A lower bound for the CO-irredundance number of a graph

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Abstract

This paper establishes a necessary and sufficient condition for a CO-irredundant set of vertices of a graph to be maximal and shows that the smallest cardinality of a maximal CO-irredundant set in an \( n \) vertex graph with maximum degree \( \Delta \) is bounded below by \( \frac{n}{2} \) for \( \Delta = 2 \), \( \frac{4n}{13} \) for \( \Delta = 3 \) and \( \frac{2n}{(3\Delta - 3)} \) for \( \Delta \geq 4 \). This result is best possible and extremal graphs are characterised for \( \Delta \geq 3 \).

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1. Introduction

The closed (open) neighbourhood of the vertex \( x \) of a simple graph \( G = (V, E) \) is denoted by \( N[x] \) \( (N(x)) \) and as usual, for a vertex subset \( X \subseteq V \), \( N[X] = \bigcup_{x \in X} N[x] \) and \( N(X) = \bigcup_{x \in X} N(x) \).

A set \( X \) is irredundant, if for every \( s \in X \), \( N[s] - N[X - \{s\}] \neq \emptyset \). Irredundant sets are sometimes called CC-irredundant since they are defined by the existence of a non-empty difference of two closed neighbourhoods. Cockayne, Hedetniemi and Miller introduced these sets in [5] which became of interest due to the following theorem.
Theorem 1 (Cockayne [5]). (i) A dominating set $D$ is minimal dominating if and only if $D$ is irredundant.

(ii) If $D$ is minimal dominating, then $D$ is maximal irredundant.

The reader is referred to Haynes et al. [13] for an extensive bibliography on irredundant sets.

A set $X$ is CO-irredundant, iff or every $x \in X$, $N[x] - N(X - \{x\}) \neq \emptyset$. Farley and Schacham [10] introduced these sets and termed them CO-irredundant because the neighbourhoods in the above definition are closed and open. In 1998, Simmons [14] provided an analogous result to Theorem 1 for CO-irredundant and total dominating sets.

Theorem 2 (Simmons [14]). (i) A total dominating set $D$ is a minimal total dominating if and only if $D$ is CO-irredundant.

(ii) If $D$ is minimal total dominating, then $D$ is maximal CO-irredundant.

It is easily seen that $X$ is CO-irredundant if, and only if, each $x \in X$ has at least one of the three types of $X$-private neighbour ($X$-pn), which we now formally define.

For $x \in X$, the vertex $y$ is an:

(i) $X$-self private neighbour ($X$-spn) of $x$ if $y = x$ and $x$ is an isolated vertex of $G[X]$,  
(ii) $X$-internal private neighbour ($X$-ipn) of $x$ if $y \in X - \{x\}$ and $N(y) \cap X = \{x\}$, 
(iii) $X$-external private neighbour ($X$-epn) of $x$ if $y \in V - X$ and $N(y) \cap X = \{x\}$.

Given $x \in X \subseteq V$ let

$$epn(x, X) = \{x\} \text{ if } x \text{ is a } X\text{-spn},$$
$$ipn(x, X) = \emptyset \text{ otherwise}$$

and

$$pn(x, X) = spn(x, X) \cup ipn(x, X) \cup epn(x, X).$$

Then $X$ is CO-irredundant if for every $x \in X$, $pn(x, X)$ is nonempty.

Let COIR($G$) (coir($G$)) be the largest (smallest) cardinality of a maximal CO-irredundant set. We abbreviate these notations to COIR and coir whenever possible. Nordhaus-Gaddum type results [7] and NP-completeness results [11] have been established for COIR. A set $X \subseteq V$ is called 1-dependent if every vertex of $X$ has an $X$-spn or an $X$-ipn. In [12] it is shown that for any bipartite graph $G$, COIR = $\beta^1(G)$ (the cardinality of the largest 1-dependent set of $G$). CO-irredundant Ramsey numbers were introduced in [6] and also appear in [9,14]. In [2,4,11] CO-irredundance has been embedded in classifications of graph theoretic properties based on the existence of private neighbours.

The main result of this paper (found in Section 3) is a lower bound for coir in terms of the maximum degree $\Delta(G)$ and the order $n(G)$. Similar bounds have been found for irredundance [1,8], open irredundance (also called OC-irredundance) [3] and domination [15]. In Section 2, a necessary and sufficient condition for a CO-irredundant set to be maximal is established and in Section 4 it is shown that the bound found in Section 3 is attained and extremal graphs are characterised for $\Delta \geq 3$. 
2. Maximal CO-irredundant sets

In this section we establish necessary and sufficient conditions for a CO-irredundant set to be maximal.

Lemma 1. Let $X \subseteq V$, $x \in X$ and $v \in V - X$.

(i) $pn(v, X \cup \{v\}) = N[v] - N(X)$.
(ii) $pn(x, X \cup \{v\}) = pn(x, X) - N(v)$.

Proof. If $y \in pn(v, X \cup \{v\})$ then either $y \in spn(v, X \cup \{v\})$, $y \in ipn(v, X \cup \{v\})$ or $y \in epn(v, X \cup \{v\})$. In each case $y \in N[v]$ but $y \notin N(X)$, thus $y \in N[v] - N(X)$. If $y \in N[v] - N(X)$, then either $y = v$ and $v$ is an $(X \cup \{v\})$-spn or $y$ is adjacent to $v$ but no other vertex in $X \cup \{v\}$ (i.e., $y$ is either an $(X \cup \{v\})$-ipn or an $(X \cup \{v\})$-epn). Thus $y \in pn(v, X \cup \{v\})$. A similar proof may be used to prove part (ii) of the lemma. □

Theorem 3. Let $X$ be a CO-irredundant set of $G$ and $S = V - N(X)$. Then $X$ is a maximal CO-irredundant set if and only if for every $v \in N[S] - X$, there exists an $x_v \in X$ such that $pn(x_v, X) \subseteq N(v)$.

Proof. Let $X$ be a maximal CO-irredundant set and suppose $v \in V - X$. Since $X \cup \{v\}$ is not CO-irredundant, there is an $x_v \in X \cup \{v\}$ such that $pn(x_v, X \cup \{v\}) = \emptyset$. If $x_v = v$, then by Lemma 1(i), $N[v] \subseteq N(X)$ and thus $v \notin N[S]$. Otherwise $x_v \in X$ and by Lemma 1(ii), $pn(x_v, X) \subseteq N(v)$.

Conversely suppose $X$ is not a maximal CO-irredundant set, then there exists $v \in V - X$ such that $X \cup \{v\}$ is a CO-irredundant set. Since $pn(v, X \cup \{v\}) \neq \emptyset$, by Lemma 1(i), $N[v] - N(X) \neq \emptyset$ and thus $v \in N[S] - X$. However, for any $x \in X$, $pn(x, X \cup \{v\}) \neq \emptyset$, and so by Lemma 1(ii), $pn(x, X) \subseteq N(v)$. □

Let $x \in X$ and $v \in N[S] - X$ (where $S = V - N(X)$). If $pn(x, X) \subseteq N(v)$, then we say $v$ annihilates (or is an annihilator of) $x$.

3. The bound

For a given $n$ and $A$ let $G$ be any edge-minimal graph with $n(G) = n$, $A(G) \subseteq A$ and $coir$ being minimum. Let $X$ be a maximal CO-irredundant set of $G$ with $|X| = coir$. The set $X$ induces the following partition of the vertex set:

- $Y_0 = \{x \in X | |N(x) \cap X| = 0\}$
- $Y_1 = \{x \in X | |N(x) \cap X| = 1 \text{ and } x \text{ has an } X-\text{ipn}\}$
- $Y_2 = \{x \in X | |N(x) \cap X| \geq 2 \text{ and } x \text{ has an } X-\text{ipn}\}$
- $Z_1 = \{x \in X | |N(x) \cap X| = 1\} - Y_1$
- $Z_2 = \{x \in X | |N(x) \cap X| \geq 2\} - Y_2$
\[ B = \bigcup_{x \in X} \text{epn}(x, X) \]

\[ C = N(X) - (B \cup X) \]

\[ R = V - N[X]. \]

Let \(|Y_i| = y_i\) for \(i = 0, 1, 2\). Notice that \(Y_0, Y_1, Y_2, Z_1\) and \(Z_2\) form a partition of \(X\) and that the set \(S\) defined in Theorem 3 is equal to \(Y_0 \cup R\), so that \(N[S] - X = N[R] \cup N(Y_0)\).

The following four preliminary results will be used in the proof of a lower bound for \(\text{coir}\).

**Lemma 2.** If \(v \in B\) annihilates \(x \in X\), then \(x \notin N(v)\).

**Proof.** This follows directly from the definition of \(B\), the definition of the word annihilates and the fact that \(v \notin N(v)\). \(\square\)

**Lemma 3.** If \(v \in R\) annihilates \(x \in X\), then \(x\) has no \(X\)-spn and no \(X\)-ipn.

**Proof.** If \(x\) is an \(X\)-spn or has an \(X\)-ipn, then \(v\) is adjacent to some \(y\) (possibly \(x\)) in \(X\) and so \(v \notin R\), contradiction. \(\square\)

The next result follows directly from the definition of \(Z_1\).

**Lemma 4.** If \(z \in Z_1\), then \(N(z) \cap X = \{y\}\) where \(y \in Y_2\).

**Lemma 5.** If \(w \in B\) annihilates \(y \in Y_2\) and \(N(w) \cap X = \{z\}\), then \(z\) is the only \(X\)-ipn of \(y\) and \(z \in Z_1\). Further, \(z\) is annihilated by some vertex of \(R\). If in addition \(y\) has an \(X\)-epn, then \(z\) is annihilated by no more than \(\Delta - 2\) vertices in \(R\).

**Proof.** From the definitions of sets \(B\) and \(Y_2\) it is clear that \(z \in Z_1\) and that \(z\) is the only \(X\)-ipn of \(y\). Since \(w\) annihilates \(y\), \(w \in N[R] \cup N(Y_0)\) and thus \(w\) is adjacent to some \(v \in R\). If \(v\) does not annihilate \(z\), then \(v\) annihilates some other vertex of \(X\) and therefore has degree at least two. Consider \(G^* = G - uv\). Clearly, \(N_{G^*}[R] \cup N_{G^*}(Y_0) \subseteq N_G[R] \cup N_G(Y_0)\) and each vertex of \(N_{G^*}[R] \cup N_{G^*}(Y_0)\) annihilates a vertex of \(X\) in \(G^*\). Thus \(X\) is a maximal CO-irredundant set of \(G^*\), \(\text{coir}(G^*) \leq \text{coir}(G)\) and \(G^*\) has fewer edges than \(G\), a contradiction which shows that \(v\) annihilates \(z\).

If \(y\) has an \(X\)-epn, then \(w\) is adjacent to \(z\), the \(X\)-eps of \(y\) and hence to at most \(\Delta - 2\) vertices of \(R\). \(\square\)

**Theorem 4.** For \(\Delta = 2\), \(\text{coir} \geq n/2\), for \(\Delta = 3\), \(\text{coir} \geq 4n/13\) and for \(\Delta \geq 4\), \(\text{coir} \geq 2n/(3\Delta - 3)\).

**Proof.** By Theorem 3 and Lemma 3, each vertex of \(R\) annihilates at least one vertex of \(Z_1 \cup Z_2\). Let \(r_z\) be the number of vertices in \(R\) that annihilate \(z \in Z_1 \cup Z_2\). Then,

\[ |R| \leq \sum_{z \in Z_1 \cup Z_2} r_z. \] (1)
Define

\[ A_1^* = \{ z \in Z_1 | r_z \geq A-1 \}, \]
\[ A_1 = \{ z \in Z_1 | 0 < r_z < A-1 \}, \]
\[ A_2 = \{ z \in Z_2 | r_z > 0 \}, \]
\[ A_3^* = \{ z \in Z_1 | r_z = 0 \text{ and } |N(z) \cap B| = A-1 \}, \]
\[ A_3 = \{ z \in Z_1 | r_z = 0 \} - A_3^*, \]
\[ A_4 = \{ z \in Z_2 | r_z = 0 \}. \]

Let \(|A_i| = a_i\) for \(i \in \{1, 2, 3, 4\}\) and \(|A_i^*| = a_i^*\) for \(i \in \{1, 3\}\). It is clear that \(Z_1 = A_1^* \cup A_1 \cup A_3^* \cup A_3\) (disjoint union) and \(Z_2 = A_2 \cup A_4\) (disjoint union).

For each \(z \in A_1^*\) and \(w \in pn(z, X)\), \(w\) is adjacent to \(z\) and \(r_z\) vertices of \(R\). Since \(deg(w) \leq A\), this implies

\[ \sum_{z \in A_1^*} r_z = \sum_{z \in A_1^*} (A-1) = (A-1)a_1^*. \] (2)

For each \(z \in A_2\) and \(w \in pn(z, X)\), \(w\) is adjacent to \(z\), \(r_z\) vertices of \(R\) and at least one other vertex (as \(w\) annihilates some \(y \in X(y \neq z)\) by Theorem 3 and Lemma 2). Thus,

\[ \sum_{z \in A_2} r_z \leq \sum_{z \in A_2} (A-2) \leq (A-2)a_2. \] (3)

Therefore from (1)–(3) and the definition of \(A_1\),

\[ |R| \leq \sum_{z \in Z_1 \cup Z_2} r_z \]
\[ \leq \sum_{z \in A_1^*} r_z + \sum_{z \in A_1} r_z + \sum_{z \in A_2} r_z + \sum_{z \in A_3^* \cup A_3 \cup A_4} r_z \]
\[ \leq (A-1)a_1^* + (A-2)(a_1 + a_2) + 0. \] (4)

Let

\[ B_1 = B \cap N(A_1^* \cup A_1 \cup A_2 \cup Y_0), \]
\[ B_2 = B \cap N(A_3^* \cup A_3 \cup A_4), \]
\[ B_3 = B \cap N(Y_2), \]
\[ B_4 = B \cap N(Y_1). \]

Notice that \(B_1, B_2, B_3\) and \(B_4\) form a partition of \(B\). Let \(b_z\) be the number of vertices of \(B_1\) that annihilate \(z \in Z_1 \cup Z_2 \cup Y_2\).
Since each element of $B_1$ is in $N[R] \cup N(Y_0)$, it annihilates some $z \in Z_1 \cup Z_2 \cup Y_2$ and thus,

$$|B_1| \leq \sum_{z \in Z_1 \cup Z_2 \cup Y_2} b_z. \quad (5)$$

Now partition $Y_2$ into the following four sets:

$$D = \{y \in Y_2 | b_y = 0\},$$
$$D_0 = \{y \in Y_2 - D | N(y) \cap B = \emptyset\},$$
$$D_1 = \{y \in Y_2 - D | N(y) \cap B = \{z\} \} \text{ and}$$
$$D_2 = \{y \in Y_2 - D | N(y) \cap B \geq 2\}.$$ 

Let $d_1 = |D_1|$ and $d = |D|.$

For $z \in Z_1 \cap N(D_1 \cup D_2)$ let $d_z$ be the number of X-epns of $y$, where $\{y\} = N(z) \cap (D_1 \cup D_2).$ If $z \in ((Z_1 \cup Z_2) - (Z_1 \cap N(D_1 \cup D_2)))$, let $d_z = 0.$

Suppose that $d_z \neq 0.$ Then $z \in Z_1 \cap N(D_1 \cup D_2)$ and so $z \in Z_1$ is an X-ipn of $y \in (D_1 \cup D_2) \subseteq Y_2.$ The definition of $D_1 \cup D_2$ implies that $b_y > 0.$ Therefore some $w \in B$ annihilates $y \in Y_2$ and so $N(w) \cap X = \{z\}.$ By Lemma 5, $1 \leq r_z \leq A - 2$ and we conclude that $z \in A_1.$ Hence,

$$d_1 + 2d_2 \leq \sum_{z \in Z_1 \cup Z_2} d_z = \sum_{z \in A_1} d_z. \quad (6)$$

If $d_z \neq 0$, then the vertex $w$ defined in the previous paragraph is adjacent to $z$, $r_z$ vertices of $R$, $b_z$ vertices of $B_1$ and $d_z$ vertices of $B_3.$ For $z \in Z_1 \cup Z_2$ with $d_z = 0$ any X-epn $w$ of $z$ has these adjacencies. Using $\deg(w) \leq A$ we conclude $b_z \leq (A - 1) - r_z - d_z.$ Hence,

$$\sum_{z \in Z_1 \cup Z_2 - A_1^*} b_z \leq \sum_{z \in Z_1 \cup Z_2 - A_1^*} [(A - 1) - r_z - d_z]. \quad (7)$$

If, in addition, $z \in A_1^*$ and $v \in B$ annihilates $z$, then $v$ is adjacent to the $A - 1$ X-epns of $z$ and to some $y \in X.$ Since $\deg(v) \leq A$, $v \notin N[R]$ and thus $v \in N(Y_0).$ This implies

$$\sum_{z \in A_1^*} b_z \leq \min \left( A y_0, \sum_{z \in A_1^*} [(A - 1) - r_z - d_z] \right)$$
$$= \min(A y_0, (A - 1)a_3^*). \quad (8)$$

If $z \in Y_2$ and $b_z \neq 0$ then by Lemma 5, $z$ has exactly one X-ipn, $w \in Z_1.$ However, $w$ is adjacent to $z$ and at most $A - 1$ other vertices. Thus $b_z \leq (A - 1).$ Hence from
inequalities (1), (5)–(8),

\[ |B_1| \leq \sum_{z \in Z_1 \cup Z_2 \cup Y_2} b_z \]
\[ = \sum_{z \in A_1^2 \cup A_1 \cup A_2} b_z + \sum_{z \in A_1^2} b_z + \sum_{z \in Y_2} b_z \]
\[ \leq (\Delta - 1)(a_1^* + a_1 + a_2 + a_3 + a_4) - \sum_{z \in Z_1 \cup Z_2} r_z - \sum_{z \in A_1} d_z \]
\[ + \min(\Delta y_0, (\Delta - 1)a_3^*) + (\Delta - 1)(d_0 + d_1 + d_2) \]
\[ \leq (\Delta - 1)(a_1^* + a_1 + a_2 + a_3 + a_4 + d_0) + (\Delta - 2)d_1 \]
\[ + (\Delta - 3)d_2 + \min(\Delta y_0, (\Delta - 1)a_3^*) - |R|. \] (9)

Each vertex of \( A_4 \) is adjacent to at least two vertices in \( X \) and thus is adjacent to at most \( \Delta - 2 \) vertices in \( B \). Hence,

\[ |B_2| = \sum_{z \in A_3^2} |N(z) \cap B| \]
\[ = \sum_{z \in A_3} |N(z) \cap B| + \sum_{z \in A_4} |N(z) \cap B| \]
\[ \leq (\Delta - 1)a_3^* + (\Delta - 2)a_3 + (\Delta - 2)a_4. \] (10)

For \( y \in Y_2 \) let \( k_y = |N(y) \cap Z_1| \) and let \( l_y = |N(y) \cap (Y_2 \cup Z_2)| \). By Lemmas 4 and 5, \( d_0 + d_1 + \sum_{y \in D \cup D_2} k_y = \sum_{y \in Y_2} k_y \geq |Z_1| \). If \( z \in D_2 \), then by Lemma 5, \( z \) is adjacent to at least one \( y \in Y_2 \cup Z_2 \). Thus \( \sum_{z \in D_2} l_z \geq d_2 \). It now follows that,

\[ |B_3| \leq |D|D \cup D_2| + |D_1| - \sum_{y \in D \cup D_2} k_y - \sum_{y \in D \cup D_2} l_y \]
\[ \leq \Delta d + d_0 + 2d_1 - |Z_1| - d_2 \]
\[ = \Delta d + d_0 + 2d_1 + (\Delta - 1)d_2 - (a_1^* + a_1 + a_3^* + a_3). \] (11)

Furthermore, \( |B_4| \leq (\Delta - 1)|Y_1| \). Therefore, by inequalities (9)–(11):

\[ |B| + |R| \leq (\Delta - 1)(a_1^* + a_1 + a_2 + a_3 + a_4 + d_0) \]
\[ + (\Delta - 2)d_1 + (\Delta - 3)d_2 + (\Delta - 1)a_3^* \]
\[ + (\Delta - 2)a_3 + (\Delta - 2)a_4 + \Delta d + d_0 \]
\[ + 2d_1 + (\Delta - 1)d_2 - (a_1^* + a_1 + a_3^* + a_3) \]
\[ + (\Delta - 1)|Y_1| + \min(\Delta y_0, (\Delta - 1)a_3^*) \]
\[ = (\Delta - 1)y_1 + (\Delta - 2)a_1^* + (\Delta - 2)a_1 \]
\[ + (\Delta - 1)a_2 + (\Delta - 2)a_3^* + (2\Delta - 4)a_3 \]
\[ + (2\Delta - 3)a_4 + \Delta d + \Delta d_0 + \Delta d_1 \]
\[ + (2\Delta - 4)d_2 + \min(\Delta y_0, (\Delta - 1)a_3^*). \] (12)
The number \( \eta \) of edges incident with a vertex in \( C \) and a vertex in \( X \), satisfies

\[
2|C| \leq \eta \leq |A| |Y_0| + (A - 1)(|Y_1| + |Z_1|) + (A - 2)(|Y_2| + |Z_2|) - |B|. \tag{13}
\]

Therefore, by inequalities (4), (12) and (13),

\[
2n = 2|Y_0| + 2|Y_1| + 2|Y_2| + 2|Z_1| + 2|Z_2| + 2|B| + 2|R| + 2|C|
\leq (A + 2)|Y_0| + (A + 1)(|Y_1| + |Z_1|) + A(|Y_2| + |Z_2|) + |B| + 2|R| \quad \text{(by (13))}
\leq (A + 2)y_0 + (A + 1)y_1 + Ay_2 + 2Aa_1^* + (2A - 1)a_1
+ (2A - 2)a_2 + (A + 1)a_3 + (A - 1)a_3 + Aa_4 + |B| + |R| \quad \text{(by (4))}
\leq (A + 2)y_0 + 2Ay_1 + (3A - 2)a_1^* + (3A - 3)a_1
+ (3A - 3)a_2 + (2A - 1)a_3^* + (3A - 3)a_3
+ (3A - 3)a_4 + 2Ad_0 + 2Ad_1 + (3A - 4)d_2
+ \min(|A|y_0, (A - 1)a_3^*). \quad \text{(by (12)).}
\]

By re-ordered the terms on the right hand side, we obtain

\[
2n \leq 2Ad_0 + (3A - 2)a_1^* + (A + 2)y_0 + (2A - 1)a_3^*
+ \min(|A|y_0, (A - 1)a_3^*) + 2A(d + d_1 + y_1)
+ (3A - 3)(a_1 + a_2 + a_3 + a_4) + (3A - 4)d_2. \tag{14}
\]

Let \( z \in A_1^* \) and \( w \in pn(z, X) \), then \( w \) is adjacent to \( z \) and to \( A - 1 \) vertices in \( R \). Since \( w \in N[R] \) by Theorem 3, there exists a \( y_w \in X \) such that \( pn(y_w, X) \subseteq N(w) \). Clearly \( pn(y_w, X) = \{ z \} \) and by Lemmas 4 and 5, \( y_w \in D_0 \). This implies that \( y_w \) is adjacent to exactly one vertex of \( Z_1 \cup Y_1 \) (namely \( z \)) and thus \( a_1^* \leq d_0 \). Let \( x_1 = d_0 - a_1^* \) and \( x_2 = 2a_1^* \).

\[
x_1, x_2 \geq 0,
\]

\[
x_1 + x_2 = a_1^* + d_0,
\]

and

\[
2Ad_0 + (3A - 2)a_1^* = 2Ax_1 + \left( \frac{5}{2}A - 1 \right)x_2. \tag{15}
\]

From (14) and (15) we deduce

\[
2n \leq (A + 2)y_0 + (2A - 1)a_3^* + \min(|A|y_0, (A - 1)a_3^*) + (3A - 3)(a_1 + a_2 + a_3 + a_4)
+ (3A - 4)d_2 + 2A(d + d_1 + y_1 + x_1) + \left( \frac{5}{2}A - 1 \right)x_2. \tag{16}
\]

We now make further substitutions which depend on the minimum included in (16).
Case 1: If $\Delta y_0 \geq (\Delta - 1)a_3^*$, then let $x_3 = y_0 - \frac{\Delta - 1}{\Delta} a_3^*$ and $x_4 = \frac{2\Delta - 1}{\Delta} a_3^*$.

Case 2: If $\Delta y_0 \leq (\Delta - 1)a_3^*$, then let $x_3 = a_3^* - \frac{\Delta}{\Delta - 1} y_0$ and $x_4 = \frac{2\Delta - 1}{\Delta - 1} y_0$.

Then

(i) in both Cases 1 and 2, $x_3, x_4 \geq 0$ and $x_3 + x_4 = a_3^* + y_0$
and

(ii) $(\Delta + 2)y_0 + (2\Delta - 1)a_3^* + \min(\Delta y_0, (\Delta - 1)a_3^*)$

$$= \left(\frac{4\Delta^2 - \Delta - 2}{2\Delta - 1}\right)x_4 + \begin{cases} (\Delta + 2)x_3 & \text{(Case 1)} \\ (2\Delta - 1)x_3 & \text{(Case 2)}. \end{cases}$$ (17)

From (16) and (17) we obtain

$$2n \leq (3\Delta - 3)(a_1 + a_2 + a_3 + a_4) + (3\Delta - 4)d_2 + 2\Delta(d_1 + y_1 + x_1) + \left(\frac{5}{2}\Delta - 1\right)x_2 + \max(\Delta + 2, 2\Delta - 1)x_3 + \left(\frac{4\Delta^2 - \Delta - 2}{2\Delta - 1}\right)x_4.$$ (18)

Let $h(\Delta)$ be the largest coefficient on the right hand side of (18). Since $x_1 + x_2 + x_3 + x_4 = d_0 + a_1^* + a_3^* + y_0$ (by (15) and (17)), it follows from (18) that

$$2n \leq h(\Delta)[y_0 + y_1 + (d + d_0 + d_1 + d_2) + (a_1^* + a_3^* + a_3 + a_3^*) + (a_2 + a_4)]$$

$$= h(\Delta)(y_0 + y_1 + y_2 + z_1 + z_2)$$

and therefore

$$2n \leq h(\Delta)|X|.$$ (19)

It is easily seen that

$$h(\Delta) = \begin{cases} 4 & \text{if } \Delta = 2 \\ 13 & \text{if } \Delta = 3 \\ 2 & \text{if } \Delta \geq 4 \end{cases}$$

and so the result follows immediately from (19). □

4. Extremal graphs

For $n$ even (resp. odd) let $X$ be an $\lceil n/2 \rceil$ vertex subset of $C_n$ whose induced subgraph contains no edge (resp. one edge). Further, for $n$ odd let $X$ be an independent set of $P_n$ of cardinality $\lceil n/2 \rceil$. 
In each case (by Theorem 3) \( X \) is a maximal CO-irredundant set and so \( C_n \) (and \( P_n \) for \( n \) odd) are extremal graphs for the bound (and its obvious improvement for \( n \) odd) of Theorem 4 in the case \( \Delta = 2 \).

Now suppose that \( H \) is an edge-minimal graph which attains the bound of Theorem 4 for some \( n \) and \( \Delta \geq 3 \) and let \( X \) be a maximal CO-irredundant set of \( H \) with \( |X| = \text{coir} \).

**Lemma 6.** The partition of \( V(H) \) induced by \( X \) (developed in Section 3) satisfies:

(a) \( D = D_1 = D_2 = Y_0 = Y_1 = A_3^* = \emptyset \),
(b) \( |A_4^*| = |D_0| \),
(c) \( A_1 = A_3 = A_4 = \emptyset \),
(d) (i) \( |B| = |B_1| = (\Delta - 1)|A_1^*| + |A_2| \),
    (ii) \( 2|C| = (\Delta - 2)|A_1^*| + (\Delta - 3)|A_2| \),
    (iii) \( |R| = (\Delta - 1)|A_1^*| + (\Delta - 2)|A_2| \),
(e) Each \( z \in A_1^* \) joins \( \Delta - 1 \) vertices of \( B \) and a vertex in \( D_0 \). Further, each member of \( N(z) \cap B \) joins \( z \) and each member of a vertex subset \( S \subseteq R \), where \( |S| = \Delta - 1 \),
(f) Each vertex of \( D_0 \) is adjacent to one vertex of \( A_1^* \), one vertex of \( A_2 \cup D_0 \) and \( \Delta - 2 \) vertices in \( C \),
(g) Each \( z \in A_2 \) joins one vertex of \( B \), \( w_z \), two vertices in \( A_2 \cup D_0 \) and \( \Delta - 3 \) vertices of \( C \). Further, \( w_z \) joins \( z \), one other vertex of \( N(A_2) \cap B \) and \( \Delta - 2 \) vertices of \( R \),
(h) Each vertex of \( C \) is adjacent to exactly two vertices of \( A_2 \cup D_0 \),
(i) Each vertex of \( R \) annihilates exactly one vertex of \( X \),
(j) If \( \Delta = 3 \), then \( A_2 = \emptyset \) and if \( \Delta \geq 5 \), then \( A_1^* = D_0 = \emptyset \).

**Proof.** Since \( H \) attains the bound, we have equality in all the inequalities used in the proof of Theorem 4. Therefore,

all variables in (18) with coefficients strictly less than \( h(\Delta) \), are zero. \( \tag{20} \)

i.e. for \( \Delta \geq 3 \),

\[
d = d_1 = d_2 = y_1 = x_1 = x_3 = x_4 = 0.
\]

Now \( x_3 = x_4 = 0 \) implies that \( a_3^* = y_0 = 0 \) and \( d_0 - a_1^* = x_1 = 0 \). Therefore (a) and (b) are established.

Now (c) is shown to be true. If \( \Delta = 3 \), then (20) yields \( a_1 = a_3 = a_4 = 0 \). Therefore we need only consider the case \( \Delta \geq 4 \).

Suppose (contrary to the result) that \( z \in A_4 \). Equality in (7) implies that \( b_z = \Delta - 1 \geq 1 \) and thus there is a \( w \in B_1 \) which annihilates \( z \). From (a) \( y_0 = 0 \) and so \( w \) is adjacent to some \( y \in A_1 \cup A_1^* \cup A_2 \). Equality in (2)–(4) yield \( r_y \geq \Delta - 2 \). Since \( w \in pn(y, X) \), it follows that \( w \) is adjacent to at least \( \Delta - 2 \) vertices of \( R \). Further, from (10) we deduce \( |N(z) \cap B | = \Delta - 2 \). Since \( w \) annihilates \( z \), this implies \( w \) is adjacent to at least \( \Delta - 2 \) vertices in each of \( B \) and \( R \) and to \( y \). Thus \( \text{deg}(w) \geq 2\Delta - 3 > \Delta \), a contradiction which shows \( A_4 = \emptyset \).

Suppose that \( z \in A_1 \cup A_3 \). By Lemma 4 and (a), \( z \) is adjacent to some \( y \in Y_2 = D_0 \). Using (b), we deduce \( A_1^* \neq \emptyset \). For each \( v \in A_1^* \) choose \( w_v \in pn(v, X) \). In view of Lemma 4 and (a), let \( N(v) \cap D_0 = \{ y_v \} \) and \( D^* = \{ y_v | v \in A_1^* \} \). Now \( w_v \in N[R] \) and hence annihilates
some \( u \in X \). By definition of \( A_1^* \), \( w_v \) is adjacent only to \( v \) and to \( \Delta - 1 \) vertices of \( R \). It follows that \( u = y_v \) and by Lemma 5,

\[
N(y_v) \cap Z_1 = \{ v \}.
\] (21)

Eq. (21) implies that \(|D^*| \geq |A_1^*|\) and so from (b) we deduce that \( D^* = D_0 \). This equality and (21) show that \( y \) cannot exist, a contradiction which proves that \( A_1 \cup A_3 = \emptyset \). Thus (c) is established.

Observe that (a) and (c) imply that \( X = A_1^* \cup A_2 \cup D_0 \). This fact will be used in the remainder of this proof without mention.

Equality in each of (4), (9), (12), (13) together with (a), (b) and (c) imply (d).

We now establish (e)–(h). From the definition of \( D_0 \) and Lemma 3 we deduce for \( y \in D_0 \), \( r_y = 0 \). Thus equality in (7) implies that \( b_y = \Delta - 1 \). Hence by Lemma 5, \( y \) is adjacent to exactly one vertex of \( A_1^* \), \( z_y \). Since \( b_y = \Delta - 1 \), \( z_y \) is adjacent to the \( \Delta - 1 \) annihilators of \( y \) in \( B \) and to \( y \). It follows from (b) and the definition of \( D_0 \) that each \( z \in A_1^* \) is adjacent to \( \Delta - 1 \) vertices of \( B \) and one vertex of \( D_0 \) and that each \( y \in D_0 \) is adjacent to a vertex of \( A_1^* \) and a vertex of \( A_2 \cup D_0 \). It is easily seen that,

\[
|N(A_1^*) \cap B| = (\Delta - 1)|A_1^*|.
\] (22)

Equality in (2) implies that each \( z \in A_1^* \) is annihilated by \( \Delta - 1 \) vertices in \( R \). Thus if \( w \in N(z) \cap B \), \( w \) is adjacent to \( z \) and to the \( \Delta - 1 \) vertices of \( R \) which annihilate \( z \). Hence (e) is established.

By the definition of a CO-irredundant set, each vertex of \( A_2 \) has at least one \( X \)-epn. Together (d), (22) and the definition of \( B \) imply \(|N(A_2) \cap B| = |A_2|\) and thus each vertex of \( A_2 \) is joined to exactly one vertex of \( B \). Hence each vertex of \( A_2 \) is joined to two vertices of \( A_2 \cup D_0 \) and one vertex of \( B \). Now each vertex of \( A_1^* \) has degree \( \Delta \), each vertex of \( A_2 \) has degree three and each vertex of \( D_0 \) has degree two. Therefore from (d) and the definition of \( C \), it follows that each vertex of \( A_1^* \), \( A_2 \) and \( D_0 \) is adjacent to 0, \( \Delta - 3 \) and \( \Delta - 2 \) vertices of \( C \), respectively and each vertex of \( C \) is adjacent to two vertices of \( D_0 \cup A_2 \) (establishing (f) and (h)). For each \( z \in A_2 \), equality in (3) (resp. in (7)) implies that \( z \) is annihilated by \( \Delta - 2 \) vertices of \( R \) (resp. one vertex of \( B \)). Since each vertex of \( N(A_1^*) \cap B \) has degree \( \Delta \) this implies, for each \( y \in (N(A_2) \cap B) \), \( y \) is joined to exactly one \( z \in A_2 \), one vertex of \( B \cap A_2 \) and the \( \Delta - 2 \) vertices of \( R \) which annihilate \( z \). Thus (g) is established.

Together, Theorem 3 and equality in (1), imply (i). Part (j) follows directly from (20).

\textbf{Theorem 5.} Let \( G = (V, E) \) be a graph with \( \Delta(G) \geq 3 \). Then \( G \) attains the bound established in Theorem 4 if, and only if, for some subset \( X \) of \( V \), the partition of \( G \) induced by \( X \) (developed in Section 3) satisfy conditions (a)–(j) in Lemma 6 and (k) any edge \( uv \) in \( G \), which is not required by conditions (a)–(j), is such that \( \{u, v\} \) is a subset of \( C \) or \( R \).

\textbf{Proof.} Let \( H \) be an edge-minimal spanning subgraph of \( G \) with maximum degree \( \Delta(G) \) and \( \text{coir}(H) = \text{coir}(G) \). Then, by Lemma 6, \( H \) has CO-irredundant set \( X \) with cardinality
coir($H$) and the partition of $V$ in $H$ induced by $X$ satisfies conditions (a)–(j). Thus each vertex of $X \cup B$ (in this partition) has degree $\Delta$ in $G$. It follows that the partition of $V$ in $G$ induced by $X$ is the same partition of $V$ in $H$ induced by $X$ and this partition satisfies conditions (a)–(i) in $G$. Condition (k) follows from Theorem 3 and the fact that no vertex of $C$ annihilates a vertex of $X$.

Let $G$ be a graph with CO-irredundant set $X$ whose partition of $G$ induced by $X$ (developed in Section 3) satisfy conditions (a)–(k). Theorem 3 shows $X$ is a maximal CO-irredundant set. It is easy to check that $|X|$ attains the bound established in Theorem 4. □

Figs. 1–3 show examples of extremal graphs. In each case, vertices in $D_0$ are coloured black and vertices in $A_2$ are coloured gray. Vertices in $X$ which are coloured white belong to $A_1^*$. 
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References