ELSEVIER
Available online at www.sciencedirect.com
ScienceDirect
Advances in Mathematics 221 (2009) 217–250
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

Unitary local systems, multiplier ideals, and polynomial periodicity of Hodge numbers

Nero Budur [1]

Department of Mathematics, The University of Notre Dame, IN 46556, USA

Received 13 February 2007; accepted 2 December 2008

Available online 15 January 2009

Communicated by Tony Pantev

Abstract

The space of unitary local systems of rank one on the complement of an arbitrary divisor in a complex projective algebraic variety can be described in terms of parabolic line bundles. We show that multiplier ideals provide natural stratifications of this space. We prove a structure theorem for these stratifications in terms of complex tori and convex rational polytopes, generalizing to the quasi-projective case results of Green–Lazarsfeld and Simpson. As an application we show the polynomial periodicity of Hodge numbers $h^{q,0}$ of congruence covers in any dimension, generalizing results of E. Hironaka and Sakuma. We extend the structure theorem and polynomial periodicity to the setting of cohomology of unitary local systems. In particular, we obtain a generalization of the polynomial periodicity of Betti numbers of unbranched congruence covers due to Sarnak–Adams. We derive a geometric characterization of finite abelian covers, which recovers the classic one and the one of Pardini. We use this, for example, to prove a conjecture of Libgober about Hodge numbers of abelian covers.

MSC: 32S20

Keywords: Local systems; Parabolic line bundles; Multiplier ideals; Generic vanishing; Finite abelian covers; Polynomial periodicity of Hodge numbers

---

E-mail address: nbudur@nd.edu.

[1] The author was supported in part by the NSF grant DMS-0700360 and by the Institute of Advanced Study.

doi:10.1016/j.aim.2008.12.006

1. Introduction

Motivation. This work arose out of an effort to understand global invariants of singularities of divisors in nonsingular complex projective varieties. One of the most versatile tools that measures the complexity of singularities is the multiplier ideal, developed by Nadel [34], Esnault and Viehweg [14], Demailly [9], Ein and Lazarsfeld [12], Siu [2] (see [23] for more on them). Their nature and power still remains somewhat mysterious. Locally, there is a better understanding of multiplier ideals through different interpretations (resolution of singularities, plurisubharmonic functions, positive characteristic methods, D-modules) and through connections with many other different notions (log-canonical threshold, jet-schemes, monodromy on Milnor fibers, action of the Galois group on the homology of infinite abelian covers of complements to isolated non-normal crossings divisors, Bernstein–Sato polynomial, Hodge filtration of some Hodge modules, etc.). However, some of the most striking applications of multiplier ideals are to global problems: Minimal Model Program, invariance of plurigenera, linear series, etc. (see [23]). The key feature that makes the multiplier ideals so powerful is the vanishing of spaces of the type

$$H^q\big(X, \omega_X \otimes L \otimes \mathcal{J}(D)\big), \tag{1}$$

where X is a nonsingular complex projective variety, ω_X the canonical sheaf, $\mathcal{J}(D)$ the multiplier ideal of an effective $\mathbb{Q}$-divisor D on X, and L is a line bundle such that $L - D$ enjoys some positivity. In this paper we adopt an opposite point of view. Namely, when the support of D is fixed and $L - D$ is not positive anymore we regard the dimensions of the spaces (1) as global invariants of the singularities of the support of D. This paper is focused on these invariants and tries to develop a natural setting for them. Not surprisingly, in light of work of Esnault and Viehweg [13,14], the natural setting we found most natural is the theory of local systems of rank one. As a by-product of work aimed at multiplier ideals, we are able to prove a few other general facts about local systems.

Local systems of rank one. Let X be a nonsingular projective variety over $\mathbb{C}$. Let M be the moduli space of complex local systems of rank one on X. Then M has three interpretations, that is it has three different complex structures on the same real analytic group (see [45]):

$$\begin{aligned}
M_B &= \operatorname{Hom}\big(\pi_1(X), \mathbb{C}^*\big) = \operatorname{Hom}\big(H_1(X,\mathbb{Z}), \mathbb{C}^*\big) = H^1(X, \mathbb{C}^*),\\
M_{DR} &= \big\{(L, \nabla)\colon\ L \in \operatorname{Pic}(X),\ \nabla : L \to L \otimes \Omega^1_X \text{ integrable connection}\big\},\\
M_{Dol} &= \big\{(E, \theta)\colon \text{ Higgs line bundle}\big\} = \operatorname{Pic}^\tau(X) \times H^0\big(X, \Omega^1_X\big),
\end{aligned}$$

where $\operatorname{Pic}^\tau(X) = \ker[c_1 : \operatorname{Pic}(X) \to H^2(X, \mathbb{R})]$. Let $U \subset M$ be the unitary local systems. We have $U_B = \operatorname{Hom}(H_1(X, \mathbb{Z}), S^1)$, where $S^1 \subset \mathbb{C}^*$ is the unit circle, and $U_{Dol} = \operatorname{Pic}^\tau(X)$. U_B is a totally real subgroup of M_B, whereas U_{Dol} is a finite disjoint union of copies of the Picard variety $\operatorname{Pic}^0(X)$. For example, for a curve of genus g, we have $U_B = (S^1)^{2g}$ and $U_{Dol} = \operatorname{Jac}(X)$, the Jacobian of X.

Notation. Let $D = (D_i)_{i \in S}$ be a finite tuple of distinct irreducible and reduced codimension one subvarieties of X. We use the following notation:

$$\alpha+\beta=(\alpha_i+\beta_i)_{i\in S}, \qquad \alpha\cdot D=\sum_{i\in S}\alpha_i\cdot D_i,$$

$$\llcorner\alpha\lrcorner=(\llcorner\alpha_i\lrcorner)_{i\in S}, \qquad \alpha\cdot[D]=\sum_{i\in S}\alpha_i\cdot[D_i],$$

$$\{\alpha\}=\big(\{\alpha_i\}\big)_{i\in S}, \qquad \mu^*(\alpha\cdot D)=\sum_{i\in S}\alpha_i\cdot\mu^*D_i,$$

where $\alpha,\beta\in\mathbb{R}^S$, $[D_i]$ are the cohomology classes of D_i (it will be clear from context which cohomology), $\mu: Z\to X$ is a map of varieties, $\llcorner.\lrcorner$ is the round-down function, $\{.\}$ is the fractional part. We also denote by D the divisor $\bigcup_{i\in S}D_i$. By $H_*(X)$ ($H^*(X)$) we always mean the (co)homology with integral coefficients. We will use the term *polytope* to mean a usual polytope that does not necessarily include all its faces.

Definition 1.1. (a) Define the set of *realizations of boundaries of X on D* as

$$\operatorname{Pic}^\tau(X,D):=\big\{(L,\alpha)\in\operatorname{Pic}(X)\times[0,1)^S\colon\ c_1(L)=\alpha\cdot[D]\in H^2(X,\mathbb{R})\big\}.$$

(b) Define the set of *realizable boundaries of X on D* as

$$B(X,D):=\big\{\alpha\in[0,1)^S\colon\ c_1(L)=\alpha\cdot[D]\in H^2(X,\mathbb{R})\text{ for some }L\in\operatorname{Pic}(X)\big\}.$$

It is common in the literature to call the realizations of boundaries on X *parabolic line bundles* of parabolic degree 0. They were introduced by Seshadri in [42]. The terminology is influenced by the use of the term boundary in birational geometry. However, from our point of view it is natural to exclude the boundaries for which some coefficient is 1.

The set of realizations of boundaries $\operatorname{Pic}^\tau(X,D)$ is an abelian group under the operation

$$(L,\alpha)\cdot(L',\alpha')=\big(L\otimes L'\otimes\mathcal{O}_X(-\llcorner\alpha+\alpha'\lrcorner\cdot D),\{\alpha+\alpha'\}\big).$$

The operation of taking the fractional part of the sum induces a group structure on the set of realizable boundaries $B(X,D)$.

Theorem 1.2. *Let X be a nonsingular complex projective variety, D a divisor on X, and let $U=X-D$. There is a natural canonical group isomorphism*

$$RH:\operatorname{Pic}^\tau(X,D)\xrightarrow{\sim}\operatorname{Hom}\big(H_1(U),S^1\big)$$

between realizations of boundaries of X on D and unitary local systems of rank one on U.

The original result is the case of X being a projective curve and for higher rank local systems and is due to Mehta and Seshadri [32] (announced in [42]). In this paper, for simplicity we only consider rank one local systems. The theorem is not new, it is known to differential geometers at least in the simple normal crossings case, but we could not find a reference for a simple self-contained proof. The isomorphism *RH* follows for example from T. Mochizuki's extension in [33] of the Kobayashi–Hitchin correspondence from [44] to the open case and probably from other partial results leading to this extension (see Jost and Zuo [22], Simpson [43], Biquard [5], J. Li [24], Steer and Wren [47]). We will reproduce a self-contained proof of Theorem 1.2, showed to us by T. Mochizuki, which is fairly elementary, requiring only standard tools from differential geometry. This shortens the exposition from an earlier preprint in which we derived Theorem 1.2 from the powerful Theorem 1.1 of [33]. We show the isomorphism RH is natural (Proposition 4.1). For $N > 1$, the character group of $H_1(U, \mathbb{Z}_N)$ is the N-torsion part of $\mathrm{Pic}^\tau(X, D)$ (Lemma 4.3). The group of realizations of boundaries depends only on the complement U; see Proposition 3.3 for the explicit isomorphism between the groups of realizations of boundaries on a pair (X, D) and on a resolution (Z, E) isomorphic above U. On $\mathrm{Pic}^\tau(X, D)$ there is a mixture of real and complex structures: $\mathrm{Pic}^\tau(X, D)$ maps onto $B(X, D) \subset \mathbb{R}^S$ with fibers $\mathrm{Pic}^\tau(X)$ consisting of finitely many disjoint copies of the complex torus $\mathrm{Pic}^0(X)$. The purely real part $B(X, D)$ splits canonically into a finite disjoint union of convex rational polytopes in $\mathbb{R}^S$ (see Section 4).

Canonical stratifications. The space of unitary local systems of rank one on a quasi-projective nonsingular variety U has canonical stratifications. Given U, X, and D as above define

$$V_i^q(X, D) := \{(L, \alpha) \in \mathrm{Pic}^\tau(X, D)\colon\ h^q\big(X, \omega_X \otimes L \otimes \mathcal{J}(\alpha \cdot D)\big) \geqslant i\},$$

where $\mathcal{J}(\alpha \cdot D)$ denotes the multiplier ideal of the $\mathbb{R}$-divisor $\alpha \cdot D$. Under the isomorphism of Theorem 1.2, $V_i^q(X, D)$ defines a subset $V_i^q(U)$ of the space of unitary local systems of rank one on U independent of X and D (Lemma 6.3). $V_i^q(U)$ can also be interpreted via the initial piece of the Hodge filtration (see [49]) on the cohomology of U with unitary local systems as coefficients (Proposition 6.4).

We prove a structure result about these canonical sets. The original theorem when $D = 0$ is due to Green and Lazarsfeld [16], [17]. For different proofs see also Arapura [3], Pink and Roessler [38], Simpson [45]. The most general theorem over projective varieties is due to Simpson [45]. Over quasi-projective varieties, Arapura [4] has a similar result for the canonical sets defined via the cohomology of the local systems under the assumptions of D having normal crossings and $H^1(X, \mathbb{C}) = 0$. Using Simpson's version and Theorem 1.2, we give a generalization in a different direction to quasi-projective varieties which holds without additional hypotheses.

Theorem 1.3. *With the notation of Theorem* 1.2*, there exists a decomposition of the set of realizable boundaries $B(X, D) \subset \mathbb{R}^S$ into a finite number of rational convex polytopes P such that for every q and i the subset $V_i^q(X, D)$ of* $\mathrm{Pic}^\tau(X, D)$ *is a finite union of sets of the form $P \times \mathcal{U}$, where $\mathcal{U}$ is a torsion translate of a complex subtorus of* $\mathrm{Pic}^\tau(X)$*. Any intersection of sets $P \times \mathcal{U}$ is also of this form. Pointwise, the subset of $V_i^q(X, D)$ corresponding to $P \times \mathcal{U}$ consists of the realizations $(L + M, \alpha)$ with $\alpha \in P$ and $M \in \mathcal{U}$, for some line bundle L depending on P which can be chosen such that (L, α) is torsion for some $\alpha \in P$.*

The decomposition into polytopes of $B(X, D)$ is not the canonical one mentioned before, but it is any of the ones induced by log-resolutions of (X, D). When D is a simple normal crossing divisor, the decomposition of $B(X, D)$ is the canonical one.

The proof of Theorem 1.3 extends to cover a more general result regarding loci given by dimensions of pieces of the Hodge filtration on cohomology of unitary local systems on U. Define

$$W_i^{p,q}(U) := \{\mathcal{V} \in \mathrm{Hom}\big(H_1(U), S^1\big) \mid \dim \mathrm{Gr}_F^p H^{p+q}(U, \mathcal{V}) \geqslant i\}.$$

Then $V_i^q(U) = W_i^{0,n-q}(U)^\vee := \{\mathcal{V}^\vee \mid \mathcal{V} \in W_i^{0,n-q}(U)\}$, see Proposition 6.4. More generally, let

$$W_i^{p,q}(U,\mathcal{W}) = \bigl\{\mathcal{V} \in \operatorname{Hom}\bigl(H_1(U), S^1\bigr) \bigm| \dim \operatorname{Gr}_F^p H^{p+q}(U, \mathcal{W}\otimes\mathcal{V}) \geqslant i\bigr\}$$

for a fixed unitary local system $\mathcal{W}$ on U of arbitrary rank. In this notation, $W_i^{p,q}(U) = W_i^{p,q}(U, \mathbb{C}_U)$.

Theorem 1.4. *The statement of Theorem* 1.3 *holds for the image of* $W_i^{p,q}(U,\mathcal{W})$ *in* $\operatorname{Pic}^\tau(X,D)$ *under the assumption that either* $\mathcal{W}$ *is the restriction of a local system from a nonsingular compactification of* U*, or there are no nontrivial unitary rank one local systems on a such compactification.*

The decomposition of $B(X,D)$ into polytopes needed for Theorem 1.4 is in general a refinement of the decomposition needed for Theorem 1.3 and is given by Remark 8.2(d).

Corollary 1.5. *With the assumptions of Theorem* 1.4*, the statement of Theorem* 1.3 *holds for the image of*

$$\bigl\{\mathcal{V} \in \operatorname{Hom}\bigl(H_1(U), S^1\bigr) \bigm| \dim H^m(U, \mathcal{W}\otimes\mathcal{V}) \geqslant i\bigr\}$$

in $\operatorname{Pic}^\tau(X,D)$.

When $\mathcal{W} = \mathbb{C}_U$ and $H^1(X,\mathbb{C}) = 0$, the loci of Corollary 1.5 (without the restriction to the unitary case) are considered in the first theorem of [4] in terms of maps from U to products complex tori. Our method does not allow us to replace in Corollary 1.5 S^1 by $\mathbb{C}^*$ (i.e. unitary by arbitrary). One could ask if the polytopes in the decomposition from Corollary 1.5, under the embedding $B(X,D) \subset (S^1)^S$, are translates of real subtori $(S^1)^r$. Our method does not answer this question.

One problem with removing the assumption on $\mathcal{W}$ in Theorem 1.4 and Corollary 1.5 is that we do not know the answer of the following question (see Remark 8.4):

Question 1.6. With the notation as above, fix integers m and i. Suppose $\mathcal{W}$ is a unitary local system on U. Is the image of the set

$$\bigl\{\mathcal{V} \in \operatorname{Hom}\bigl(H_1(X), S^1\bigr) \bigm| \dim IH^m(X, \mathcal{W}\otimes\mathcal{V}_{|U}) \geqslant i\bigr\}$$

in $\operatorname{Pic}^\tau(X)$ a finite union of torsion translates of complex subtori of $\operatorname{Pic}^\tau(X)$?

Connection with other recent works. (a) Professor Libgober has kindly informed us that for the case when $q \geqslant 1$, D having isolated non-normal crossings singularities, the irreducible components D_i being ample, and $H_1(X) = 0$, the polytopes in the description of V_i^q are determined (see [28]) by the local polytopes of quasiadjunction introduced in [27], extending the case $X = \mathbb{P}^2$ of [25]. In [28], under the same assumptions, the sets V_i^q were related to the homotopy groups $\pi_q(U)$. See [25,28,29] for properties and applications of local and global polytopes of quasiadjunction. Even if the end result is the same as ours in this special case, the methods are different (see Example 6.6). We do not know how to extend to method of polytopes of quasiadjunction to recover the more general case we consider. For a streamlined version of the method of this article for a special case of Theorem 1.4, see Theorem 2.1 of the recent preprint [30] which appeared before this revision was made.

(b) The size of sets similar-looking to V_i^q have been considered from the point of view of the Fourier–Mukai transform by Pareschi and Popa [37]. For example, for X a smooth projective variety of dimension n and Albanese dimension $n-k$, [37, Theorem A] gives that the codimension in $\mathrm{Pic}^\tau(X)$ of $V_1^q(L,\alpha)$ (see Lemma 6.2) is $\geqslant q-k+\kappa$, where κ is the Iitaka dimension of $L-\alpha\cdot D$ along a generic fiber of the Albanese map of X. If, in Theorem 1.3 for the decomposition of $V_1^q(X,D)$, a set $P\times\mathcal{U}$ consists of realizations $\{(L+M,\alpha)\}$, then the codimension of $\mathcal{U}$ equals the codimension of $V_1^q(L,\alpha)$ for some $\alpha\in P$ (cf. proof of Theorem 1.3).

Congruence covers and polynomial periodicity of Hodge numbers. As an application of the above results, we use counting of lattice points in rational convex polytopes to show polynomial periodicity of the Hodge numbers $h^{q,0}$ of congruence covers and, more generally, of the Hodge ranks $h^{p,q}$ (see definition below) of the cohomology of the pullbacks of a unitary local system to the unbranched congruence covers.

Definition 1.7. A function f on the set $\{1,2,3,\dots\}$ is a *quasi-polynomial* if there exist a natural number M and polynomials $f_i(x)\in\mathbb{Q}[x]$ for $1\leqslant i\leqslant M$ such that $f(N)=f_i(N)$ if $N\equiv i \pmod M$.

Let X be a nonsingular complex projective variety, $D=\bigcup_{i\in S}D_i$ a divisor on X with irreducible components D_i, and let $U=X-D$. The canonical surjections

$$H_1(U,\mathbb{Z})\to H_1(U,\mathbb{Z}_N),$$

for $N>1$, give canonical normal abelian covers $X_N\to X$ called congruence covers. Let $h^q(N)$ denote the Hodge numbers $h^{q,0}=h^{0,q}$ of any nonsingular model of X_N. These are birational invariants hence $h^q(N)$ is well defined.

Theorem 1.8. *With the notation as above, for every q the function $h^q(N)$ is a quasi-polynomial in N.*

The case when X is a surface was done by E. Hironaka [20,21], and Sakuma [40], generalizing previous results of Zariski, Libgober, Vacquie (see [20] for the history). Even the case of $X=\mathbb{P}^n$ for $n>2$ is new to our knowledge. The original result is of Sarnak and Adams [41] who proved in every dimension the polynomial periodicity of the Betti numbers of the unbranched congruence covers $U_N\to U$ (see also [7]). For a survey of applications and related results in the case of hyperplane arrangements see [48].

The proof of Theorem 1.8, adapted to the situation of Theorem 1.4, yields the following generalization. Let $g_N:U_N\to U$ be the unbranched congruence cover and fix a unitary local system $\mathcal{W}$ on U. Define the *Hodge rank* $h^{p,q}_{\mathcal{W}}(N)$ as the dimension of $\mathrm{Gr}_F^p H^{p+q}(U_N,g_N^*\mathcal{W})$. This definition should not be confused with the usual definition of Hodge numbers of a mixed Hodge structure which involves the weight filtration as well. Typically $H^k(U_N,g_N^*\mathcal{W})$ does not have weight k and it is not pure. However, we do not worry about the weight filtration in this article. If $\mathcal{W}=\mathbb{C}_U$ and X_N is a nonsingular compactification of U_N with a complement E_N which is a simple normal crossings divisor, then $h^{p,q}_{\mathcal{W}}(N)=h^q(X_N,\Omega^p_{X_N}(\log E_N))$ and $h^{0,q}_{\mathcal{W}}(N)=h^q(N)$ as in Theorem 1.8.

We assume that $\mathcal{W}$ is the restriction of a local system from a nonsingular compactification of U, or that there are no nontrivial unitary rank one local systems on a such compactification. The following generalizes the result of Sarnak and Adams [41] on the Betti numbers of U_N:

Theorem 1.9. *For every pair of integers p and q, the Hodge rank function $h^{p,q}_{\mathcal{W}}(N)$ is a quasi-polynomial in N.*

Finite abelian covers. The subject of finite abelian covers is well-known and has been studied by many people. To our knowledge, the geometric characterization of abelian covers has two versions: a classical one [35,51] and the one of [36]. The isomorphism of Theorem 1.2 recovers both characterizations offering a common generalized approach. This does not seem to be well known to algebraic geometers since as a consequence we are able to prove some results about abelian covers (Corollaries 1.12 and 1.13) which, to our knowledge, appear in their published version only under some additional hypotheses. We employ these results in deriving the theorems mentioned earlier. From Theorem 1.2 we derive the following geometric characterization, essentially proven in [35].

Corollary 1.10. *Let X be a nonsingular complex projective variety, $D = \bigcup_{i\in S} D_i$ a divisor on X with irreducible components D_i, and let $U = X - D$. Let G be a finite abelian group. The equivalence classes of normal G-covers (Definition* 5.1*) of X unramified above U are into one-to-one correspondence with the subgroups $G^* \subset \mathrm{Pic}^\tau(X, D)$.*

Here $G^* = \mathrm{Hom}(G, \mathbb{C}^*)$ is the dual group of G. We show that Corollary 1.10 also recovers via the isomorphism *RH* the geometric characterization of finite abelian covers due to Pardini [36]. In particular, as expected, one has the following:

Corollary 1.11. *With the notation as in Corollary* 1.10*, let $\pi : \widetilde{X} \to X$ be a normal G-cover of X unramified above U corresponding to an inclusion $G^* = \{(L_\chi, \alpha_\chi)\colon\ \chi \in G^*\} \subset \mathrm{Pic}^\tau(X, D)$. Then*

$$\pi_*\mathcal{O}_{\widetilde{X}} = \bigoplus_{\chi\in G^*} L_\chi^{-1},$$

where G acts on L_χ^{-1} via the character χ.

The following computation was done by Esnault and Viehweg [14] in the case when G is cyclic.

Corollary 1.12. *With the notation as in Corollary* 1.11*, suppose $\mu : Z \to X$ is a log resolution of (X, D) which is an isomorphism above U. Let $\rho : \widetilde{Z} \to Z$ be the corresponding normal G-cover unramified above U. Then we have the eigensheaf decomposition*

$$\rho_*\mathcal{O}_{\widetilde{Z}} = \bigoplus_{\chi\in G^*} \mu^* L_\chi^{-1} \otimes \mathcal{O}_Z\bigl(\llcorner \mu^*(\alpha_\chi \cdot D)\lrcorner\bigr),$$

where the round-down of a divisor means rounding-down of the coefficients of its irreducible components.

The following computation was done by Libgober (see [28]) in the case $X = \mathbb{P}^N$ and proposed as Conjecture 6.4 in [28] for the case of $H_1(X) = 0$, D having ample irreducible components and isolated non-normal crossings, and $q = n$:

Corollary 1.13. *With the notation as in Corollary* 1.11*, let $H^0(Y, \Omega_Y^q)$ denote the space of global q-forms on a nonsingular model Y of $\widetilde{X}$. Then*

$$H^0\big(Y, \Omega_Y^q\big) \cong \bigoplus_{\chi \in G^*} H^{n-q}\big(X, \omega_X \otimes L_\chi \otimes \mathcal{J}(\alpha_\chi \cdot D)\big),$$

where n is the dimension of X.

Layout. In Section 2 we review multiplier ideals, local systems, and the details of analysis needed for the next section. In Section 3 we give a self-contained proof of Theorem 1.2. In Section 4 we show the naturality of this isomorphism and describe the structure of the spaces of realizations of boundaries and of realizable boundaries. In Section 5 we derive the geometric characterization and computations for finite covers described in the introduction. In Section 6 we prove Theorem 1.3. In Section 7 we prove Theorem 1.8. In the last section we prove generalizations in terms of local systems of the above results, namely Theorem 1.4, Corollary 1.5, and Theorem 1.9.

2. Basic notions

In this paper, all algebraic varieties are defined over the field of complex numbers. The underlying analytic variety of an algebraic variety X will be denoted by the same notation. This section recalls the definition of the basic objects we work with in this paper and it can be skipped by the knowledgeable reader.

Multiplier ideals. We review the definition of multiplier ideals and the local vanishing theorem [23]. For a nonsingular variety X, let ω_X denote the canonical sheaf. Let $\mu : Y \to X$ be a proper birational morphism. The exceptional set of μ is denoted by $Ex(\mu)$. This is the set of points $\{y \in Y\}$ where μ is not biregular. We say that μ is a resolution if Y is nonsingular. Let D be an effective $\mathbb{R}$-divisor on X. We say that a resolution μ is a *log resolution* of (X, D) if $\mu^{-1}(D) \cup Ex(\mu)$ is a divisor with simple normal crossings. Such a resolution always exists, by Hironaka. Denote by $\omega_{Y/X}$ the relative canonical sheaf $\omega_Y \otimes (\mu^*\omega_X)^{\otimes -1}$. If $D = \sum \alpha_i D_i$ is an effective $\mathbb{R}$-divisor on X, where D_i are the irreducible components of D and $\alpha_i \in \mathbb{R}$, the round down of D is the integral divisor $\llcorner D \lrcorner = \sum \llcorner \alpha_i \lrcorner D_i$.

Definition 2.1. Let D be an effective $\mathbb{R}$-divisor on a nonsingular complex variety X, and fix a log resolution $\mu : Y \to X$ of (X, D). Then the *multiplier ideal sheaf* of D is defined to be $\mathcal{J}(D) = \mu_*(\omega_{Y/X} \otimes \mathcal{O}_Y(-\llcorner \mu^* D \lrcorner))$.

This is a sheaf of ideals in $\mathcal{O}_X$ independent of the log resolution. Analytically, the multiplier ideals are defined locally to consist of holomorphic functions f such that $|f|^2 / \prod |g_i|^{2\alpha_i}$ is locally integrable, where g_i are fixed equations of the irreducible components of D and $\alpha_i \in \mathbb{R}_{>0}$ are the corresponding coefficients in D.

Theorem 2.2. *(Local vanishing [23, Theorem 9.4.1].) Let D be an effective $\mathbb{R}$-divisor on a nonsingular complex variety X, and $\mu : Y \to X$ of (X, D) a log resolution of (X, D). Then $R^j \mu_*(\omega_{Y/X} \otimes \mathcal{O}_Y(-\llcorner \mu^* D \lrcorner)) = 0$ for $j > 0$.*

Metrics, connections, and curvature. Recall the following basic facts from differential geometry [18]. Let X be a complex manifold and L a complex line bundle on X. A trivialization of L over an open subset Ω of X is the same as a frame u of L above Ω. Fix frames u for L. A *hermitian metric* $(.,.)$ on L is equivalent to a C^∞ real-valued positive function $h(x) = (u(x), u(x))$. One also writes $h = \exp(-2\phi)$, with $\phi \in C^\infty(\Omega, \mathbb{R})$. A *connection* on L is a linear map

$$\mathbb{D} : \mathcal{A}^0(L) \to \mathcal{A}^1(L),$$

where $\mathcal{A}^p(L)$ is the sheaf of C^∞ L-valued p-forms on X such that $\mathbb{D}(\sigma u) = (d\sigma)u + \sigma \mathbb{D}(u)$ for C^∞ functions σ where d is the usual exterior derivative. If L is a holomorphic line bundle, there is a natural $(0, 1)$-operator $\bar\partial$ on $\mathcal{A}^0(L)$ defined locally by $\bar\partial(\sigma u) = (\bar\partial \sigma)u$ for a holomorphic frame u. In this case there exists a unique connection $\mathbb{D}$ on L such that it is compatible with the hermitian metric h and with the complex structure. That is $d(v, w) = (\mathbb{D}v, w) + (v, \mathbb{D}w)$ and, if $\mathbb{D} = \mathbb{D}' + \mathbb{D}''$ is the decomposition of $\mathbb{D}$ into the $(1, 0)$ and $(0, 1)$ parts, then $\mathbb{D}'' = \bar\partial$. $\mathbb{D}$ is called the *metric connection* of h. If u is a holomorphic frame for L and $\mathbb{D}$ is the metric connection of h, then

$$\mathbb{D}u = \frac{\partial h}{h} \cdot u.$$

A connection on a holomorphic line bundle L is a *holomorphic connection* if it compatible with the complex structure. A connection $\mathbb{D}$ extends to operators $\mathbb{D} : \mathcal{A}^p(L) \to \mathcal{A}^{p+1}(L)$ by forcing $\mathbb{D}(\sigma u) = (d\sigma)u + (-1)^p \sigma \wedge \mathbb{D}u$ for all C^∞ p-forms σ. The operator $\mathbb{D}^2 : \mathcal{A}^0(L) \to \mathcal{A}^2(L)$ is linear in the C^∞ functions on X and $\mathbb{D}^2 u = \Theta u$ for a closed globally defined 2-form Θ, called the *curvature* of $\mathbb{D}$. If u is a holomorphic frame for L and $\mathbb{D}$ is the metric connection of $h = \exp(-2\phi)$, then

$$\frac{i}{2\pi} \Theta = \frac{i}{\pi} \partial \bar\partial \phi. \tag{2}$$

If in addition X is compact, then the real cohomology class defined by $i(2\pi)^{-1}\Theta$ on X is the same as the first Chern class $c_1(L)$.

Local systems. A *complex local system* $\mathcal{V}$ on a complex manifold X is a locally constant sheaf of finite dimensional complex vector spaces [11]. The rank of $\mathcal{V}$ is the dimension of a fiber of $\mathcal{V}$. Local systems of rank one on X are equivalent with representations $H_1(X) \to \mathbb{C}^*$. Unitary local systems of rank one on X correspond to representations $H_1(X) \to S^1$, where S^1 is the unit circle in $\mathbb{C}$. Let L be a complex line bundle on X. If $\mathbb{D}^2 = 0$ for a connection $\mathbb{D}$ on L, then $\mathbb{D}$ is called *flat*. It can be showed then that there is exactly one holomorphic structure on L such that the $(0, 1)$-part of $\mathbb{D}$ is compatible with it. The functor taking a complex bundle L on X with a flat connection $\mathbb{D}$ to the kernel of $\mathbb{D}$ defines an equivalence between the categories of complex vector bundles with flat connections and local systems. Equivalently, there is an equivalence between the categories of holomorphic vector bundles with flat holomorphic connections and local systems. Under this equivalence, a local system is unitary if and only if the corresponding flat connection is the metric connection for some hermitian metric [10, V]. It follows from (2) that in this case the flatness condition on $\mathbb{D}$ reads, in terms of a holomorphic frame u for L and metric $h = \exp(-2\phi)$, as $\partial\bar{\partial}\phi = 0$.

The canonical Deligne extension. Let X be a compact complex manifold and U an open subset such that the complement is a divisor with simple normal crossings. Let L_U be a holomorphic line bundle on U with a holomorphic flat connection $\mathbb{D}$ defining a unitary local system on U. Let D_i be the irreducible components of $X - U$ and let $\alpha_i \in [0,1)$ be such that the monodromy of $(L_U, \mathbb{D})$ around a general point of D_i is multiplication by $\exp(-2\pi i\alpha_i)$. There exists an extension $(L, \mathbb{D})$ to a holomorphic line bundle with a flat logarithmic connection on X uniquely characterized by the condition that the eigenvalue of the residue of $(L, \mathbb{D})$ around D_i is α_i [8]. $(L, \mathbb{D})$ is called the *canonical Deligne extension* of $(L_U, \mathbb{D})$. We recall next how the line bundle L is constructed. Explicitly, let u_i be a frame of the multi-valued flat sections of $(L_U, \mathbb{D})$ around a general point of D_i. We can assume that a local equation for D_i is given by the local coordinate z_i. Let $v_i = \exp(\log z_i \cdot \alpha_i) \cdot u_i$. Then v_i is a single-valued holomorphic section of L_U. Let L_0 be the line bundle on $X - W$ obtained by declaring v_i to be a local holomorphic frame, where W is the closed set of singular points of $\bigcup_i D_i$. Since the codimension of W is $\geqslant 2$, L_0 extends uniquely to a line bundle L on X.

Proposition 2.3. *(See [13, (B.3)].) With the notation as above, the first Chern class of L is given by $c_1(L) = -\sum_{i\in S}\alpha_i[D_i]$ in $H^2(X,\mathbb{R})$.*

3. Unitary local systems and realizations of boundaries

Proof of Theorem 1.2 for the simple normal crossings case. With the notation of Theorem 1.2, assume in addition that D is a simple normal crossings divisor. We define a map

$$RH : \mathrm{Pic}^{\tau}(X,D) \to \mathrm{Hom}\big(H_1(U), S^1\big)$$

as follows, by an argument showed to us by T. Mochizuki. This shortens the exposition from an earlier preprint in which we derived the map RH from the powerful correspondence of [33, Theorem 1.1]. The responsibility for any faults of the exposition lies with the author.

Let (L, α) be a realization of a boundary. To attach a unitary local system to (L, α) we will find a hermitian metric h on the restriction of L to U such that the curvature of the metric connection $\mathbb{D}_h$ is zero. Then, by Section 2, $(L_{|U}, \mathbb{D}_h)$ gives a unitary local system of rank one on U.

Fix an open cover $\mathcal{C}$ of X and trivializations above $\Omega \in \mathcal{C}$ of L. Let h_0 be a hermitian metric on L. Let σ_i be a nonzero holomorphic global section of the line bundle $\mathcal{O}_X(D_i)$ vanishing exactly at D_i. Let $|\sigma_i| : X \to \mathbb{R}$ be the norm of σ_i for a fixed hermitian metric H_i on $\mathcal{O}_X(D_i)$. Let $H_{i,U}$ be the restriction of H_i to U. Remark that $\sigma_{i,U} = \sigma_{i\,|U}$ is a frame for $\mathcal{O}_X(D_i)$ above U. Hence we can assume $H_{i,U} = |\sigma_{i,U}|^2$. Define $h_{1,U\cap\Omega} = h_{0,U\cap\Omega}\prod_{i\in S}|\sigma_{i,U\cap\Omega}|^{-2\alpha_i}$, where the appearance of $U \cap \Omega$ in the index means restriction of the corresponding function to $U \cap \Omega$. Then $h_{1,U\cap\Omega}$ are $\mathbb{R}_{>0}$-valued smooth functions on $U \cap \Omega$. For $\Omega, \Omega' \in \mathcal{C}$, denote by $g_{\Omega,\Omega'}$ the transition functions for the trivializations of L above $\Omega \cap \Omega'$. Then, on $U \cap \Omega \cap \Omega'$, we have $h_{1,U\cap\Omega'} = |g_{\Omega,\Omega'}|^2 \cdot h_{1,U\cap\Omega}$. In other words, the $h_{1,U\cap\Omega}$ patch up to form a hermitian metric on $L_{|U}$. We call this metric $h_{1,U}$. Remark that, unless all the α_i are 0, the same process does not define a metric on L.

By Section 2, there is an equality of $(1,1)$-forms globally defined on U

$$\frac{i}{2\pi}\Theta_{h_{1,U}} = \frac{i}{2\pi}\Theta_{h_{0,U}} - \sum_{i\in S}\alpha_i \frac{i}{2\pi}\bar{\partial}\partial \log|\sigma_{i,U}|^2, \tag{3}$$

where Θ_* is the curvature of the metric connection of the metric $*$. The closed $(1,1)$-form $i(2\pi)^{-1}\Theta_{h_{0,U}}$ is the restriction to U of the closed $(1,1)$-form $i(2\pi)^{-1}\Theta_{h_0}$ which represents the first Chern class of L. The closed $(1,1)$-form $i(2\pi)^{-1}\bar{\partial}\partial \log|\sigma_{i,U}|^2$ is the restriction to U of the closed $(1,1)$-form on X given by $i(2\pi)^{-1}\Theta_{H_i}$ which represents the cohomology class of D_i. Define a global $(1,1)$-form on X by

$$\frac{i}{2\pi}\Theta_{h_1} = \frac{i}{2\pi}\Theta_{h_0} - \sum_{i\in S}\alpha_i \frac{i}{2\pi}\Theta_{H_i}.$$

Then $\Theta_{h_{1,U}}$ is the restriction to U of Θ_{h_1}. By assumption $c_1(L) = \alpha\cdot[D]$ in $H^2(X,\mathbb{R})$. Hence the cohomology class of $i(2\pi)^{-1}\Theta_{h_1}$ is zero. By the $\bar{\partial}\partial$-Poincaré lemma [18, p. 387], $i(2\pi)^{-1}\Theta_{h_1} = i(2\pi)^{-1}\bar{\partial}\partial\chi$ for some real-valued function χ. Define a new hermitian metric on U by

$$h = h_{1,U}\exp(-\chi_{|U}).$$

Then the curvature of the metric connection $\mathbb{D}_h$ of h is zero. Hence $(L_{|U},\mathbb{D}_h)$ is flat connection over U.

The map *RH* is well defined. A different choice for the metric h_0 gives a new metric h' which differs from h by multiplication by a constant, hence $\mathbb{D}_h$ and $\mathbb{D}_{h'}$ define the same local system. It is straight forward that *RH* is a group homomorphism.

The monodromy of the local system attached to (L,α) around a general point of D_i is multiplication by $\exp(2\pi i\alpha_i)$. To observe this let $\Omega\subset X$ be a small neighborhood around a general point of D_i. We can assume that $\Omega = \Delta^n$, where n is the dimension of X and Δ is a small disc in $\mathbb{C}$, with coordinates $z_1,\dots,z_n$. We can further assume that z_1 is a local equation of $D_{|\Omega} = (D_i)_{|\Omega}$. Let Ω^* be $\Omega - D$, which we can assume to be $\Delta_1^*\times\Delta^{n-1}$, where Δ_1^* is the punctured disc in $\mathbb{C}$ with coordinate z_1. Then $L_{|\Omega^*} = \mathcal{O}_{\Omega^*}\cdot u$, where u is a fixed holomorphic frame for L above Ω^* which we can assume to be orthonormal for $h_0\prod_{j\neq i}H_j^{-\alpha_j}\exp(-\chi)$. Then $h = |z_1|^{-2\alpha_i}$. The metric connection of h is given by

$$\mathbb{D}u = \frac{\partial h}{h}\cdot u = (-\alpha_i)\frac{dz_1}{z_1}\cdot u.$$

The flat sections of $L_{|\Omega^*}$ are of the form σu such that

$$0 = \mathbb{D}(\sigma u) = \left(d\sigma - \alpha_i\sigma\frac{dz_1}{z_1}\right)u. \tag{4}$$

Thus a multi-valued frame for the local system corresponding to the solution space of (4) is $u_i = \exp(\log z_1\cdot\alpha_i)\cdot u$ (see also [8, II.1.17.1]). Let T_i be the monodromy around D_i of this local system. Then

$$T_i u_i = \exp\bigl((\log z_1 + 2\pi i)\cdot\alpha_i\bigr)\cdot u = \exp(2\pi i\alpha_i)u_i,$$

which is what we wanted to show.

The map RH is injective. Indeed, if $\mathbb{C}_U$ is the resulting local system, its only extension to X as a local system is $\mathbb{C}_X$. On the other hand, since the monodromy around the D_i's is trivial, we must have $\alpha_i = 0$ for all $i \in S$. Thus $h = h_0 \exp(-\chi)$ and $(L_{|U}, \mathbb{D}_h)$ is the restriction of $(L, \mathbb{D}_h)$ to U. Hence $L = \mathcal{O}_X$ and $RH^{-1}(1) = (\mathcal{O}_X, 0)$.

The map RH is surjective, and thus an isomorphism. Indeed, let ρ be a unitary character of $H_1(U)$ corresponding to a line bundle with a flat connection $(L_U, \mathbb{D})$ on U. We define (L, α) as follows. Let $\alpha_i \in [0, 1)$ be such that the monodromy of $(L_U, \mathbb{D})$ around a general point of D_i is multiplication by $\exp(2\pi i \alpha_i)$. Let $(L_U^\vee, \mathbb{D}^\vee)$ denote the dual connection of $(L_U, \mathbb{D})$. Let $(L^\vee, \mathbb{D}^\vee)$ denote the canonical Deligne extension of $(L_U^\vee, \mathbb{D}^\vee)$ to a line bundle with a flat connection with logarithmic singularities along D. Then L is the dual of $L^\vee$. By Proposition 2.3, $c_1(L^\vee) = -\sum_{i\in S} \alpha_i [D_i]$ in $H^2(X, \mathbb{R})$. Hence (L, α) is a realization of the boundary $\alpha \cdot D$ such that $RH(L, \alpha) = (L_U, \mathbb{D})$. □

Comparison with log resolutions. Let X be a nonsingular complex projective variety, $D = \bigcup_{i\in S} D_i$ a divisor on X with irreducible components D_i, and let $U = X - D$. Fix a log resolution $\mu : Z \to X$ of (X, D) which is an isomorphism above U. Let $E = Z - U$ with irreducible decomposition $E = \bigcup_{j\in S'} E_j$. Let $\mu^* D_i = \sum_{j\in S'} e_{ij} \cdot E_j$. Let $\mathcal{V}$ be a unitary local system on U and denote by T_j the monodromy of $\mathcal{V}$ around E_j given by a small loop in U centered at a general point of E_j. Since $\mathcal{V}$ is unitary, T_j is multiplication by $\exp(-i2\pi\beta_j)$ for some $\beta_j \in [0, 1)$. If E_j is the strict transform of some D_i, we denote by $T(i)$ the monodromy T_j and we let $\alpha_i = \beta_j$. Let γ_j and δ_i be the images in $H_1(U)$ of a small loop in U centered at a general point of E_j and, respectively, D_i.

Lemma 3.1. *With the notation as above, for all $j \in S'$,*

(a) $\gamma_j = \sum_{i\in S} e_{ij} \delta_i \in H_1(U)$.
(b) $T_j = \prod_{i\in S} T(i)^{e_{ij}}$.
(c) $\beta_j = \{\sum_{i\in S} e_{ij} \alpha_i\}$, *the fractional part.*

Proof. By definitions, (a) implies (b) which implies (c). We show (a). By duality (e.g. [6, VI. Theorem 8.3]), we have canonical isomorphisms

$$H^{2n-2}(D) = H_2(X, U),$$
$$H^{2n-2}(E) = H_2(Z, U),$$

where n is the dimension of X. Since the singular locus D_{sing} of D has codimension at least 2 in X,

$$H^{2n-2}(D) = H^{2n-2}(D, D_{sing}) = H_0(D - D_{sing}) = \mathbb{Z}^S.$$

Indeed, $D - D_{sing}$ has one path-connected component for each D_i. Call η_i this generator of $H^{2n-2}(D)$, so that this last space is $\bigoplus_{i\in S} \mathbb{Z}\eta_i$. Similarly, $H^{2n-2}(E) \cong \bigoplus_{j\in S'} \mathbb{Z}\xi_j$, where ξ_j corresponds to E_j. The images of η_i and ξ_j in $H_1(U)$ by the boundary maps $H_2(X, U) \to H_1(U)$ and, respectively $H_2(Z, U) \to H_1(U)$, are δ_i and γ_j, respectively. Hence (a) follows if we show that, under the natural direct image $\mu_* : H_2(Z, U) \to H_2(X, U)$, there is an equality $\mu_*(\xi_j) = \sum_{i\in S} e_{ij} \eta_i$.

Since $H_2(X,U)$ and $H_2(Z,U)$ are free abelian groups, by the Universal Coefficient Theorem [6, V. Corollary 7.2] the map μ_* is the adjoint of the pullback map $\mu^*: H^2(X,U)\to H^2(Z,U)$. Let $Z^2_D(X)$ and $Z^2_E(Z)$ be the groups of 2-codimensional cycles on X and Z with supports in D and E, respectively. There are cycle maps [15, p. 380] cl^X and cl^Z making the following diagram commutative [15, Proposition 19.2; see also Example 19.2.6]:

$$\begin{tikzcd} Z^2_D(X) \arrow[r, "cl^X"] \arrow[d, "\mu^*"'] & H^2(X,U) \arrow[d, "\mu^*"'] \\ Z^2_E(Z) \arrow[r, "cl^Z"] & H^2(Z,U) \end{tikzcd}$$

By definition in [15], $cl^X(D_i)=\eta_i^\vee$, under the isomorphism

$$H^2(X,U)=H_{2n-2}(D)=\bigoplus_i H_{2n-2}(D_i)=\bigoplus_i H^{2n-2}(D_i)^\vee.$$

Similarly, $cl^Z(E_j)=\xi_j^\vee$. Since the cycle pullback μ^*D_i is $\sum_{j\in S'}e_{ij}E_j$, the claim follows. $\square$

The canonical generators η_i and $\eta_i^\vee$ of $H_2(X,U)$ and $H^2(X,U)$ will be called $[D_i]^\vee$ and $[D_i]$, respectively, when no confusion arises.

Lemma 3.2. *With the notation as in Lemma* 3.1*, let* $(M_U,\mathbb{D})$ *be a flat connection on* U *corresponding to the local system* $\mathcal{V}$*. Denote by* $(M,\mathbb{D})$ *the canonical Deligne extension to* Z*. There exists a line bundle* L *on* X *such that the canonical Deligne extension is*

$$M=\mu^*L\otimes\mathcal{O}_Z(\llcorner e\lrcorner\cdot E),$$

where $e=(e_j)_{j\in S'}$ *with* $e_j=\sum_{i\in S}e_{ij}\alpha_i$.

Proof. Recall the local description of M. Let z_j be local coordinates of E_j. Around a general point on E_j, let u_j be a frame of the multi-valued flat sections of $(M_U,\mathbb{D})$. Then $v_j=\exp(\log z_j\cdot\beta_j)\cdot u_j$ is a single-valued holomorphic section of M_U. Indeed,

$$T_jv_j=\exp\bigl((\log z_j+2\pi i)\beta_j\bigr)\cdot\exp(-2\pi i\beta_j)u_j=v_j.$$

Then M is the line bundle obtained by declaring v_j to be a local holomorphic frame on Z and gluing. Define

$$w_j=\exp(\log z_j\cdot e_j)\cdot u_j,$$

so that by Lemma 3.1(c), $v_j=\exp(\log z_j\cdot(-\llcorner e_j\lrcorner))\cdot w_j$. Then

$$T_jw_j=\exp(2\pi i e_j)\cdot\exp(-2\pi i\beta_j)\cdot w_j=w_j.$$

Hence w_j is also single-valued. Moreover, w_j is the restriction to an appropriate open subset of Z of $w'\circ\mu$, where $w'=\exp(\sum_{i\in S}\log f_i(y)\cdot\alpha_i)\cdot u'$ with f_i a local equation on X for D_i

and u' a local frame for the multi-valued flat sections of $(M, \mathbb{D})$ with respect to a chart (with coordinates y) in X. Hence w' defines, locally in charts on X, a holomorphic frame for M_U on U. We let L to be the line bundle on X obtained by declaring a local frame to be w' and then gluing. The lemma follows. □

Proposition 3.3. *With the notation as in Lemma* 3.1*, the map*

$$\mu_{par}^{*}: \mathrm{Pic}^{\tau}(X, D) \to \mathrm{Pic}^{\tau}(Z, E)$$

given by $(L, \alpha) \mapsto (\mu^* L - \llcorner e \lrcorner \cdot E, \{e\})$ *is an isomorphism, where* $e \in \mathbb{R}^{S'}$ *is given by* $\mu^*(\alpha \cdot D) = e \cdot E$.

Proof. It is clearly an injective group homomorphism. To show that μ_{par}^* is surjective, let $(M^{-1}, \beta) \in \mathrm{Pic}^{\tau}(Z, E)$ for some line bundle M on Z. Let α be defined in terms of β as in Lemma 3.1. By the proof of Theorem 1.2 for the simple normal crossings case, (M^{-1}, β) corresponds under the isomorphism *RH* to the flat connection $(M^{-1}_{|U}, \mathbb{D})$ such that M is the canonical Deligne extension of the dual connection. By Lemma 3.2, $M = \mu^* L \otimes \mathcal{O}_Z(\llcorner e \lrcorner \cdot E)$ for some line bundle L on X. By Lemma 3.1(c), (L^{-1}, α) lies in $\mathrm{Pic}^{\tau}(X, D)$. Thus $\mu_{par}^*(L^{-1}, \alpha) = (M^{-1}, \beta)$. □

Proof of Theorem 1.2 for the general case. It follows by the simple normal crossings case plus Proposition 3.3. □

4. Naturality and structure

Naturality. We show now that the canonical isomorphism *RH* is natural in the sense which we describe below. Let X be a nonsingular complex projective variety, $D = \bigcup_{i \in S} D_i$ a divisor on X with irreducible components D_i, and let $U = X - D$. Consider the long exact sequence of homology with integral coefficients

$$\cdots \to H_2(X) \xrightarrow{\psi} H_2(X, U) \to H_1(U) \xrightarrow{\phi} H_1(X) \to 0, \tag{5}$$

where the maps ψ and ϕ are defined as illustrated. Because S^1 is injective, there is a short exact sequence

$$0 \to \mathrm{Hom}\big(H_1(X), S^1\big) \to \mathrm{Hom}\big(H_1(U), S^1\big) \to \mathrm{Hom}\big(\ker(\phi), S^1\big) \to 0. \tag{6}$$

On the other hand, there is a short exact sequence

$$0 \to \mathrm{Pic}^{\tau}(X) \to \mathrm{Pic}^{\tau}(X, D) \to B(X, D) \to 0, \tag{7}$$

where the injective morphism is $L \mapsto (L, 0)$.

Proposition 4.1. *With the notation as above, the sequences* (6) *and* (7) *are compatible via the isomorphism RH.*

Proof. Let $H_2(X,U)=\bigoplus_{i\in S}\mathbb{Z}[D_i]^\vee$ as in the proof of Lemma 3.1. Let K be the group $\mathrm{Hom}(\ker(\phi),S^1)$. Then K is the kernel of the map $\mathrm{Hom}(H_2(X,U),S^1)\to\mathrm{Hom}(\mathrm{Im}(\psi),S^1)$, hence

$$K=\left\{\alpha\in[0,1)^S\colon \sum_{i\in S}\beta_i\alpha_i\in\mathbb{Z} \text{ whenever } \sum_{i\in S}\beta_i[D_i]^\vee\in\mathrm{Im}(\psi),\ \beta_i\in\mathbb{Z}\right\}.$$

Consider the map $\mathrm{Pic}^\tau(X,D)\to K$ given by $(L,\alpha)\mapsto\alpha$. By Lemma 4.2 below $B(X,D)\subset K$, i.e. this map is well defined. The kernel is $\mathrm{Pic}^\tau(X)$. This gives a morphism (i.e. commuting diagram) from the exact sequence (7) to (6), where the first two vertical maps are RH (the first map is for the case $D=0$), and the last vertical map is the inclusion $B(X,D)\subset K$. By the snake lemma, $B(X,D)\cong K$ canonically, i.e. we obtain a short exact sequence

$$0\to\mathrm{Pic}^\tau(X)\to\mathrm{Pic}^\tau(X,D)\to K\to 0. \qquad \square \tag{8}$$

Lemma 4.2. *With the notation as in Proposition* 4.1*, let $[D_i]$ be the real cohomology classes of the divisors D_i, $[D_i]^\vee$ the canonical generators of the integral homology group $H_2(X,U)$. Let $a=\sum_{i\in S}\alpha_i[D_i]$ be an element in the image of $H^2(X,\mathbb{Z})$ in $H^2(X,\mathbb{R})$, with $\alpha_i\in\mathbb{R}$ (hence, actually $\alpha_i\in\mathbb{Q}$). Let $b=\sum_{i\in S}\beta_i[D_i]^\vee$ be an element in the image of ψ, with $\beta_i\in\mathbb{Z}$. Then $\sum_{i\in S}\beta_i\alpha_i\in\mathbb{Z}$ and depends only on a and b.*

Proof. This is a corollary of the existence of the cap product. Consider the cap product for integral coefficients

$$H^2(X)\otimes H_2(X)\xrightarrow{\cap}H_0(X)\cong\mathbb{Z}.$$

If $t\in H^2(X)$ is a torsion class, then $mt=0$ for some $m\in\mathbb{Z}_{>0}$. Hence $m(t\cap b)$ and $t\cap b$ are 0 for all $b\in H_2(X)$. The group $H^2(X)$ modulo torsion can be identified with the image of $H^2(X)\to H^2(X,\mathbb{R})$. Let A be the free finitely generated subgroup of $H^2(X)/\text{torsion}$ given by the intersection with the subspace V generated by the $[D_i]$ ($i\in S$) in $H^2(X,\mathbb{R})$. Then the cap product induces a bilinear map

$$A\otimes H_2(X)\to\mathbb{Z}.$$

If u is in the image of $H_2(U)\to H_2(X)$ and $a\in A$, then $a\cap u=0$. Hence we have a map

$$A\otimes B\to\mathbb{Z}, \tag{9}$$

where B is the quotient of $H_2(X)$ by the image of $H_2(U)$. From (6), B can be identified with the image of ψ. This proves the statement of the lemma for the case when α_i are integers.

It remains to show that the bilinear map (9) extends to a bilinear map

$$l^{-1}(A)\otimes B\to\mathbb{Z},$$

where $l:\mathbb{R}^S\to V$ is the linear map $\alpha\mapsto\alpha\cdot[D]$. Let W be the subspace $l^{-1}(0)$ of $\mathbb{R}^S$. Then it is enough to show that the pairing induced by linearity between W and B is zero. By (9) this is true for the subgroup $W\cap\mathbb{Z}^S$ in W. The claim follows since $W\cap\mathbb{Z}^S$ is a lattice in W. $\square$

Lemma 4.3. *With the notation as in Proposition* 4.1*, let* $N > 1$ *be an integer. There is a natural isomorphism between the group of characters of* $H_1(U, \mathbb{Z}_N)$ *and the subgroup of* N*-torsion elements of* $\mathrm{Pic}^\tau(X, D)$.

Proof. By *RH*, the subgroup of N-torsion elements of $\mathrm{Pic}^\tau(X, D)$ can be identified with $\mathrm{Hom}(H_1(U), \mu_N)$, where $\mu_N = \mathrm{Hom}(\mathbb{Z}_N, S^1)$. By adjunction of functors, the last group is naturally isomorphic with $\mathrm{Hom}(H_1(U) \otimes_{\mathbb{Z}} \mathbb{Z}_N, S^1)$. $\square$

The realizable boundaries. We consider the set of realizable boundaries $B(X, D)$. By the previous subsection, $B(X, D)$ is the direct sum of a finite group with $(S^1)^{r(U)}$, where $r(U) := b_1(U) - b_1(X)$. Indeed, $B(X, D)$ is canonically isomorphic with $\mathrm{Hom}(\ker(\phi), S^1)$ by Proposition 4.1, and $\ker(\phi)$ is a finitely generated abelian group of rank $r(U)$. Remark that the first Betti number of X, $b_1(X)$, is a birational invariant since it equals twice the Hodge number $h^{1,0}(X)$. By definition, $B(X, D)$ has an inclusion in $\mathbb{R}^S \cap [0, 1)^S$.

Lemma 4.4. *With the notation as in Proposition* 4.1*,* $B(X, D)$ *is a finite disjoint union of rational convex polytopes in* $\mathbb{R}^S$*. More precisely, for the linear map* l *of* $\mathbb{R}^S$ *onto* $\mathbb{R}^{|S|-r(U)}$ *defined in the proof of Lemma* 4.2*, there is a finite set of points* $p_k \in \mathbb{R}^{|S|-r(U)}$ *such that* $B(X, D)$ *is the union of the polytopes* $P_k = l^{-1}(p_k) \cap [0, 1)^S$.

Proof. With the notation of the proof of Lemma 4.2,

$$B(X, D) = l^{-1}(\Lambda) \cap [0, 1)^S,$$

where Λ is the subset of V consisting only of cohomology classes which are realized as first Chern classes of line bundles on X. Then Λ is a free finitely generated abelian group and $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} = V$. That is Λ is a lattice in V. There are only finitely many lattice points in V in the image of $[0, 1)^S$ under l. $\square$

Structure. It follows from the short exact sequence (8) and the construction of the map l of Lemma 4.2 that, topologically, $\mathrm{Pic}^\tau(X, D)$ has a finite disjoint decomposition

$$\mathrm{Pic}^\tau(X, D) = \coprod_k P_k \times \mathrm{Pic}^\tau(X), \tag{10}$$

where P_k are rational convex polytopes in $\mathbb{R}^S \cap [0, 1)^S$. Pointwise, the subset of $\mathrm{Pic}^\tau(X, D)$ corresponding to $P_k \times \mathrm{Pic}^\tau(X)$ consists of realizations $(L_k + M, \alpha)$ of boundaries $\alpha \in P_k$, where $M \in \mathrm{Pic}^\tau(X)$, for some line bundle L_k.

Proposition 4.5. *For each* k*,* L_k *can be chosen such that* (L_k, α_k) *is torsion, for some* $\alpha_k \in P_k$.

Proof. Let $\alpha \in P_k$ be a rational point. It is enough to show that there exists some $(L, \alpha) \in \mathrm{Pic}^\tau(X, D)$ which is torsion. Fix L such that $(L, \alpha) \in \mathrm{Pic}^\tau(X, D)$. It is enough to show that $(L, \alpha) \cdot \mathrm{Pic}^\tau(X)$ contains a torsion element.

From the exponential exact sequence we have the short exact sequence

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}^\tau(X) \to H^2(X, \mathbb{Z})_{tor} \to 0.$$

Hence we can write

$$\mathrm{Pic}^{\tau}(X)=\coprod_{E} E\cdot \mathrm{Pic}^{0}(X),$$

where the disjoint union is taken over a fixed set of representatives for $H^2(X,\mathbb{Z})_{tor}$ in $\mathrm{Pic}^{\tau}(X)$. Thus

$$(L,\alpha)\cdot \mathrm{Pic}^{\tau}(X)=\coprod_{E}(L+E,\alpha)\cdot \mathrm{Pic}^{0}(X).$$

It is enough for our purpose to show that $(L+E,\alpha)\cdot \mathrm{Pic}^0(X)$ contains a torsion element of $\mathrm{Pic}^{\tau}(X,D)$.

Let $g=(L+E,\alpha)$ and $H=\mathrm{Pic}^0(X)$. Since $\alpha\in\mathbb{Q}^S$ and $H^2(X,\mathbb{Z})_{tor}$ is finite, there exists $N>1$ such that $g^N\in H$. Since H is divisible, there exists $h\in H$ such that $h^N=(g^{-1})^N$. Thus $(gh)^N=1$ and gh is torsion. □

Example 4.6. Consider the case of $(\mathbb{P},D)$, where $\mathbb{P}=\mathbb{P}^2$ and D is the union of 5 distinct lines D_i $(i=1,\dots,5)$ intersecting in one point in $\mathbb{P}$. Here $\mathrm{Pic}^{\tau}(\mathbb{P})=\{\mathcal{O}\}$ and $\mathrm{Pic}^{\tau}(\mathbb{P},D)=B(\mathbb{P},D)$. The map l from Lemma 4.4 is $l(\alpha)=(\sum_{i=1,\dots,5}\alpha_i)[D_1]$, $p_k=k[D_1]$. The canonical decomposition of $B(\mathbb{P},D)$ is $\coprod_{k=0,\dots,4}P_k$, where $P_k=[0,1)^5\cap l^{-1}(p_k)$ with $k=0,\dots,4$.

5. Finite abelian covers

The subject of finite abelian covers is well known and has been studied by many people. To our knowledge, the geometric characterization of abelian covers has two versions which appear in their final form in [35] and [36]. The isomorphism of Theorem 1.2 recovers both characterizations and allows us to prove some results which, to our knowledge, appear in their published version only under some additional hypotheses.

Let X be a nonsingular complex projective variety, $D=\bigcup_{i\in S}D_i$ a divisor on X with irreducible components D_i, and let $U=X-D$. Let G be a finite abelian group.

Definition 5.1. A map $Y\to X$ is a *G-cover* if it is a finite map together with a faithful action of G on Y such that the map exhibits X as the quotient of Y via G. Two covers are said two be equivalent if there is an isomorphism between them commuting with the cover maps.

Recall the following topological characterization (see [51, Appendix 1 to Chapter VIII], or [35]). The morphisms of $H_1(U)$ onto G are in one-to-one correspondence with the equivalence classes of unramified G-covers of U. These, in turn, are in one-to-one correspondence with equivalence classes of normal G-covers of X unramified above U. The group G is recovered as the group of automorphisms of the cover commuting with the cover map.

Proof of Corollary 1.10. By the topological characterization and by the canonical isomorphism RH, it is enough to show there is a one-to-one equivalence between surjections $H_1(U)\to G$ and subgroups $G^*\subset \mathrm{Hom}(H_1(U),S^1)$. This is a standard exercise in duality. □

Remark 5.2. It is well known that the local systems corresponding to $G^* \subset \mathrm{Hom}(H_1(U), S^1)$ in Corollary 1.10 can also be obtained as follows. Let $\pi_U : V \to U$ be the unramified cover induced by a surjection $H_1(U) \to G$. Then $(\pi_U)_* \mathbb{C}_V$ is a higher rank local system on U which decomposes into rank one unitary local systems $\mathcal{V}_\chi$ ($\chi \in G^*$). Moreover, there is an eigensheaf decomposition of $(\pi_U)_* \mathcal{O}_V$ into line bundles $\mathcal{M}_\chi$ ($\chi \in G^*$), and a natural derivative ∇ such that $(\mathcal{M}_\chi, \nabla)$ is the flat connection corresponding to $\mathcal{V}_\chi$.

Recovering the geometric characterization of [36]. Let X, D, U, and G be as above. Pardini [36] gives the following geometric characterization of finite abelian covers which also follows from Theorem 1.2.

Definition 5.3. The data $\{(L_\chi, H_i, \psi_i)\colon\ \chi \in G^*,\ i \in S\}$ is called *building data on X with branch D* if H_i is a cyclic subgroup of G, ψ is a generator of H_i^*, L_χ are line bundles satisfying the linear relation

$$L_\chi + L_{\chi'} = L_{\chi\chi'} + \varepsilon_{\chi,\chi'} \cdot D, \tag{11}$$

where $\varepsilon_{\chi,\chi'} \in \{0,1\}^S$ with

$$(\varepsilon_{\chi,\chi'})_i = \begin{cases} 0 & \text{if } \iota_{\chi,i} + \iota_{\chi',i} < m_i, \\ 1 & \text{otherwise.} \end{cases}$$

Here m_i is the order of H_i, and $\iota_{\chi,i} \in \{0,\dots,m_i - 1\}$ is given by $\chi_{|H_i} = \psi_i^{\iota_{\chi,i}}$.

Corollary 5.4. *There is a one-to-one correspondence between the set of equivalence classes of normal G-covers of X unramified above U and building data on X with branch D. This correspondence is the same as the one of* [36].

Pardini's correspondence is as follows. Suppose we start with a normal G-cover $\pi : Y \to X$ unramified above D. Then there is an eigensheaf decomposition

$$\pi_* \mathcal{O}_Y = \bigoplus_{\chi \in G^*} L_\chi^{-1}.$$

Let H_i be the inertia group of (any = all) components T of $\pi^{-1}(D)$. Let ψ_i be the representation of H_i on $\mathbf{m}/\mathbf{m}^2$ induced by the cotangent map, where $\mathbf{m}$ is the maximal ideal of the local ring $\mathcal{O}_{Y,T}$. Then (L_χ, H_i, ψ_i) is the corresponding building data. Conversely, suppose we start with building data (L_χ, H_i, ψ_i). Define the $\mathcal{O}_X$-linear multiplication maps $\mu_{\chi,\chi'} : L_\chi^{-1} \otimes L_{\chi'}^{-1} \to L_{\chi\chi'}^{-1}$ by setting

$$\mu_{\chi,\chi'} = \prod_{i \in S} \sigma_i^{(\varepsilon_{\chi,\chi'})_i},$$

viewed as a global section of $L_\chi \otimes L_{\chi'} \otimes L_{\chi\chi'}^{-1}$, where σ_i are sections of $\mathcal{O}_X(D_i)$ vanishing on D_i. The corresponding G-cover is then defined (up to equivalence due to freedom of rearranging the χ's around) as

$$Y = \mathbf{Spec}_{\mathcal{O}_X}\left(\bigoplus_{\chi\in G^*} L_\chi^{-1}\right). \tag{12}$$

Proof of Corollary 5.4. By Corollary 1.10, the normal G-covers of X unramified above U up to equivalence are into one-to-one correspondence with subgroups $G^* \subset \mathrm{Pic}^\tau(X, D)$. Start with $G^* = \{(L_\chi, \alpha_\chi)\colon \chi \in G^*\} \subset \mathrm{Pic}^\tau(X, D)$, corresponding to the G-cover $\pi : Y \to X$ and to the epimorphism $\rho : H_1(U) \to G$. To this we attach building data (L_χ, H_i, ψ_i) as follows.

Let δ_i be homology class of a small loop in U centered at a general point of D_i. Let H_i be the cyclic subgroup of G generated by $\rho(\delta_i)$. Let ψ_i be the character of H_i taking $\rho(\delta_i)$ to $\exp(i2\pi(1/m_i))$, where m_i is the order of H_i. Then (L_χ, H_i, ψ_i) is a building data. Indeed, (L_χ, α_χ) corresponds to the local system given by $\chi \circ \rho$. Hence, by the description of the isomorphism RH, χ sends $\rho(\delta_i)$ to $\exp(i2\pi\alpha_{\chi,i})$. Thus $\iota_{\chi,i} = m_i\alpha_{\chi,i}$ and the linear relation (11) follows from the group operation on $\mathrm{Pic}^\tau(X, D)$.

Let $\pi' : Y' \to X$ be the G-cover corresponding to the building data (L_χ, H_i, ψ_i) via Pardini's correspondence. It is enough to show that π' is equivalent to π. It is actually enough to show that the corresponding unramified covers above U are equivalent. This follows from Remark 5.2 and the explicit description of the isomorphism RH. $\square$

Proof of Corollary 1.11. It follows from Corollary 5.4 and (12). $\square$

Proof of Corollary 1.12. Let $E = Z - U$ and E_j $(j \in S')$ be its irreducible components. We have obtained in Proposition 3.3 that the map

$$\mu_{par}^* : \mathrm{Pic}^\tau(X, D) \to \mathrm{Pic}^\tau(Z, E)$$

given by $(L, \alpha) \mapsto (\mu^* L - \llcorner e \lrcorner \cdot E, \{e\})$ is an isomorphism, where $e \in \mathbb{R}^{S'}$ is given by $\mu^*(\alpha \cdot D) = e \cdot E$. Therefore Corollary 1.12 follows from Corollaries 1.10 and 1.11. $\square$

Proof of Corollary 1.13. The space of global q-forms is a birational invariant of nonsingular projective varieties. Hence we can choose any desingularization Y of $\widetilde{X}$. Let Y and maps $\eta : Y \to \widetilde{X}$, $\nu : Y \to \widetilde{Z}$ be a common G-equivariant desingularization [1,39] of $\widetilde{X}$ and $\widetilde{Z}$, so that we have a commutative diagram

$$\begin{tikzcd} & Y \arrow[dl, "\nu"'] \arrow[dr, "\eta"] \arrow[ddr, bend left, "f"] & \\ \widetilde{Z} \arrow[d, "\rho"'] & & \widetilde{X} \arrow[d, "\pi"] \\ Z \arrow[rr, "\mu"] & & X \end{tikzcd}$$

where f is the induced map from Y to X. Corollary 1.13 follows from Proposition 5.5 for the case when $M = \mathcal{O}_X$ and by Hodge duality. $\square$

Proposition 5.5. *With the notation as in the proof of Corollary* 1.13*, let M be a line bundle on X. Let $\chi \in G^*$ correspond to $(L_\chi, \alpha_\chi) \in \operatorname{Pic}^\tau(X, D)$. Then*

$$H^q\big(Y, f^*M^{-1}\big)_\chi \cong H^{n-q}\big(X, \omega_X \otimes M \otimes L_\chi \otimes \mathcal{J}(\alpha_\chi \cdot D)\big),$$

where the subscript χ denotes the χ-eigenspace and n is the dimension of X.

Proof. By [14, Lemma 3.24], $\widetilde{Z}$ has quotient singularities. Hence, by [50, Proposition 1], $\widetilde{Z}$ has rational singularities. That is $\nu_*\mathcal{O}_Y = \mathcal{O}_{\widetilde{Z}}$ and $R^i\nu_*\mathcal{O}_Y = 0$ for $i > 0$. Then, by the projection formula [19, Ex. III.8.3], $\nu_*(f^*M^{-1}) = \rho^*\mu^*M^{-1}$ and $R^i\nu_*(f^*M^{-1}) = 0$ for $i > 0$. Hence, by the equivariant version of the Leray spectral sequence [23, Proposition B.1.1], there is a G-equivariant isomorphism

$$H^q\big(Y, f^*M^{-1}\big) \cong H^q\big(\widetilde{Z}, \rho^*\mu^*M^{-1}\big)$$

for all q. Now, ρ is a finite morphism, so $R^i\rho_*\mathcal{O}_{\widetilde{Z}} = 0$ for $i > 0$. By the projection formula again, $R^i\rho_*(\rho^*\mu^*M^{-1})$ vanishes for $i > 0$ and is equal to $\rho_*\mathcal{O}_{\widetilde{Z}} \otimes \mu^*M^{-1}$ for $i = 0$. Then the equivariant version of [23, Proposition B.1.1] gives again a G-equivariant isomorphism

$$H^q\big(\widetilde{Z}, \rho^*\mu^*M^{-1}\big) \cong H^q\big(Z, \rho_*\mathcal{O}_{\widetilde{Z}} \otimes \mu^*M^{-1}\big)$$

for all q. Hence, by Corollary 1.12, we have an isomorphism

$$H^q\big(\widetilde{Z}, \rho^*\mu^*M^{-1}\big)_\chi \cong H^q\big(Z, \mu^*\big(M^{-1} \otimes L_\chi^{-1}\big) \otimes \mathcal{O}_Z\big(\llcorner \mu^*(\alpha_\chi \cdot D) \lrcorner\big)\big) = \star.$$

By Serre duality on Z,

$$\star \cong H^{n-q}\big(Z, \omega_Z \otimes \mu^*(M \otimes L_\chi) \otimes \mathcal{O}_Z\big(-\llcorner \mu^*(\alpha_\chi \cdot D) \lrcorner\big)\big).$$

By Theorem 2.2, we are again in the situation of [23, Proposition B.1.1] and obtain

$$\star \cong H^{n-q}\big(X, \omega_X \otimes M \otimes L_\chi \otimes \mathcal{J}(\alpha_\chi \cdot D)\big). \qquad \square$$

6. Canonical stratifications

To prove Theorem 1.3 we use the following version due to C. Simpson [45] which builds on a previous result of Green and Lazarsfeld [17] and D. Arapura [3].

Theorem 6.1 *(Simpson). Let $f : Y \to X$ be a morphism between two nonsingular complex projective varieties. Let G be a finite abelian group acting by automorphisms on Y, trivially on X, and suppose f is G-equivariant. Fix $\chi \in G^*$ and integers q, i. Then the set*

$$V_i^q(f, \chi) := \big\{M \in \operatorname{Pic}^\tau(X)\colon\ h^q(Y, f^*M)_\chi \geqslant i\big\}$$

is a finite union of torsion translates of complex subtori of $\operatorname{Pic}^\tau(X)$*, and so is any intersection of these translates.*

This is the "Higgs field equals zero" case (see [45, Section 5]) of a stronger equivariant version which is not explicitly stated in [45] but which follows straight-forwardly from the results there. Indeed, one only needs the equivariant version of [45, Proposition 7.9] to construct the appropriate absolute functor (in the terminology of [45]) and, thus, the appropriate absolute closed subset of the moduli space of rank one local systems on X. By the definition in [45, Section 6], intersections of absolute closed subsets are again absolute closed subsets. Then [45, Theorem 6.1] applies and the restriction to $\mathrm{Pic}^\tau(X)$ gives Theorem 6.1 as stated. See also Theorem 8.3. For sake of completeness, we give a less conceptual proof of Theorem 6.1 based on the trick of [3] (see also [45, Section 5]) where we replace Hodge decomposition by eigenspace decomposition.

Proof of Theorem 6.1. Let $V_i^q(f)=\{M\in \mathrm{Pic}^\tau(X)\colon\ h^q(Y,f^*M)\geqslant i\}$. By [45, Section 5], the conclusion of the theorem holds for $V_i^q(f)$. Since $h^q(Y,f^*M)=\sum_{\chi\in G^*}h^q(Y,f^*M)_\chi$, one has the equality

$$V_i^q(f)=\bigcup_{\substack{P:G^*\to\mathbb{N}\\ \sum_\chi P(\chi)=i}}\Bigl[\bigcap_{\chi\in G^*}V^q_{P(\chi)}(f,\chi)\Bigr].$$

Let V be an irreducible component of the closed subvariety $V_i^q(f,\chi)$ of $\mathrm{Pic}^\tau(X)$. To finish the proof, it is enough to show that V is an irreducible component of a set of the type $V_I^q(f)$ for some I. Define a function $P:G^*\to\mathbb{N}$, dependent on q, by $P(\psi)=\max\{j\mid V\subset V_j^q(f,\psi)\}$. Then V is an irreducible component of $\bigcap_{\psi\in G^*}V^q_{P(\psi)}(f,\psi)$ since $V^q_{P(\chi)}(f,\chi)\subset V_i^q(f,\chi)$. But V is not included in $\bigcap_{\psi\in G^*}V^q_{P'(\psi)}(f,\psi)$ for any other function $P':G^*\to\mathbb{N}$ with $\sum_\psi P'(\psi)=\sum_\psi P(\psi)=:I$. Hence V is an irreducible component of $V_I^q(f)$. $\square$

Lemma 6.2. *Let X be a nonsingular complex projective variety, $D=\bigcup_{i\in S}D_i$ a divisor on X with irreducible components D_i. Let (L,α) be a torsion element in* $\mathrm{Pic}^\tau(X,D)$. *Then the set*

$$V_i^q(L,\alpha):=\bigl\{M\in \mathrm{Pic}^\tau(X)\colon\ h^q\bigl(X,\omega_X\otimes M\otimes L\otimes\mathcal{J}(\alpha\cdot D)\bigr)\geqslant i\bigr\}$$

is a finite union of torsion translates of complex subtori of $\mathrm{Pic}^\tau(X)$, *and so is any intersection of these translates.*

Proof. Choose any finite group G with an embedding $G^*\subset \mathrm{Pic}^\tau(X,D)$ (hence G is abelian) such that $(L,\alpha)\in G^*$. Let $\chi\in G^*$ be the character corresponding to (L,α) and construct f as in Proposition 5.5. By Proposition 5.5,

$$V_i^q(L,\alpha)=\bigl\{M^{-1}\colon\ M\in V_i^{n-q}(f,\chi)\bigr\}.$$

The lemma follows then from Theorem 6.1. $\square$

Let X be a nonsingular complex projective variety of dimension n, $D=\bigcup_{i\in S}D_i$ a divisor on X with irreducible components D_i, and let $U=X-D$. Let G be a finite abelian group. Define

$$V_i^q(X,D):=\bigl\{(L,\alpha)\in \mathrm{Pic}^\tau(X,D)\colon\ h^q\bigl(X,\omega_X\otimes L\otimes\mathcal{J}(\alpha\cdot D)\bigr)\geqslant i\bigr\},$$

where $\mathcal{J}(\alpha \cdot D))$ denotes the multiplier ideal of the $\mathbb{R}$-divisor $\alpha \cdot D$. Whenever the context leaves no room for ambiguity, we will just write V_i^q instead of $V_i^q(X, D)$.

Lemma 6.3. *With the notation as above, the set $V_i^q(U)$ of unitary rank one local systems on U corresponding to $V_i^q(X, D)$ under the isomorphism RH depends only on U and not on (X, D).*

Proof. Let $\mu : (Z, E) \to (X, D)$ be a log-resolution which is an isomorphism above U, where $E = \bigcup_{j \in S'} E_j$ is the inverse image $\mu^{-1}(D)$. The map

$$\mu_{par}^* : \mathrm{Pic}^\tau(X, D) \to \mathrm{Pic}^\tau(Z, E)$$

given by $(L, \alpha) \mapsto (\mu^* L - \llcorner e \lrcorner \cdot E, \{e\})$ is an isomorphism, where $e \in \mathbb{R}^{S'}$ is given by $\mu^*(\alpha \cdot D) = e \cdot E$. By Theorem 2.2, $R^j \mu_*(\omega_{Y/X} \otimes \mathcal{O}_Y(-\llcorner \mu^*(\alpha \cdot D) \lrcorner)) = 0$ for $j > 0$, and it follows that under the map μ_{par}^* the sets $V_i^q(X, D)$ and $V_i^q(Z, E)$ are into one-to-one equivalence. Since the map μ_{par}^* commutes with the isomorphisms RH, $V_i^q(X, D)$ and $V_i^q(Z, E)$ induce the same subset of the space of unitary rank one local systems on U. The case of two different divisorial compactifications of U is reduced to the above case by considering a common log resolution. $\square$

We can actually describe $V_i^q(U)$ purely in terms of local systems. For a unitary local system $\mathcal{V}$ on U, let F be the Hodge filtration on $H^*(U, \mathcal{V})$ constructed by Timmerscheidt [49].

Proposition 6.4. *With the notation as in Lemma* 6.3*,*

$$V_i^q(U) = \bigl\{\mathcal{V} \in \mathrm{Hom}\bigl(H_1(U), S^1\bigr) \bigm| \dim \mathrm{Gr}_F^0 H^{n-q}\bigl(U, \mathcal{V}^\vee\bigr) \geqslant i\bigr\}.$$

Proof. By Lemma 6.3, we may assume $U = X - D$ with D having simple normal crossings. Let $\mathcal{V} \in \mathrm{Hom}(H_1(U), S^1)$ and let $(L, \alpha) \in \mathrm{Pic}^\tau(X, D)$ be the corresponding realization of boundary. Recall from the proof of Theorem 1.2 that L^{-1} is the line bundle of the canonical Deligne extension to X of the connection on U given by the dual local system $\mathcal{V}^\vee$. We have by [49, 2nd Theorem, part (a)],

$$\dim \mathrm{Gr}_F^0 H^{n-q}\bigl(U, \mathcal{V}^\vee\bigr) = \dim H^{n-q}\bigl(X, \Omega_X^0(\log D) \otimes L^{-1}\bigr) = \star,$$

where $\Omega_X^p(\log D)$ denotes the sheaf of p-differential forms with logarithmic poles along D, so that $\Omega_X^0(\log D) = \mathcal{O}_X$ by definition. Then, by Serre duality,

$$\star = \dim H^q(X, \omega_X \otimes L)$$

and this last space is the same as $H^q(X, \omega_X \otimes L \otimes \mathcal{J}(\alpha \cdot D))$ since $\mathcal{J}(\alpha \cdot D) = \mathcal{O}_X$ in this case. $\square$

Proof of Theorem 1.3. From (10) and Proposition 4.5, we can write V_i^q as a disjoint union

$$\begin{aligned} V_i^q &= \coprod_k V_i^q \cap \bigl[P_k \times \mathrm{Pic}^\tau(X)\bigr] \\ &= \coprod_k V_i^q \cap \bigl[\bigl\{(L_k, \alpha)\colon \alpha \in P_k\bigr\} \cdot \mathrm{Pic}^\tau(X)\bigr], \end{aligned}$$

where P_k are rational convex polytopes in $\mathbb{R}^S$, and (L_k, α_k) is torsion in $\mathrm{Pic}^\tau(X, D)$ for some $\alpha_k \in P_k$.

The simple normal crossings case. Assume now that D has simple normal crossings. Then $V_i^q(L_k, \alpha_k) = V_i^q(L_k, \alpha)$ for all $\alpha \in P_k$ since $\mathcal{J}(\alpha \cdot D) = \mathcal{O}_X$. We will call this set $V_i^q(L_k)$. Hence,

$$V_i^q = \coprod_k P_k \times V_i^q(L_k) = \coprod_k \{(L_k, \alpha)\colon \alpha \in P_k\} \cdot V_i^q(L_k).$$

Then Theorem 1.3 follows in this case from Lemma 6.2.

General case. Assume now that (X, D) is as in the statement of the theorem. Let $\mu : (Z, E) \to (X, D)$ be a log-resolution which is an isomorphism above U, where $E = \bigcup_{j \in S'} E_j$ is the inverse image $\mu^{-1}(D)$. By above, $V_i^q(Z, E)$ is a finite union of sets of the form $P' \times \mathcal{U}'$, where $\bigcup_{P'} P'$ is the canonical decomposition of $B(Z, E)$ into rational convex polytopes in $\mathbb{R}^{S'}$ from Lemma 4.4, and $\mathcal{U}' \subset \mathrm{Pic}^\tau(Z, E)$ is a torsion translate of a complex subtorus of $\mathrm{Pic}^\tau(Z)$. By Lemma 6.3, there is a one-to-one correspondence between $V_i^q(Z, E)$ and $V_i^q(X, D)$ under the isomorphism of Proposition 3.3. The inverse image under this correspondence of $P' \times \mathcal{U}'$ is $P \times \mathcal{U}$ obtained as follows. P is the image of P' under the projection of $\mathbb{R}^{S'}$ onto $\mathbb{R}^S$ given by the coefficients of the strict transforms of the D_i. Conversely, a realizable boundary α of X on D maps to the realizable boundary $\{e\}$ of Z on E, where $e \cdot E = \mu^*(\alpha \cdot D)$. Hence $\bigcup_P P$ form a decomposition of $B(X, D)$ into rational convex polytopes which is a refinement of the canonical decomposition. Under the isomorphism μ_{par}^*, subtori of $\mathrm{Pic}^\tau(Z)$ correspond to subtori of $\mathrm{Pic}^\tau(X)$ since $\mathrm{Pic}^\tau(Z)$ consists of finitely many copies of $\mathrm{Pic}^\tau(X)$. Similarly, torsion elements in $\mathrm{Pic}^\tau(Z, E)$ also correspond to torsion elements in $\mathrm{Pic}^\tau(X, D)$. Therefore $\mathcal{U}'$ corresponds to $\mathcal{U}$ which is a torsion translate of a subtorus in $\mathrm{Pic}^\tau(X)$. □

Lemma 6.5. *With the same notation is in Theorem* 1.3*, fix q. Then there exists i_0 such that $i > i_0$ implies $V_i^q = \emptyset$.*

Proof. It is enough to restrict to the case when D has simple normal crossings. For a fixed k, it is enough to show that $V_i^q(L_k) = \emptyset$ for $i \gg 0$. But $V_i^q(L_k)$ are closed subsets in the Zariski topology of $\mathrm{Pic}^\tau(X)$ and $V_i^q(L_k) \supset V_{i+1}^q(L_k)$. Hence $V_i^q(L_k) = \emptyset$ for $i \gg 0$ and this proves the lemma. □

Example 6.6. (a) Consider the case of Example 4.6. Let $\mu : (\widetilde{\mathbb{P}}, E) \to (\mathbb{P}, D)$ be the log resolution given by the blow-up of the intersection point of the D_i's. Let E_i be the strict transform of D_i $(i = 1, \dots, 5)$, and E_6 be the exceptional divisor. The decomposition of $B(\widetilde{\mathbb{P}}, E)$ is $\coprod_{k=0,\dots,4} Q_k$, where $Q_k = [0, 1)^6 \cap l_{\widetilde{\mathbb{P}}}^{-1}(q_k)$ with $l_{\widetilde{\mathbb{P}}}(e) = (\sum_{i=1,\dots,5} e_i)[\mu^*\mathcal{O}_{\mathbb{P}}(1)] + (e_6 - \sum_{i=1,\dots,5} e_i)[E_6]$, $q_k = [L_k]$, $L_k = \mu^*\mathcal{O}_{\mathbb{P}}(k) \otimes \mathcal{O}_{\widetilde{\mathbb{P}}}(-kE_6)$, for $k = 0, \dots, 4$. This induces a decomposition of $B(\mathbb{P}, D)$ via the projection $\mathbb{R}^6 \to \mathbb{R}^5$ given by $e \mapsto (e_1, e_2, e_3, e_4, e_5)$. In this case, the induced decomposition on $B(\mathbb{P}, D)$ is the same as the canonical one, the image of Q_k is P_k. To check for membership of P_k in V_i^q we need $h^q(\widetilde{\mathbb{P}}, \omega_{\widetilde{\mathbb{P}}} \otimes L_k) \geqslant i$. We have $h^q(\widetilde{\mathbb{P}}, \omega_{\widetilde{\mathbb{P}}} \otimes L_k) = h^q(\widetilde{\mathbb{P}}, \mu^*\mathcal{O}_{\mathbb{P}}(k - 3) \otimes \mathcal{O}_{\widetilde{\mathbb{P}}}((1 - k)E_6))$ and we can check that the non-trivial V_i^q's are $V_1^1 = P_2 \cup P_3 \cup P_4$, $V_2^1 = P_3 \cup P_4$, $V_3^1 = P_4$, $V_1^2 = P_0$.

(b) This basic example can be computed by other means (see also [26, Example 1 in Section 5]). We included it to show the basic steps of the method of this article of obtaining the structure of V_i^q: firstly resolve singularities, secondly get a pool of candidate polytopes from the map l, lastly check which of the line bundles attached to these polytopes satisfy the conditions of V_i^q. This method is different than the method of [25,27,28] which firstly computes the global polytopes of quasiadjunction and then checks for them the conditions imposed by V_i^q. For example, P_1 is not a global polytope of quasiadjunction since the multiplier ideal sheaf $\mathcal{J}(\alpha \cdot D) = \mathcal{O}_{\mathbb{P}}$ for $\alpha \in P_1$. We do not know how to adapt the method of polytopes of quasiadjunction to prove Theorem 1.3.

7. Congruence covers and polynomial periodicity

In this section we prove Theorem 1.8. First we reduce the question to computing torsion in the canonical stratifications of Theorem 1.3. Then we use the structure of these sets described in Theorem 1.3 to further reduce the question to computing lattice points in convex rational polytopes.

Reduction to torsion in canonical stratifications. With the notation as in Theorem 1.8, denote by G_N the finite group $H_1(U, \mathbb{Z}_N)$. Denote by $\operatorname{Pic}^\tau(X, D)[N]$ the N-torsion part of $\operatorname{Pic}^\tau(X, D)$ and by V_i^q the sets $V_i^q(X, D)$. By Corollary 1.13 and Lemma 4.3,

$$\begin{aligned} h^q(N) &= \sum_{\chi \in G_N^*} h^{n-q}\big(X, \omega_X \otimes L_\chi \otimes \mathcal{J}(\alpha_\chi \cdot D)\big) \\ &= \sum_{(L,\alpha)\in \operatorname{Pic}^\tau(X,D)[N]} h^{n-q}\big(X, \omega_X \otimes L \otimes \mathcal{J}(\alpha \cdot D)\big) \\ &= \sum_{i \geqslant 1} i \cdot \#\big[\big(V_i^{n-q} - V_{i+1}^{n-q}\big)[N]\big], \end{aligned}$$

where $\#S$ denotes the number of elements of a finite set S, and $S[N]$ denotes the set of N-torsion elements of $\operatorname{Pic}^\tau(X, D)$ lying in the subset S. Since $V_{i+1}^q \subset V_i^q$,

$$h^q(N) = \sum_{i \geqslant 1} \# V_i^{n-q}[N].$$

By Lemma 6.5, this is a finite sum. By Theorem 1.3 and the inclusion–exclusion formula,

$$h^q(N) = \sum_j a_j \cdot \#\big[(\mathcal{P}_j \cdot T_j \cdot \mathcal{T}_j)[N]\big],$$

where the sum is finite, $a_j \in \mathbb{Z}$, $T_j \in \operatorname{Pic}^\tau(X, D)$ is torsion, $\mathcal{T}_j$ are subtori of $\operatorname{Pic}^\tau(X)$, and

$$\mathcal{P}_j = \big\{(L_j, \alpha) \in \operatorname{Pic}^\tau(X, D)\colon\ \alpha \in P_j\big\},$$

for some L_j, where P_j is a rational convex polytope in $\mathbb{R}^S \cap [0, 1)^S$. The data $(a_j, L_j, P_j, T_j, \mathcal{T}_j)$ depends on q but not on N. Therefore Theorem 1.8 follows from the assertion that the functions

$$\#\big[(\mathcal{P} \cdot T \cdot \mathcal{T})[N]\big]$$

are quasi-polynomials in N, where $(\mathcal{P}, T, \mathcal{T})$ is $(\mathcal{P}_j, T_j, \mathcal{T}_j)$ for some j. We will show that this assertion is equivalent to one in terms of lattice points in rational convex polytopes.

Reduction to lattice points in polytopes. Fix L, P, T, and $\mathcal{T}$ as above. The N-torsion elements of $\mathcal{P} \cdot T \cdot \mathcal{T}$ consist of realizations of boundaries $(L \otimes T \otimes M, \alpha)$ in $\mathrm{Pic}^\tau(X, D)$ such that $\alpha \in P$, $M \in \mathcal{T}$, $N\alpha \in \mathbb{Z}^S$, and

$$\mathcal{O}_X(N\alpha \cdot D) \otimes (L \otimes T)^{\otimes -N} = M^{\otimes -N}.$$

Since $\mathcal{T}$ is a torus,

$$\#\big[(\mathcal{P} \cdot T \cdot \mathcal{T})[N]\big] = N^{\mathrm{rank}(\mathcal{T})} \cdot \#\mathcal{A}_N,$$

where

$$\mathcal{A}_N = \big\{\alpha \in \mathbb{R}^S\colon\ \alpha \in P,\ N\alpha \in \mathbb{Z}^S,\ \mathcal{O}_X(N\alpha \cdot D) \otimes (L \otimes T)^{\otimes -N} \in \mathcal{T}\big\}.$$

We will show that $\#\mathcal{A}_N$ is a quasi-polynomial in N.

Let β be a point with integral coordinates in the $\mathbb{R}$-span of P. It follows from Proposition 4.5 that we can choose L such that some multiple of L is linearly equivalent to the same multiple of $\beta \cdot D$. Let $\mathcal{L} = \mathcal{O}_X(L + T - \beta \cdot D)$, so that $\mathcal{L}$ is a torsion line bundle. Let Q be the convex integral polytope $P - \beta$ in $\mathbb{R}^S$, which might be missing some of its faces if P does. Via the transformation $\alpha \mapsto \alpha - \beta$ there is a one-to-one correspondence of $\mathcal{A}_N$ with

$$\mathcal{B}_N = \big\{\alpha \in \mathbb{R}^S\colon\ \alpha \in Q,\ N\alpha \in \mathbb{Z}^S,\ \mathcal{O}_X(N\alpha \cdot D) \in \mathcal{L}^{\otimes N} \cdot \mathcal{T}\big\}.$$

We will show that $\#\mathcal{B}_N$ is a quasi-polynomial in N.

Let $W \subset \mathbb{R}^S$ be the subspace consisting of α such that $\alpha \cdot [D] = 0$ in $H^2(X, \mathbb{R})$. We can write

$$\mathcal{B}_N = \frac{1}{N}\rho^{-1}\big(\mathcal{L}^{\otimes N} \cdot \mathcal{T}\big) \cap W \cap Q,$$

where $\rho : \mathbb{Z}^S \to \mathrm{Pic}(X)$ is the group homomorphism sending α to $\mathcal{O}_X(\alpha \cdot D)$. The assertion that $\#\mathcal{B}_N$ is a quasi-polynomial follows from Lemma 7.1 below. This ends the proof of Theorem 1.8.

In what follows we do not require that a convex polytope contains all its faces.

Lemma 7.1. *Let $\mathcal{G}$ be an abelian group and $\rho : \mathbb{Z}^S \to \mathcal{G}$ a group homomorphism. Let $Q \subset \mathbb{R}^S$ be a convex integral polytope. Let $\mathcal{T}$ be a subgroup of $\mathcal{G}$. Let $g \in \mathcal{G}$ be a torsion point. Let W be a vector subspace of $\mathbb{R}^S$. Then, for $N \in \mathbb{N}$, the function*

$$F(N) = \#\left[\frac{1}{N}\rho^{-1}(Ng + \mathcal{T}) \cap W \cap Q\right]$$

is a quasi-polynomial.

Proof. Denote by Λ the inverse image under ρ of $\mathcal{T}$. Then Λ is a free finitely generated abelian subgroup of $\mathbb{Z}^S$. Fix $w \in \rho^{-1}(g + \mathcal{T})$. Then for every $N \geqslant 0$

$$\rho^{-1}(Ng + \mathcal{T}) = Nw + \Lambda.$$

Moreover, since g is torsion, a multiple of w lies in Λ. Let $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$.

Let $\Lambda' = \Lambda \cap W$ and $V' = V \cap W$. Thus Λ' is a lattice in V'. Fix $w' \in (w + \Lambda) \cap W$. Hence w' is Λ'-rational point of V'. Then

$$(Nw + \Lambda) \cap W = Nw' + \Lambda',$$

for $N \geqslant 0$. Let $Q' = Q \cap V'$, so that Q is a Λ'-rational convex polytope in V'. Then

$$\begin{aligned} F(N) &= \#\left[\frac{1}{N}(Nw' + \Lambda') \cap Q'\right] \\ &= \#\left[\Lambda' \cap N(Q' - w')\right]. \end{aligned}$$

Since $(Q' - w')$ is a Λ'-rational convex polytope in V', the lemma follows from Theorem 7.2. □

Theorem 7.2. *(E. Ehrhart, see [46, 4.6.25].) Let Q be a convex rational polytope in $\mathbb{R}^n$. Then, for $N \in \mathbb{N}$, the function*

$$f(N) = \#\left[\mathbb{Z}^n \cap NQ\right]$$

is a quasi-polynomial in N.

The result above is due to E. Ehrhart and is originally stated assuming the polytope is closed. The case of polytopes missing some faces follows since the sum and the difference of two quasi-polynomials is again a quasi-polynomial.

8. Generalizations

We show in this section how the proofs for Theorems 1.3 and 1.8 extend to give Theorems 1.4 and 1.9. Let U be a nonsingular quasi-projective variety. Let G be a finite abelian group with a surjection $H_1(U) \to G$. Let $g : V \to U$ be the corresponding unramified abelian cover.

Lemma 8.1. *With the notation as above, let $\mathcal{W}$ be a unitary local system on U.*

(a) *There is an eigenspace decomposition*

$$H^m(V, g^*\mathcal{W}) = \bigoplus_{\chi \in G^*} H^m(U, \mathcal{W} \otimes \mathcal{V}_\chi),$$

where $\mathcal{V}_\chi$ is the rank one unitary local system induced by χ.

(b) *The decomposition above is compatible with the Hodge filtration, i.e. there is an eigenspace decomposition*

$$\mathrm{Gr}_F^p H^m(V, g^*\mathcal{W}) = \bigoplus_{\chi \in G^*} \mathrm{Gr}_F^p H^m(U, \mathcal{V}_\chi \otimes \mathcal{W}).$$

Proof. (a) Since g is finite, $H^m(V, g^*\mathcal{W}) = H^m(U, g_*g^*\mathcal{W})$ which, by the projection formula, is isomorphic to $H^m(U, \mathcal{W} \otimes g_*\mathbb{C}_V)$. The claim follows since $g_*\mathbb{C}_V = \bigoplus_{\chi \in G^*} \mathcal{V}_\chi$ (see Remark 5.2).

(b) Follows from (a) by the G-equivariant version of functoriality for the mixed Hodge structure on the cohomology of unitary local systems (see [49]). $\square$

Let X be a nonsingular compactification of U with complement $D = \bigcup_{i \in S} D_i$ a divisor with simple normal crossings. For a unitary local system $\mathcal{W}$ on U, denote from now on by $\overline{\mathcal{W}}$ the vector bundle of the canonical Deligne extension of $\mathcal{W}$ to X.

Remark 8.2. (a) Let $\mathcal{V}$ be a unitary local system of rank one on U and let $(L, \alpha) \in \mathrm{Pic}^\tau(X, D)$ be the corresponding realization of boundary. The inverse of (L, α) in $\mathrm{Pic}^\tau(X, D)$ is (M, β), where $M = L^{-1} + \sum_{\alpha_i \neq 0} D_i$, and β_i is 0 if $\alpha_i = 0$ and $1 - \alpha_i$ otherwise. By the proof of Theorem 1.2, $\overline{\mathcal{V}} = M^{-1}$, hence $\overline{\mathcal{V}} = L - \sum_{\alpha_i \neq 0} D_i$.

(b) In general, the canonical extension is not compatible with $\otimes$. For example, if $\mathcal{V}$ and $\mathcal{W}$ are two rank one unitary local systems on U with corresponding realization of boundaries (L, α), respectively (M, β), then

$$\overline{\mathcal{V} \otimes \mathcal{W}} = L + M - \llcorner \alpha + \beta \lrcorner \cdot D - \sum_{\{\alpha_i + \beta_i\} \neq 0} D_i,$$

$$\overline{\mathcal{V}} \otimes \overline{\mathcal{W}} = L + M - \sum_{\alpha_i \neq 0} D_i - \sum_{\beta_i \neq 0} D_i.$$

However, if $\mathcal{W}$ is the restriction to U of a unitary local system (of arbitrary rank) on X, then $\overline{\mathcal{V} \otimes \mathcal{W}} = \overline{\mathcal{V}} \otimes \overline{\mathcal{W}}$. This follows from the construction of the canonical extension [8].

(c) If $\mathcal{V}$ and $\mathcal{W}$ are unitary local systems on U, with $\mathcal{V}$ of rank one corresponding to (L, α) and $\mathcal{W}$ being the restriction of a local system from X, by (a) and (b) above

$$\overline{\mathcal{V} \otimes \mathcal{W}} = L \otimes \mathcal{O}_X\Big(-\sum_{\alpha_i \neq 0} D_i\Big) \otimes \overline{\mathcal{W}}.$$

The last vector bundle determines polytopes in $[0, 1)^S$ such that the vector bundle remains constant when α varies within these polytopes. For example, consider the polytopes P_k of (10). Then $\overline{\mathcal{V} \otimes \mathcal{W}}$ remains constant if α varies within a fixed polytope $P_{k,S'} := P_k \cap \{\alpha \mid \alpha_i \neq 0 \text{ if } i \in S',\ \alpha_i = 0 \text{ if } i \notin S'\}$, where $S' \subset S$. The polytopes $P_{k,S'}$ form a more refined decomposition of $B(X, D)$ than (10).

(d) A similar conclusion holds for any $\mathcal{V}$ and $\mathcal{W}$ as in (c) without assuming that $\mathcal{W}$ is the restriction of a local system from X. More precisely, there exists a finer decomposition than (10) of $B(X, D)$ into rational convex polytopes P_k such that if $\mathcal{V} = (L, \alpha)$ varies with $\alpha \in P_k$ and L and k fixed, then $\overline{\mathcal{V} \otimes \mathcal{W}}$ is constant. This follows from the explicit construction of the canonical extensions. For example, let u_i be a frame of the multi-valued flat sections of $\mathcal{V}$ around a general point of D_i. Then $\overline{\mathcal{V}}$ is locally given by the holomorphic frame $v_i = \exp(\log z_i \cdot \alpha_i) u_i$, where z_i is a local equation for D_i and the monodromy of $\mathcal{V}$ around D_i is $\exp(-2\pi i \alpha_i)$ with $\alpha_i \in [0, 1)$. Fix an orthonormal frame $u_{i,j}$ $(1 \leqslant j \leqslant \mathrm{rank}(\mathcal{W}))$ of the multi-valued flat sections of $\mathcal{W}$ around a general point of D_i, such that the monodromy around D_i sends $u_{i,j}$ to $\exp(-2\pi i \alpha_{i,j}) u_{i,j}$, with $\alpha_{i,j} \in [0,1)$. Then a local holomorphic frame of $\overline{\mathcal{W}}$ is given by $v_{i,j} := \exp(\log z_i \cdot \alpha_{i,j}) u_{i,j}$. A local holomorphic frame of $\overline{\mathcal{V} \otimes \mathcal{W}}$ is given by $w_{i,j} := \exp(\log z_i \cdot \beta_{i,j}) u_i \otimes u_{i,j}$, where $\beta_{i,j} = \{\alpha_i + \alpha_{i,j}\}$ is the fractional part of $\alpha_i + \alpha_{i,j}$. The polytopes are determined by imposing the condition that β is constant (here $\alpha_{i,j}$ are fixed). Remark that $\overline{\mathcal{V}} \otimes \overline{\mathcal{W}}$ is given locally by $v_i \otimes v_{i,j} = \exp(\log z_i \cdot \llcorner \alpha_i + \alpha_{i,j} \lrcorner) w_{i,j}$.

Proof of Theorem 1.4. The proof is essentially the same as for Theorem 1.3. Assume first that D is a divisor with simple normal crossings. Then, by the decomposition (10),

$$W_i^{p,q}(U, \mathcal{W}) = \coprod_k W_i^{p,q}(U, \mathcal{W}) \cap \left[P_k \times \mathrm{Pic}^\tau(X)\right].$$

The intersection is possible via the identification of unitary local systems of rank one on U with realizations of boundaries of X on D. Pointwise, the set $W_i^{p,q}(U, \mathcal{W}) \cap [P_k \times \mathrm{Pic}^\tau(X)]$ consists of the local systems $\mathcal{V} \in \mathrm{Hom}(H_1(U), S^1)$ corresponding to $\{(L_k + M, \alpha) \mid \alpha \in P_k, M \in \mathrm{Pic}^\tau(X)\}$ such that $\dim \mathrm{Gr}_F^p H^{p+q}(U, \mathcal{V} \otimes \mathcal{W}) \geqslant i$, where L_k depends on P_k. By [49, 2nd Theorem, part (a)],

$$\dim \mathrm{Gr}_F^p H^{p+q}(U, \mathcal{V} \otimes \mathcal{W}) = h^q\left(X, \Omega_X^p(\log D) \otimes \overline{\mathcal{V} \otimes \mathcal{W}}\right).$$

By Remark 8.2(d), the last quantity is constant if α varies within a fixed polytope P_k, possibly coming from a finer decomposition of $B(X, D)$ than (10). We will work with this finer decomposition from now on and use the same notation, P_k, for its polytopes.

Fix $\alpha_k \in P_k$ and let $\mathcal{V}_k$ denote the local system corresponding to (L_k, α_k). The proof of Proposition 4.5 goes word-by-word for the finer P_k's. In other words, we can assume that (L_k, α_k) is a torsion element of $\mathrm{Pic}^\tau(X, D)$. We have

$$W_i^{p,q}(U, \mathcal{W}) = \coprod_k P_k \times W_i^{p,q}(\mathcal{V}_k \otimes \mathcal{W}),$$

where

$$W_i^{p,q}(\mathcal{V}_k \otimes \mathcal{W}) := \left\{\mathcal{V} \in \mathrm{Hom}\left(H_1(X), S^1\right) \,\middle|\, \dim \mathrm{Gr}_F^p H^{p+q}(U, \mathcal{V}_k \otimes \mathcal{W} \otimes \mathcal{V}_{|U}) \geqslant i\right\}.$$

Since $\mathcal{V}_k$ have finite order, we can find a finite abelian group G with and embedding $G^* \subset \mathrm{Pic}^\tau(X, D) \cong \mathrm{Hom}(H_1(U), S^1)$, such that $\mathcal{V}_k \in G^*$. Let χ_k be the character of G given by $\mathcal{V}_k$. Let $g : V \to U$ the unramified G-cover given by $G^* \subset \mathrm{Pic}^\tau(X, D)$. Then, by Lemma 8.1(b),

$$W_i^{p,q}(\mathcal{V}_k \otimes \mathcal{W}) = \left\{\mathcal{V} \in \mathrm{Hom}\left(H_1(X), S^1\right) \,\middle|\, \dim \mathrm{Gr}_F^p H^{p+q}\left(V, g^*(\mathcal{W} \otimes \mathcal{V}_{|U})\right)_{\chi_k} \geqslant i\right\}.$$

If $H_1(X) = 0$, then $\mathrm{Pic}^\tau(X) = \{(\mathcal{O}_X, 0)\}$, $\mathrm{Pic}^\tau(X, D) = B(X, D)$ and $W_i^{p,q}(\mathcal{V}_{k,S'} \otimes \mathcal{W})$ is either empty or $\{(\mathcal{O}_X, 0)\}$. Then the statement of Theorem 1.4 for this case is clear.

If $\mathcal{W}$ is the restriction of a local system from X, then the statement of Theorem 1.4 for this case follows from Theorem 8.3.

If D is assumed to be general, the conclusion follows from the simple normal crossings case as in the proof of Theorem 1.3. □

Let $L(X)$ denote the category of local systems on X, $M(X)$ denote the Betti realization of the moduli space of local systems on X, and $M_i(X)$ denote the component of local systems of rank i. This changes the notation from Section 1. Thus $M_1(X)$ is now what we denoted by $M_B(X)$ in Section 1. Let $U_1(X)$ denote the subspace of unitary local systems of $M_1(X)$. Note that in the notation of the following result, $\mathcal{W}$ will be a local system on X, not on U.

Theorem 8.3. *With the notation as in Theorem* 6.1*, let U be an open subset of X and such that $D = X - U$ is a simple normal crossings divisor. Let $V = f^{-1}(U)$. Let $\mathcal{W}$ be a unitary local system on X. Then for all $\chi \in G^*$, the image of the set*

$$\left\{\mathcal{V} \in U_1(X) \,\middle|\, \dim \mathrm{Gr}_F^p H^{p+q}\big(V, f^*(\mathcal{W} \otimes \mathcal{V})_{|V}\big)_\chi \geqslant i\right\} \tag{13}$$

in $\mathrm{Pic}^\tau(X)$ *is a finite union of torsion translates of complex subtori of* $\mathrm{Pic}^\tau(X)$*, and so is any intersection of these translates.*

Proof. As for Theorem 6.1, this follows from the general theory of absolute constructible sets of C. Simpson [45]. Any absolute closed subset of $M_1(X)$ is a finite union of torsion translates of triple subtori of $M_1(X)$ [45, Theorem 6.1]. A triple subtorus is a closed connected real analytic subgroup of the underlying analytic group of $M_1(X)$ such that it is an algebraic subgroup in each three complex structures Betti, de Rham, and Dolbeault (see Section 1). An intersection of absolute constructible subsets is also absolute. A way of producing absolute sets is via absolute functors between two categories of local systems of smooth projective varieties and via absolute natural transformations between absolute functors. For example, the image and the inverse image of absolute sets under absolute functors is again absolute [45, Corollary 7.3 and Lemma 7.4].

First, consider the loci

$$\left\{\mathcal{V} \in M(X) \,\middle|\, \dim H^m(U, \mathcal{V}_{|U}) \geqslant i\right\}, \tag{14}$$

where m and i are fixed. In case $U = X$, these loci are absolute closed sets [45, Corollary 7.14], due to the fact that $H^m(X, .)$ defines an absolute functor from $L(X)$ to $L(\text{point})$. We will reduce the general case to this particular case. Recall that the loci (14) are closed subsets being inverse images under the restriction morphism $M(X) \to M(U)$ of closed subsets (see [4, Corollary 2.5]).

We will not be able to prove that $H^m(U, \mathcal{V}_{|U})$ is an absolute functor from $L(X)$ to $L(\text{point})$. However, for the loci given by the dimension of $H^m(U, \mathcal{V}_{|U})$ to be absolute, it is enough to prove that there exists a filtration of $H^m(U, \mathcal{V}_{|U})$ such that the graded pieces form absolute functors. Indeed, the associated graded functor will be an absolute functor to vector spaces of same dimensions as given by $H^m(U, \mathcal{V}_{|U})$.

First, by [11, 2.4], there exists a long exact sequence

$$\cdots \to H_D^m(X, \mathcal{V}) \xrightarrow{\phi^m = \phi^m(\mathcal{V})} H^m(X, \mathcal{V}) \to H^m(U, \mathcal{V}) \to H_D^{m+1}(X, \mathcal{V}) \to \cdots. \tag{15}$$

Let $i : D \hookrightarrow X$ denote the inclusion map. Here $H_D^m(X, \mathcal{V}) = \mathbb{H}^m(D, i^!\mathcal{V})$. By [11, Proposition 3.3.7], $i^!\mathcal{V} = \mathbb{D}_D(\mathcal{V}_{|D}^\vee)[-2n]$, where $\mathbb{D}_D(.)$ is the dual complex in the bounded derived category of constructible sheaves on D, and n is the (complex) dimension of X. Thus, $H_D^m(X, \mathcal{V}) = \mathbb{H}^{m-2n}(D, \mathbb{D}(\mathcal{V}_{|D}^\vee))$, and by Poincaré–Verdier duality [11, Theorem 3.3.10], this space is naturally isomorphic with $H^{2n-m}(D, \mathcal{V}_{|D}^\vee)^\vee$. We will use a Mayer–Vietoris spectral sequence degenerating to this last space, whose E_∞ term is an absolute functor and induces filtrations on $\ker\phi^m$ and $\operatorname{coker}\phi^m$ such that the associated graded functors are absolute functors. This will suffice for our purposes since

$$\dim H^m(U,\mathcal{V}_{|U}) = \dim\bigl[\operatorname{coker}\phi^m \oplus \ker\phi^{m+1}\bigr].$$

Let ψ^p be the restriction map $H^p(X,\mathcal{V}) \to H^p(D,\mathcal{V})$. The natural transformations ϕ and ψ are related by $\phi^m(\mathcal{V})^\vee = \psi^{2n-m}(\mathcal{V}^\vee)$. We discuss first ψ and then we draw conclusions about ϕ.

Consider the Mayer–Vietoris spectral sequence

$$E_1^{p,q}(D) = H^q\bigl(D^{[p]},\mathcal{V}\bigr) \Rightarrow H^{p+q}(D,\mathcal{V}), \tag{16}$$

where $D^{[p]}$ is the disjoint union of $(p+1)$-fold intersections of distinct irreducible components of $D=\bigcup_{i\in S} D_i$. For $I\subset S$, let $D_I=\bigcap_{i\in I} D_i$. Then $D^{[p]}=\coprod_{I\subset S;\#I=p+1} D_I$. We recall next the construction of (16). Define

$$C^{p,q}(D) := \bigoplus_{\substack{I\subset S\\ \#I=p+1}} C^q(D_I,\mathcal{V}),$$

where $C^q(D_I,\mathcal{V})$ denote the $\mathcal{V}$-twisted cochains. Let $\hat{d}: C^{p,q}(D)\to C^{p+1,q}$ and $d: C^{p,q}(D)\to C^{p,q+1}(D)$ denote the Cech and cochain differentials. Then $(C^{\bullet,\bullet}(D),\hat{d},d)$ becomes a double complex. We suppressed from notation that $C^{\bullet,\bullet}(D)$ depends on the closed cover $\{D_i\}_{i\in S}$ and the local system $\mathcal{V}$. Let $Tot^m(D)=\bigoplus_{p+q=m} C^{p,q}(D)$, so that $(Tot^\bullet(D),\hat{d}+d)$ is the total complex of $(C^{\bullet,\bullet}(D),\hat{d},d)$. Then (16) is the spectral sequence $(E_r^{\bullet,\bullet}(D),d_r)$ given by the filtration of $Tot^\bullet(D)$ defined by

$$F^p Tot^m(D) = \bigoplus_{\substack{r+s=m\\ r\geqslant p}} C^{r,s}.$$

Define $C^{p,q}(X)$ to be 0 if $p\neq 0$ and $C^q(X,\mathcal{V})$ if $p=0$. Let $\hat{d}: C^{p,q}(X)\to C^{p+1,q}$ and $d: C^{p,q}(X)\to C^{p,q+1}(X)$ denote the 0-map and the cochain differential. Then $(C^{\bullet,\bullet}(D),\hat{d},d)$ becomes a double complex. This construction is similar to $C^{p,q}(D)$ in the sense that it depends on the closed cover $\{X\}$ of X and the local system $\mathcal{V}$; the Cech differential is 0 for this closed cover. The corresponding spectral sequence, denoted by $(E_r^{\bullet,\bullet}(X),d_r)$, degenerates to $H^{p+q}(X,\mathcal{V})$. Also, $E_r^{p,q}(X)=H^q(X,\mathcal{V})$ if $p=0$ and equals 0 otherwise, for $r\geqslant 1$.

Define a map of double complexes of bidegree $(0,0)$

$$\psi^{\bullet,\bullet}: \bigl(C^{\bullet,\bullet}(X),\hat{d},d\bigr) \to \bigl(C^{\bullet,\bullet}(D),\hat{d},d\bigr)$$

as follows. For $p\neq 0$, $\psi^{p,q}=0$. Define $\psi^{0,q}: C^q(X,\mathcal{V})\to\bigoplus_{i\in S} C^q(D_i,\mathcal{V})$ as the direct sum of the restriction maps. It is trivial that $\psi^{\bullet,\bullet}$ commutes with d (0 goes to 0). Note that $\psi^{0,q}$ is the composition of the restriction map $C^q(X,\mathcal{V})\to C^q(D,\mathcal{V})$ with the Cech map $C^q(D,\mathcal{V})\to\bigoplus_{i\in S} C^q(D_i,\mathcal{V})$. Hence $\psi^{\bullet,\bullet}$ commutes with $\hat{d}$ also.

The map of double complexes $\psi^{\bullet,\bullet}$ induces of morphism of filtered total complexes (which we prefer not to have a notation for, so that we do not confuse it with the cohomology restriction map ψ^p). By [31, Theorem 3.5], we have a morphism of spectral sequences

$$\psi_r^{\bullet,\bullet} : \left(E_r^{\bullet,\bullet}(X), d_r\right) \to \left(E_r^{\bullet,\bullet}(D), d_r\right) \quad (0 \leqslant r \leqslant \infty).$$

In particular, we have a map

$$\psi_\infty^m : E_\infty^m(X) = H^m(X, \mathcal{V}) \to E_\infty^m(D) = \bigoplus_{p+q=m} E_\infty^{p,q}(D),$$

where $E_\infty^{p,q}(D) = \mathrm{Gr}_L^p H^{p+q}(D, \mathcal{V})$ for some filtration L, and $E_\infty^m(D)$ is the associated graded vector space of $H^m(D, \mathcal{V})$.

(Example: If D has only two components D_1 and D_2, then $E_\infty^m(D) = \ker d_1^{0,m} \oplus \operatorname{coker} d_1^{0,m-1}$, where $d_1^{0,m} : H^m(D_1, \mathcal{V}) \oplus H^m(D_2, \mathcal{V}) \to H^2(D_1 \cap D_2, \mathcal{V})$ is the map induced by the Cech differential $\hat{d}$. Then $\psi_\infty^m : H^m(X, \mathcal{V}) \to E_\infty^m(D)$ is the map $(\psi_1^{0,m}, 0)$, where $\psi_1^{0,m} : H^m(X, \mathcal{V}) \to H^m(D_1, \mathcal{V}) \oplus H^m(D_2, \mathcal{V})$ is the direct sum of the restriction maps.)

Since $\psi^{\bullet,\bullet}$ factors through the restriction map $C^q(X, \mathcal{V}) \to C^q(D, \mathcal{V})$, the map ψ_∞^m factors as $\mathrm{Gr}_L \circ \psi^m$, where $\psi^m : H^m(X, \mathcal{V}) \to H^m(D, \mathcal{V})$ is the cohomology restriction map and $\mathrm{Gr}_L : H^m(D, \mathcal{V}) \to E_\infty^m(D)$ is taking elements to their associated graded image. Thus ker and coker of ψ_∞^m have the same dimension as ker and, respectively, coker of ψ^p.

The D_I are smooth projective and the differential d_1 of (16) is obtained from restrictions. It follows that the $E_1(D)$ terms of (16) form absolute functors from local systems on X to the point, and the differential d_1 is an absolute natural transformation (see [45, Propositions 7.8 and 7.9]). Kernels and cokernels of absolute natural transformations give absolute functors [45, Lemma 7.13], hence $(E_r(D), d_r)$ is an absolute functor with an absolute natural transformation, and so $E_\infty(D)$ is an absolute functor. Also ψ_∞^p is an absolute natural transformation since it is obtained from restriction maps. Thus $\ker \psi_\infty^p$ and $\operatorname{coker} \psi_\infty^p$ are absolute functors.

By dualizing the terms in the spectral sequences we obtain a natural transformations ϕ_∞^m such that $\phi_\infty^m(\mathcal{V})^\vee = \psi_\infty^{2n-m}(\mathcal{V}^\vee)$. We want to show that ϕ_∞^m are also absolute. It is enough to prove that $(E_1(D), d_1)$ dualizes to an absolute functor with an absolute natural transformation. Since composition of absolute natural transformation is again absolute, it is enough then to show that the duality $H^m(X, \mathcal{V}) \leftrightarrow H^{2n-m}(X, \mathcal{V}^\vee)^\vee$ (in our case we will want to replace X by any D_I) is an absolute natural transformation in both directions. The dual $\mathcal{V} \leftrightarrow \mathcal{V}^\vee$ is an absolute functor [45, Lemma 7.11], so $\mathcal{V} \mapsto (\mathcal{V}, \mathcal{V}^\vee)$ is an absolute functor from $L(X)$ to $L(X) \times L(X)$. Thus the natural transformation $H^m(X, \mathcal{V}) \otimes H^{2n-m}(X, \mathcal{V}^\vee) \to H^{2n}(X, \mathbb{C}_X)$ factors through the absolute natural transformation of cup product [45, Proposition 7.12]

$$H^m(X, \mathcal{V}_1) \otimes H^{2n-m}(X, \mathcal{V}_2) \to H^{2n}(X, \mathcal{V}_1 \otimes \mathcal{V}_2)$$

via an absolute functor, hence it is itself absolute. Therefore ϕ_∞^m, and hence $\ker \phi_\infty^m$ and $\operatorname{coker} \phi_\infty^m$, are absolute natural transformations.

Thus the loci given by the dimension of $\operatorname{coker} \phi_\infty^m(\mathcal{V}) \oplus \ker \phi_\infty^{m+1}(\mathcal{V})$ are absolute sets. This finishes the proof of the fact that the loci (14) are absolute closed sets. By [45, Corollary 6.2 and Lemma 7.1] (see also Conclusion in [45]), the restriction of the loci (14) to $M_1(X)$ are finite unions of torsion translates of subtori of $M_1(X)$.

Now, let $\mathcal{W}$ be a local system on X. The same proof as above extends to show that the loci

$$\left\{\mathcal{V}\in M(X)\;\middle|\;\dim H^m\big(U,(\mathcal{W}\otimes\mathcal{V})_{|U}\big)\geqslant i\right\}\tag{17}$$

(m, i are fixed) are absolute closed subsets. Hence, as above, the restriction of the loci (17) to $M_1(X)$ are finite unions of torsion translates of subtori of $M_1(X)$.

With f and V as in the theorem, the loci

$$\left\{\mathcal{V}\in M(X)\;\middle|\;\dim H^m\big(V,f^*(\mathcal{W}\otimes\mathcal{V})_{|V}\big)\geqslant i\right\}\tag{18}$$

are absolute closed sets since they are the intersection of the loci of local systems $\mathcal{V}'$ on Y given by $H^m(V,(f^*\mathcal{W}\otimes\mathcal{V}')_{|V})$, which are absolute by the previous discussion, with the image of the absolute functor f^* [45, Proposition 7.8]. Hence, the restriction of the loci (18) to $M_1(X)$ are finite unions of torsion translates of subtori of $M_1(X)$.

Assuming now for the first time that $\mathcal{W}$ is unitary, we prove the statement of the theorem for the loci

$$\left\{\mathcal{V}\in U_1(X)\;\middle|\;\dim \mathrm{Gr}^p_F H^m\big(V,f^*(\mathcal{W}\otimes\mathcal{V})_{|V}\big)\geqslant i\right\}.\tag{19}$$

By [49], for $\mathcal{V}$ a unitary local system on X, the dimension of $\mathrm{Gr}^p_F H^m(V,f^*(\mathcal{W}\otimes\mathcal{V})_{|V})$ is computed from the $m-p$ cohomology of the pth order logarithmic differentials twisted by the canonical extension of $f^*(\mathcal{W}\otimes\mathcal{V})_{|V}$ to some nonsingular compactification of V with boundary a simple normal crossings divisor. Thus the dimension of $\mathrm{Gr}^p_F H^m(V,f^*(\mathcal{W}\otimes\mathcal{V})_{|V})$ is a semi-continuous function in $\mathcal{V}$. Hence, as in [45, Section 5] (or [3], see also the proof of Theorem 6.1), the restrictions of the loci (18) to $U_1(X)$ give a stratification which is a refinement of the stratification given by the loci (19). Subtori of $M_1(X)$ restrict to subtori of $U_1(X)$. Thus the loci (19) are finite unions of torsion translates of subtori of $U_1(X)$.

Lastly, the statement of the theorem is the G-equivariant version of the above arguments (see Theorem 6.1). □

Remark 8.4. If one only assumes that $\mathcal{W}$ is a unitary local system on U in Theorem 8.3, then the analogue of (15) would involve the intersection cohomology $IH^m(X,\mathcal{W}\otimes\mathcal{V}_{|U})$. However we do not know how to prove that this forms an absolute functor from $L(X)$ to $L(\text{point})$. One is then led naturally to Question 1.6 of the introduction.

Proof of Corollary 1.5. The loci of the statement can be rewritten as

$$\bigcup_{\substack{(i_0,\dots,i_m)\\ i_0+\dots+i_m\geqslant i}}\;\bigcap_{0\leqslant p\leqslant m} W^{p,m-p}_{i_p}(U,\mathcal{W}).\tag{20}$$

The analog of Lemma 6.5 holds for $W^{p,q}_i(U,\mathcal{W})$ too, using the sets $W^{p,q}_i(\mathcal{V}_k\otimes\mathcal{W})$ instead of the sets $V^q_i(L_k)$. That is, the union in (20) is finite. Hence the theorem follows from Theorem 1.4. □

Proof of Theorem 1.9. With the notation as in the proof of Theorem 1.8,

$$h^{p,q}_{\mathcal{W}}(N)=\sum_{i\geqslant 1} i\cdot\#\big[\big(W^{p,q}_i(U,\mathcal{W})-W^{p,q}_{i+1}(U,\mathcal{W})\big)[N]\big],$$

by Lemmas 8.1 and 4.3. The rest of the proof is word by word as that of Theorem 1.8, with the following exceptions: V_i^q replaced by $W_i^{p,q}(U,\mathcal{W})$; Lemma 6.5 holds for this case as well; the reference to Theorem 1.3 is replaced by the Theorem 1.4 for $W_i^{p,q}(U,\mathcal{W})$; and Proposition 4.5 works as well for any refinement of the original decomposition of $B(X,D)$. $\square$

Acknowledgments

We thank L. Ein, L. Maxim, J. Song, the referees, and especially M. Saito for helpful comments and discussions. We thank T. Mochizuki for telling us how to give a self-contained proof of Theorem 1.2, shortening the exposition of an earlier preprint. We thank the Institute of Advanced Study in Princeton for their hospitality.

References

[1] D. Abramovich, J. Wang, Equivariant resolution of singularities in characteristic 0, Math. Res. Lett. 4 (2–3) (1997) 427–433.
[2] U. Angehrn, Y.T. Siu, Effective freeness and point separation for adjoint bundles, Invent. Math. 122 (2) (1995) 291–308.
[3] D. Arapura, Higgs line bundles, Green–Lazarsfeld sets, and maps of Kähler manifolds to curves, Bull. Amer. Math. Soc. 26 (2) (1992) 310–314.
[4] D. Arapura, Geometry of cohomology support loci for local systems. I, J. Algebraic Geom. 6 (3) (1997) 563–597.
[5] O. Biquard, Fibrés de Higgs et connexions intégrables: le cas logarithmique (diviseur lisse), Ann. Sci. École Norm. Sup. (4) 30 (1) (1997) 41–96.
[6] G. Bredon, Topology and Geometry, Grad. Texts in Math., vol. 139, Springer-Verlag, New York, 1993.
[7] D. Cohen, P. Orlik, Some cyclic covers of complements of arrangements, in: Arrangements in Boston: A Conference on Hyperplane Arrangements, 1999, Topology Appl. 118 (1–2) (2002) 3–15.
[8] P. Deligne, Équations différentielles à points singuliers réguliers, Lecture Notes in Math., vol. 163, Springer-Verlag, Berlin, 1970, iii+133 pp.
[9] J.P. Demailly, A numerical criterion for very ample line bundles, J. Differential Geom. 37 (2) (1993) 323–374.
[10] J.P. Demailly, Complex Analytic and Algebraic Geometry, book available on the author's website.
[11] A. Dimca, Sheaves in Topology, Universitext, Springer-Verlag, Berlin, 2004, xvi+236 pp.
[12] L. Ein, R. Lazarsfeld, Global generation of pluricanonical and adjoint linear series on smooth projective threefolds, J. Amer. Math. Soc. 6 (4) (1993) 875–903.
[13] H. Esnault, E. Viehweg, Logarithmic de Rham complexes and vanishing theorems, Invent. Math. 86 (1) (1986) 161–194.
[14] H. Esnault, E. Viehweg, Lectures on Vanishing Theorems, DMV Sem., vol. 20, Birkhäuser, Basel, 1992.
[15] W. Fulton, Intersection Theory, second ed., Ergeb. Math. Grenzgeb. (3), vol. 2, Springer-Verlag, Berlin, 1998, xiv+470 pp.
[16] M. Green, R. Lazarsfeld, Deformation theory, generic vanishing theorems, and some conjectures of Enriques, Catanese and Beauville, Invent. Math. 90 (2) (1987) 389–407.
[17] M. Green, R. Lazarsfeld, Higher obstructions to deforming cohomology groups of line bundles, J. Amer. Math. Soc. 4 (1) (1991) 87–103.
[18] P. Griffiths, J. Harris, Principles of Algebraic Geometry, Wiley Classics Lib., Wiley, New York, 1994, xiv+813 pp.
[19] R. Hartshorne, Algebraic Geometry, Grad. Texts in Math., vol. 52, Springer-Verlag, New York, 1977, xvi+496 pp.
[20] E. Hironaka, Polynomial periodicity for Betti numbers of covering surfaces, Invent. Math. 108 (2) (1992) 289–321.
[21] E. Hironaka, Intersection theory on branched covering surfaces and polynomial periodicity, Int. Math. Res. Not. (6) (1993) 185–196.
[22] J. Jost, K. Zuo, Harmonic maps of infinite energy and rigidity results for representations of fundamental groups of quasiprojective varieties, J. Differential Geom. 47 (3) (1997) 469–503.
[23] R. Lazarsfeld, Positivity in Algebraic Geometry, I & II, Ergeb. Math. Grenzgeb. (3), vols. 48, 49, Springer-Verlag, Berlin, 2004.
[24] J. Li, Hermitian–Einstein metrics and Chern number inequalities on parabolic stable bundles over Kähler manifolds, Comm. Anal. Geom. 8 (3) (2000) 445–475.

[25] A. Libgober, Characteristic varieties of algebraic curves, in: Applications of Algebraic Geometry to Coding Theory, Physics and Computation, Eilat, 2001, in: NATO Sci. Ser. II Math. Phys. Chem., vol. 36, Kluwer Acad. Publ., Dordrecht, 2001, pp. 215–254.
[26] A. Libgober, Hodge decomposition of Alexander invariants, Manuscripta Math. 107 (2) (2002) 251–269.
[27] A. Libgober, Isolated non-normal crossings, in: Real and Complex Singularities, in: Contemp. Math., vol. 354, Amer. Math. Soc., Providence, RI, 2004, pp. 145–160.
[28] A. Libgober, Homotopy groups of complements to ample divisors, in: S. Izumiya, et al. (Eds.), Proceedings of 12th MSJ–IRI Symposium Singularity Theory and Its Applications, Sapporo, 2003, in: Adv. Stud. Pure Math., vol. 43, Math. Soc. Japan, Tokyo, 2007.
[29] A. Libgober, Lectures on topology of complements and fundamental groups, in: J.P. Brasselet, A. Pichon (Eds.), Proceedings of Luminy School on Singularities, Luminy, 2005, in: Singularity Theory, World Scientific, Singapore, 2009, pp. 71–137.
[30] A. Libgober, Non-vanishing loci of Hodge numbers of local systems, preprint.
[31] J. McCleary, User's Guide to Spectral Sequences, Math. Lecture Ser., vol. 12, Publish or Perish, Wilmington, DE, 1985, xiv+423 pp.
[32] V.B. Mehta, C.S. Seshadri, Moduli of vector bundles on curves with parabolic structures, Math. Ann. 248 (3) (1980) 205–239.
[33] T. Mochizuki, Kobayashi–Hitchin correspondence for tame harmonic bundles II, arXiv:math.DG/0602266, 2006.
[34] A. Nadel, Multiplier ideal sheaves and Kähler–Einstein metrics of positive scalar curvature, Ann. of Math. (2) 132 (3) (1990) 549–596.
[35] M. Namba, Branched Coverings and Algebraic Functions, Pitman Res. Notes Math. Ser., vol. 161, Wiley, New York, 1987.
[36] R. Pardini, Abelian covers of algebraic varieties, J. Reine Angew. Math. 417 (1991) 191–213.
[37] G. Pareschi, M. Popa, GV-sheaves, Fourier–Mukai transform, and Generic Vanishing, arXiv:math.AG/0608127.
[38] R. Pink, D. Roessler, A conjecture of Beauville and Catanese revisited, Math. Ann. 330 (2) (2004) 293–308.
[39] Z. Reichstein, B. Youssin, Essential dimensions of algebraic groups and a resolution theorem for G-varieties, Canad. J. Math. 52 (5) (2000) 1018–1056.
[40] M. Sakuma, Homology of abelian coverings of links and spatial graphs, Canad. J. Math. 47 (1) (1995) 201–224.
[41] P. Sarnak, S. Adams, Betti numbers of congruence groups, Israel J. Math. 88 (1–3) (1994) 31–72.
[42] C.S. Seshadri, Moduli of vector bundles on curves with parabolic structures, Bull. Amer. Math. Soc. 83 (1) (1977) 124–126.
[43] C. Simpson, Harmonic bundles on noncompact curves, J. Amer. Math. Soc. 3 (3) (1990) 713–770.
[44] C. Simpson, Higgs bundles and local systems, Inst. Hautes Études Sci. Publ. Math. 75 (1992) 5–95.
[45] C. Simpson, Subspaces of moduli spaces of rank one local systems, Ann. Sci. École Norm. Sup. (4) 26 (3) (1993) 361–401.
[46] R. Stanley, Enumerative Combinatorics, vol. 1, Cambridge Univ. Press, Cambridge, 1997.
[47] B. Steer, A. Wren, The Donaldson–Hitchin–Kobayashi correspondence for parabolic bundles over orbifold surfaces, Canad. J. Math. 53 (6) (2001) 1309–1339.
[48] A. Suciu, Fundamental groups of line arrangements: Enumerative aspects, in: Advances in Algebraic Geometry Motivated by Physics, Lowell, MA, 2000, in: Contemp. Math., vol. 276, Amer. Math. Soc., Providence, RI, 2001, pp. 43–79.
[49] K. Timmerscheidt, Mixed Hodge theory for unitary local systems, J. Reine Angew. Math. 379 (1987) 152–171.
[50] E. Viehweg, Rational singularities of higher dimensional schemes, Proc. Amer. Math. Soc. 63 (1) (1977) 6–8.
[51] O. Zariski, Algebraic Surfaces, second suppl. ed., Ergeb. Math. Grenzgeb., vol. 61, Springer-Verlag, New York, 1971, with appendices by S.S. Abhyankar, J. Lipman, and D. Mumford.