockwise direction of x_i. Let $l = j - i$. Since the length of P_4 is $l + 2$, it is sufficient to prove that the length of the shortest path between x_i and y_j is at least $l + 2$. The claim is obvious for $l = 0$. We thus assume that $l > 0$ and the claim is true for any value of $j - i < l$. Assume for contradiction that there is a shortest path P' between x_i and y_j with length less than $l + 2$. Since y_j is to counter clockwise direction from x_i, the second vertex of the shortest path P' is either x_{i+1} or z_{2i}. If x_{i+1} is the second vertex then by induction hypothesis, the shortest path between x_{i+1} and y_j has length $l + 1$ and the length of P' is at least $l + 2$ which contradicts our assumption. Thus we assume that the second vertex is z_{2i}. Since P_4 contains the shortest path between z_{2i} and y_j by Case 3, the length of P' cannot be less than P_4 in this case also. Fig. 8(a) illustrates an example where $x_2 = u$ and $y_4 = v$. The shortest path $P_4 = x_2, z_4, y_2, y_3, y_4$. Fig. 8(b) illustrates an example where $x_2 = u$ and $y_5 = v$. The shortest path $P_4 = x_2, z_2, y_1, y_5$.

Thus we can find a shortest path between any pair of vertices of a doughnut graph. One can see that the shortest path between any pair of vertices can be found in $O(l_s)$ time where l_s is the length of the shortest path between u and v. ■

5. Topological properties of doughnut graphs

Let G be a p-doughnut graph. By Lemma 1, the number of vertices of G is $4p$ where $p(>3)$ is an integer. A p-doughnut graph is maximal fault tolerant since it is 5-regular by Lemma 1. By Lemma 2, every p-doughnut graph G has a doughnut embedding Γ where vertices of G lie on three vertex disjoint cycles C_1, C_2 and C_3 such that C_1 is the outer cycle containing p vertices, C_2 is the middle cycle containing $2p$ vertices and C_3 is the inner cycle containing p vertices. Then one can easily see that the diameter of a p-doughnut graph is $\lfloor p/2 \rfloor + 2$. Moreover, a doughnut graph admits a ring embedding since a doughnut graph is Hamilton-connected [6].

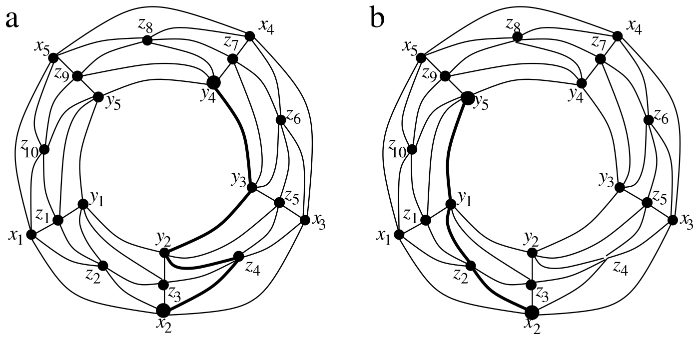

Fig. 8. Illustration for Case 4.

Table 1
Topological comparison of doughnut graphs with various Cayley graphs.

Topology	Number of nodes	Diameter	Degree	Connectivity	Fault tolerance	Hamiltonian
n-cycle	n	$\lfloor n/2 \rfloor$	2	2	Maximal	Yes
Cube-connected-cycle [10]	$d2^d$	$\lfloor 5d/2 \rfloor - 2$	3	3	Maximal	Yes
Wrapped around butterfly graph [11]	$d2^d$	$\lfloor 3d/2 \rfloor$	4	4	Maximal	Yes
d-Dimensional hypercube [12]	2^d	d	d	d	Maximal	Yes
p-doughnut graphs [1]	$4p$	$\lfloor p/2 \rfloor + 2$	5	5	Maximal	Yes

6. Conclusion

In this paper, we have studied recursive structure of doughnut graphs. We have proposed an efficient algorithm to find shortest path between any pair of vertices which exploit the structure of the graph. We have also found that doughnut graph has smaller diameter, higher degree and connectivity, maximal fault tolerance and ring embedding. There are several parameters like connectivity, degree, diameter, symmetry and fault tolerance which are considered for building interconnection networks [9]. Table 1 presents the topological comparison of various Cayley graphs, which are widely used as interconnection networks, with doughnut graphs. The table shows that topological properties of doughnut graphs are very much similar to interconnection networks. One of the limitation is the diameter which is linear but the coefficient is $1/8$. We may have an efficient routing scheme using shortest path finding algorithm. We can have a scalable interconnection network using doughnut graphs since the degree of a vertex of a doughnut graph does not change with the size of the graph. This is also important for VLSI implementation point of view as well as applications where the computing nodes in an interconnection networks only have fixed number of I/O ports. Thus doughnut graphs may find nice applications as interconnection networks.

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