Constructing Singular Functions via Farey Fractions

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To illustrate some points about continued fractions, H. Minkowski in 1904 introduced the so-called ?-function. This function and some generalizations of it are known to be singular, i.e., strictly monotone with derivative 0 almost everywhere. They can be characterized by systems of functional equations, such as

\[ f\left(\frac{x}{x+1}\right) = tf(x), \quad f\left(\frac{1}{x+1}\right) = 1 - (1 - t)f(x) \quad \text{for all } x \in [0, 1], \quad (F) \]

where \( f: [0, 1] \to \mathbb{R} \) is the unknown, and

\[ r\left(\frac{x}{x+1}\right) = tr(x), \quad r\left(\frac{1}{x+1}\right) = t + (1 - t)r(x) \quad \text{for all } x \in [0, 1]. \quad (R) \]

where \( r: [0, 1] \to \mathbb{R} \) is the unknown. In both cases, \( t \in (0, 1) \) is a given parameter.

In the present note we give a general construction of singular functions, based on the Farey fractions and including, as a special case, the Minkowski function and its generalizations. In contrast to earlier proofs, the methods presented here do not make explicit use of the theory of continued fractions.

**1. INTRODUCTION**

In [4], Hermann Minkowski introduced a function \( ?: [0, 1] \to [0, 1] \) (the question mark is Minkowski’s notation). This function is defined with the use of Farey fractions in the following way. Set

\[ ?(0) := 0 \quad \text{and} \quad ?(1) := 1. \]

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Now assume that \( \frac{a}{b} \) and \( \frac{c}{d} \) are two consecutive fractions in the Farey sequence and that \( \frac{a}{b} \) and \( \frac{c}{d} \) are already defined. Then we define \( \phi \) on the mediant \( \frac{a+c}{b+d} \) by

\[
\phi \left( \frac{a+c}{b+d} \right) = \frac{1}{2} \left( \phi \left( \frac{a}{b} \right) + \phi \left( \frac{c}{d} \right) \right).
\]

In this way, the function is defined on all rational numbers between 0 and 1. It can then be shown that it can be continuously extended to the whole interval \([0, 1]\), thus leading to the continuous, strictly monotone function \( \phi : [0, 1] \to [0, 1] \).

It is clear that this function is closely connected to continued fractions, because the Farey fractions are. In fact, Minkowski introduced \( \phi \) in order to illustrate Lagrange’s condition for quadratic irrationals: *A real number is a quadratic irrational if and only if its continued fraction expansion is infinite and periodic; it is a rational if and only if its continued fraction expansion is finite.* Minkowski’s function leads to the following criterion: \( x \) is a quadratic irrational if and only if \( \phi(x) \) is a non-dyadic rational; \( x \) is rational if and only if \( \phi(x) \) is a dyadic rational.

\( \phi(x) \) has the additional property that it is singular; i.e., it is strictly monotone but has derivative 0 almost everywhere. This was proved by Arnaud Denjoy [1, 2] for a more general family of continuous functions \( r \), (where \( 0 < t < 1 \)) with the use of continued fractions; Minkowski’s function is the special case \( r = r_{1/2} \). (R. Salem later gave a short proof of the singularity of Minkowski’s function \( \phi \) in [7].) The easiest way to define \( r \) is to characterize it as the only bounded solution of the two functional equations

\[
r \left( \frac{x}{x+1} \right) = tr(x), \quad r \left( \frac{1}{2-x} \right) = t + (1-t)r(x)
\]

for all \( x \in [0, 1] \). (R)

The system (R) is due to Georges de Rham (see [5], where it is given in a slightly different form and only for \( t = 1/2 \)).

In the present note, we will also consider another family of functional equations, generalizing Minkowski’s \( \phi \)-function in a different way. In Section 2, we will show that for each \( t \in (0, 1) \), the system

\[
f \left( \frac{x}{x+1} \right) = tf(x), \quad f \left( \frac{1}{x+1} \right) = 1 - (1-t)f(x)
\]

for all \( x \in [0, 1] \). (F)

has a unique bounded solution \( f : [0, 1] \to \mathbb{R} \). In the case \( t = 1/2 \), this solution is again the Minkowski function \( \phi \).
The main thrust of this note will then be a general construction of singular functions which involves Farey fractions and which contains the solutions of the functional equations \( F \) and \( R \) as special cases. This construction could be viewed as a generalization of the methods given by R. Salem in [7]. But while Salem uses the theory of continued fractions in his arguments, the main interest in the present paper is to see how the structure of the functional equations forces the solution to be singular. We will therefore avoid continued fractions. (A good reference for continued fractions, however, is [6].)

2. CHARACTERIZING THE \( \psi \)-FUNCTION

We introduce what we call (after Masayoshi Hata [3]) modified Farey fractions. The \( n \)th modified Farey sequence \( G_n \) is defined inductively: \( G_0 = [0/1, 1/1] \), and if \( a/b, c/d \) are two consecutive elements of \( G_n \), then \( a/b, (a + c)/(b + d), c/d \) are consecutive elements of \( G_{n+1} \). The first four modified Farey sequences are therefore

\[
\begin{align*}
G_0: & \quad \frac{0}{1}, \frac{1}{1} \\
G_1: & \quad \frac{0}{1}, \frac{1}{2}, \frac{1}{1} \\
G_2: & \quad \frac{0}{1}, \frac{1}{3}, \frac{2}{3}, \frac{1}{1} \\
G_3: & \quad \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{1}{2}, \frac{3}{3}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}.
\end{align*}
\]

(The sequence \( G_n \) is called a modified Farey sequence to distinguish it from the usual \( n \)th Farey sequence, which is that subsequence of \( G_n \) whose elements have a denominator smaller than or equal to \( n \).)

Now, to make the definition of the modified Farey sequences more precise, we write \( G_n = [a_{0,n}/b_{0,n}, \ldots, a_{2^n,n}/b_{2^n,n}] \). Then the numerators \( a_{i,n} \) and the denominators \( b_{i,n} \) are defined by the recursion

\[
\begin{align*}
\begin{cases}
a_{i,n+1} = a_{[i/2],n} + a_{[i/2]+1,n} & \text{for } i \text{ odd,} \\
a_{i,n} & \text{for } i \text{ even,}
\end{cases}
\end{align*}
\begin{align*}
\begin{cases}
b_{i,n+1} = b_{[i/2],n} + b_{[i/2]+1,n} & \text{for } i \text{ odd,} \\
b_{i,n} & \text{for } i \text{ even,}
\end{cases}
\end{align*}
\]

and \( a_{0,0} = 0, a_{1,0} = b_{0,0} = b_{1,0} = 1. \) (As usual, \([x] = \max\{k \leq x \mid k \in \mathbb{Z}\} \).)
We will now show that the modified Farey sequences also satisfy another recursion, which will be quite useful to us.

**Lemma 1.** The sequences $a_{i,n}$ and $b_{i,n}$ satisfy the recursion

$$a_{i,n+1} = \begin{cases} a_{i,n} & \text{for } i = 0, \ldots, 2^n, \\ b_{2^{n+1-i},n} & \text{for } i = 2^n, \ldots, 2^{n+1}, \end{cases}$$

$$b_{i,n+1} = \begin{cases} a_{i,n} + b_{i,n} & \text{for } i = 0, \ldots, 2^n, \\ a_{2^{n+1-i},n} + b_{2^{n+1-i},n} & \text{for } i = 2^n, \ldots, 2^{n+1}, \end{cases}$$

and $a_{0,0} = 0$, $a_{1,0} = b_{0,0} = b_{1,0} = 1$.

**Proof.** The definition of the $a_{i,n}$ and $b_{i,n}$, as given above, is equivalent to

$$a_{2i,n+1} = a_{i,n} \quad \text{for } i = 0, \ldots, 2^n,$$

$$a_{2i+1,n+1} = a_{i,n} + a_{i,n+1} \quad \text{for } i = 0, \ldots, 2^n - 1,$$

$$b_{2i,n+1} = b_{i,n} \quad \text{for } i = 0, \ldots, 2^n,$$

$$b_{2i+1,n+1} = b_{i,n} + b_{i+1,n} \quad \text{for } i = 0, \ldots, 2^n - 1,$$

for $n = 0, 1, \ldots$. From this, the following set of identities can be proved by induction,

$$a_{i,n} = a_{2i,n} \quad \text{for } i = 0, \ldots, 2^{n-1},$$

$$a_{i,n} = b_{2^{n+1-2i},n} \quad \text{for } i = 2^{n-1}, \ldots, 2^n,$$

$$b_{i,n} = a_{i,n} + b_{2i,n} \quad \text{for } i = 0, \ldots, 2^{n-1},$$

$$b_{i,n} = a_{2^{n+1-2i},n} + b_{2^{n+1-2i},n} \quad \text{for } i = 2^{n-1}, \ldots, 2^n,$$

for $n = 1, 2, \ldots$. For example, the induction step for the first identity is ($i \in \{0, \ldots, 2^n\}$):

for $i$ even: $a_{i,n+1} = a_{i/2,n} = a_{i,n} = a_{2i,n+1},$

for $i$ odd: $a_{i,n+1} = a_{(i-1)/2,n} + a_{(i+1)/2,n} = a_{i-1,n} + a_{i+1,n} = a_{i,n} = a_{2i,n+1}.$

Now, identities (1) and (2) together imply the assertion. For example,

$$a_{i,n+1} = a_{2i,n+1} = a_{i,n} \quad \text{for } i = 0, \ldots, 2^n.$$

In the next lemma we note some of the basic properties of the modified Farey sequences. We omit the proof here, because these properties follow
easily from the corresponding properties of the usual Farey sequences; it is also not difficult to prove the lemma directly by induction.

**Lemma 2.** (a) For each fraction \( a_{i,n}/b_{i,n} \in G_n \), we have \( \gcd(a_{i,n}, b_{i,n}) = 1 \).

(b) \( \bigcup_{n=0}^{\infty} G_n = \mathbb{Q} \cap [0,1] \).

(c) \( a_{i+1,n} \cdot b_{i,n} - a_{i,n} \cdot b_{i+1,n} = 1 \), and therefore \( a_{i+1,n}/b_{i+1,n} - a_{i,n}/b_{i,n} = 1/b_{i,n} \cdot b_{i+1,n} \).

We are now ready to prove that the functional equations (F) have a unique bounded solution, which in the case \( t = 1/2 \) is the Minkowski function. Almost exactly the same proof works for the system (R).

**Theorem 1.** (a) The system (F) has a unique bounded solution which is, moreover, continuous.

(b) If \( t = 1/2 \), then this solution is the Minkowski function \( ?(x) \).

**Proof.** Part (a) is an application of Banach's fixed point theorem and quite standard. We give a brief outline here without many details. Define \( \mathcal{L} := \{ u : [0,1] \to \mathbb{R} \mid u \text{ bounded} \} \) and an operator \( T : \mathcal{L} \to \mathcal{L} \) by

\[
(Tu)(y) := \begin{cases} 
  tu \left( \frac{y}{1-y} \right) & \text{for } y \in [0, \frac{1}{2}], \\
  1 - (1-t)u \left( \frac{1-y}{y} \right) & \text{for } y \in \left( \frac{1}{2}, 1 \right].
\end{cases}
\]

Then the solutions of (F) are exactly the fixed points of \( T \). Moreover, \( T \) is a contraction with respect to the supremum norm \( \| \cdot \|_\infty \) (its contraction factor is \( \max(t, 1-t) \)). Thus, Banach's fixed point theorem implies that \( T \) has exactly one fixed point in \( \mathcal{L} \). To prove that the solution is continuous, consider the space \( \mathcal{L}' := \{ u \in C[0,1] \mid u(0) = 0, u(1) = 1 \} \) and observe that \( T \) maps \( \mathcal{L}' \) into \( \mathcal{L}' \).

To prove part (b), note that \( ? \) is constructed in such a way that it satisfies \( ?(a_{i,n}/b_{i,n}) = i/2^n \). From this and Lemma 1 we get the two identities

\[
\frac{1}{2} \left( \frac{a_{i,n}}{b_{i,n}} \right) = \frac{i}{2^{n+1}} = \left( \frac{a_{i,n+1}}{b_{i,n+1}} \right)
\]

\[
= ? \left( \frac{a_{i,n}}{a_{i,n} + b_{i,n}} \right) = ? \left( \frac{a_{i,n}/b_{i,n}}{a_{i,n}/b_{i,n} + 1} \right)
\]

for \( i = 0, \ldots, 2^n \),
\[
1 - \frac{1}{2} \left( \frac{a_{i,n}}{b_{i,n}} \right) = 1 - \frac{1}{2} \frac{i}{2^n} = \frac{a_{i+1,n+1}}{b_{i+1,n+1}}
\]
\[
= \frac{b_{i,n}}{a_{i,n} + b_{i,n}} = \frac{1}{a_{i,n}/b_{i,n} + 1}
\]
for \( i = 2^n, \ldots, 2^{n+1} \).

This means that \( \psi \) satisfies the two functional equations (with \( t = 1/2 \)) on the Farey fractions. Since \( \psi \) is continuous and the Farey fractions are dense in \([0, 1]\), \( \psi \) must satisfy the equations \((F)\) on the whole interval. It is therefore the unique bounded solution of \((F)\).

3. SINGULARITY OF \( f_t \)

In this section, we will give a general construction for singular functions which involves Farey fractions. We will then show that the solutions of \((F)\) and \((R)\) fit into this scheme. As mentioned in the Introduction, A. Denjoy and R. Salem proved singularity of the functions \( r_t \) by making use of the theory of continued fractions. While it would be possible to use this theory for our purpose, we prefer to work from a real-variable point of view. This has the advantage that we can see how the structure of the functional equations forces the solution to be singular, so that it would be possible to use the same method later for other constructions. The disadvantage is, however, that we have to rederive one known theorem about continued fractions in a different setting. Lemma 3 below is equivalent to the statement that the continued fraction expansion of almost every real number is unbounded; see, for example, Theorem V.1.1 in [6].

We need some notation. Consider any \( x \in [0, 1) \). For each \( n \in \mathbb{N}_0 \), define \( x_n, y_n \) by

\[
x_n := \frac{a_{i,n}}{b_{i,n}} \leq x < \frac{a_{i+1,n+1}}{b_{i+1,n+1}} =: y_n.
\]

We always have \( x_{n-1} \leq x_n < y_n \leq y_{n-1} \); if \( x_{n-1} = x_n \) then \( y_n < y_{n-1} \), and if \( y_{n-1} = y_n \) then \( x_n > x_{n-1} \). Thus there are two mutually exclusive cases for each \( n \in \mathbb{N} \): either \( x_{n-1} = x_n \) or \( y_{n-1} = y_n \). For notational convenience, we defined the sequence \( \xi_n \subseteq (0, 1) \) by

\[
\xi_n := \begin{cases} 
0 & \text{if } x_{n-1} = x_n, \\
1 & \text{if } y_{n-1} = y_n.
\end{cases}
\]

(Of course, the \( \xi_n \)'s depend on \( x \), but we will not explicitly denote this.)
As a side note, we mention that if such a sequence $\xi_n$ is given, then one can find the $x \in [0, 1]$ generating that sequence by setting $x_n := a_{i_n,n}/b_{i_n,n}$ where $i_n = \sum_{k=1}^{n} \xi_k 2^{n-k}$, and then taking $x := \lim_{n \to \infty} x_n$. With this interpretation, we could write $x = (0.\xi_1 \xi_2 \xi_3 \ldots)_F$ and would have a “Farey expansion” of $x$ which behaves in some ways as an analogon to the usual dyadic expansion. Rational numbers would have two different such expansions, one where only finitely many $\xi_n$'s are 0, and one where only finitely many $\xi_n$'s are 1. For example, $x = 1/3$ would have the two expansions $x = (0.010000 \ldots)_F = (0.001111 \ldots)_F$. Moreover, quadratic irrationals would have a periodic “Farey expansion”; for example, the golden mean $\phi = (\sqrt{5} - 1)/2$ would have the expansion $x = (0.10101010 \ldots)_F$. One could re-interpret the Minkowski function as mapping $x = (0.\xi_1 \xi_2 \xi_3 \ldots)_F$ to $? = (0.\xi_1 \xi_2 \xi_3 \ldots)_2$ where the index 2 denotes the dyadic expansion. Of course, the “Farey expansion” is also closely connected to continued fractions; roughly, a block of precisely $N$ consecutive 0's or 1's in the Farey expansion corresponds to an entry $N$ in the continued fraction expansion. If we were to explore this relationship more closely, we could deduce Lemma 3 below from Theorem V.1.1 in [6]; but as explained above, we want to avoid the theory of continued fractions in the present note.

For any $N \in \mathbb{N}$, define the set $S_N$ to consist of those $x \in [0, 1)$ for which the number of consecutive 0's or 1's in the Farey expansion is bounded by $N$. Formally,

$$S_N := \{x \in [0, 1) \mid \text{for all } n \in \mathbb{N}; \xi_n = \xi_{n+1} = \cdots = \xi_{n+M-1} \Rightarrow M \leq N\}.$$ 

When we show that functions such as $f_i$ or $r_i$ are singular, we have to show that their derivatives are 0 except on a set of Lebesgue measure 0. This exceptional set will be $\bigcup_{N \in \mathbb{N}} S_N$, so we will now prove that each $S_N$ has measure 0.

**Lemma 3.** For each $N \in \mathbb{N}$, $m(S_N) = 0$, where $m$ stands for Lebesgue measure.

**Proof.** For each $k \in \mathbb{N}$, define the set $S_N(k)$ recursively by $S_N(0) := [0, 1)$ and

$$S_N(k) := \{x \in S_N(k-1) \mid \text{0, 1 \in } \{\xi_{N+1(k-1)+1}, \ldots, \xi_{N+1(k)}\}\}.$$ 

(For example, $x = 1/3$ is in $S_2(1)$ but not in $S_2(2)$.) Since every $x$ which has no more than $N$ consecutive 0's or 1's in its Farey expansion must have a 0 and a 1 in any block of length $N + 1$, we have $S_N \subseteq S_N(1) \subseteq S_N(k-1)$. The lemma is proved if we show that $\lim_{k \to \infty} m(S_N(k)) = 0$. 


Note that the above definition of $S_N(k)$ is equivalent to

$$S_N(k) := \{ x \in S_N(k-1) \mid x_{(N+1)(k-1)} \neq x_{(N+1)k}, y_{(N+1)(k-1)} \neq y_{(N+1)k} \}.$$

$S_N(k)$ is a collection of intervals of the form

$$I = \left[ \frac{a_i, (N+1)k}{b_i, (N+1)k}, \frac{a_i+1, (N+1)k}{b_i+1, (N+1)k} \right].$$

Moreover,

$$I \cap S_N(k+1) = \left[ \frac{a_2^{(N+1)i+1, (N+1)(k+1)}}{b_2^{(N+1)i+1, (N+1)(k+1)}}, \frac{a_2^{(N+1)i+1, (N+1)(k+1) - 1, (N+1)(k+1)}}{b_2^{(N+1)i+1, (N+1)(k+1) - 1, (N+1)(k+1)}} \right],$$

the latter set consists of $2^{N+1} - 2$ of the intervals which make up $S_N(k+1)$.

Now,

$$m(I) = \frac{1}{b_i, (N+1)k \cdot b_{i+1, (N+1)k}},$$

while

$$m(I \cap S_N(k+1)) = m(I) - m \left[ \frac{a_2^{(N+1)i, (N+1)(k+1)}}{b_2^{(N+1)i, (N+1)(k+1)}}, \frac{a_2^{(N+1)i+1, (N+1)(k+1)}}{b_2^{(N+1)i+1, (N+1)(k+1)}} \right]$$

$$- m \left[ \frac{a_2^{(N+1)i+1, (N+1)(k+1) - 1, (N+1)(k+1)}}{b_2^{(N+1)i+1, (N+1)(k+1) - 1, (N+1)(k+1)}}, \frac{a_2^{(N+1)i+1, (N+1)(k+1)}}{b_2^{(N+1)i+1, (N+1)(k+1)}} \right]$$

$$= \frac{1}{b_i, (N+1)k \cdot b_{i+1, (N+1)k}} - \frac{1}{b_i, (N+1)k \cdot b_2^{(N+1)i+1, (N+1)(k+1)}} - \frac{1}{b_2^{(N+1)i+1, (N+1)(k+1) - 1, (N+1)(k+1)} \cdot b_{i+1, (N+1)k}}.$$

From the definition of the modified Farey fractions we get that

$$b_2^{(N+1)i+1, (N+1)(k+1)} = (N + 1) b_i, (N+1)k + b_{i+1, (N+1)k},$$

$$b_2^{(N+1)i+1, (N+1)(k+1) - 1, (N+1)(k+1)} = b_i, (N+1)k + (N + 1) b_{i+1, (N+1)k}.$$
Therefore,
\[
\frac{m(I \cap S_N(k + 1))}{m(I)} = 1 - \frac{b_{i+1,(N+1)k}}{(N+1)b_{i,(N+1)k} + b_{i+1,(N+1)k}} - \frac{b_{i,(N+1)k}}{b_{i,(N+1)k} + (N+1)b_{i+1,(N+1)k}}
\]
\[
= 1 - \frac{1}{(N+1)(b_{i,(N+1)k}/b_{i+1,(N+1)k}) + 1} - \frac{1}{(N+1)(b_{i+1,(N+1)k}/b_{i,(N+1)k}) + 1}
\]
\[
\leq 1 - \frac{2}{N+2},
\]
because the function \( x \rightarrow 1/((N+1)x+1) + 1/((N+1)/x+1) \) has a unique minimum on the positive halfline at \( x = 1 \). Therefore, \( m(S_N(k + 1)) \leq (1 - 2/(N + 2)) \cdot m(S_N(k)) \), which implies \( \lim_{k \to \infty} m(S_N(k)) = 0 \).

With the use of this lemma, we can now give a fairly general construction of singular functions which uses the modified Farey fractions.

**Theorem 2.** Let \( f \) be a continuous function \( f:[0,1] \to \mathbb{R} \) with \( f(0) = 0 \) and \( f(1) = 1 \) (without loss). Suppose that there exists a set \( T \subset (0,1) \) which is bounded away from 0 or from 1 (i.e., \( \text{dist}(T,0) > 0 \) or \( \text{dist}(T,1) > 0 \)) such that for any two consecutive Farey fractions \( a_{i,n}/b_{i,n}, a_{i+1,n}/b_{i+1,n} \in G_n \) there exists a \( t_{i,n} \in T \) with
\[
f\left( \frac{a_{i+1,n+1}}{b_{i+1,n+1}} \right) = t_{i,n}f\left( \frac{a_{i,n}}{b_{i,n}} \right) + (1 - t_{i,n})f\left( \frac{a_{i+1,n}}{b_{i+1,n}} \right). \tag{3}
\]
Then \( f \) is strictly monotone, and its derivative is 0 almost everywhere. In other words, \( f \) is then singular.

**Proof.** (a) \( f \) is strictly monotone. This follows from the denseness of \( \bigcup_{N \in \mathbb{N}} G_N \) and the continuity of \( f \) if we show that \( f(a_{i,n}/b_{i,n}) < f(a_{i+1,n}/b_{i+1,n}) \) for all \( n \in \mathbb{N}_0 \) and \( i = 0,\ldots,2^n - 1 \). The latter assertion is easily proved by induction on \( n \): For \( n = 0 \), we have \( f(0) = 0 < 1 = f(1) \); and if \( f(a_{i,n}/b_{i,n}) < f(a_{i+1,n}/b_{i+1,n}) \) is already proved, then
\[
f\left( \frac{a_{i+1,n+1}}{b_{i+1,n+1}} \right) = f\left( \frac{a_{i,n}}{b_{i,n}} \right) < t_{i,n}f\left( \frac{a_{i,n}}{b_{i,n}} \right) + (1 - t_{i,n})f\left( \frac{a_{i+1,n}}{b_{i+1,n}} \right)
\]
\[
= f\left( \frac{a_{i+1,n+1}}{b_{i+1,n+1}} \right) = f\left( \frac{a_{i+1,n+1}}{b_{i+1,n+1}} \right) = f\left( \frac{a_{2i+2,n+1}}{b_{2i+2,n+1}} \right)
\]
with some \( t_{i,n} \in T \).
(b) $f' = 0$ almost everywhere. Since $f$ is monotone, its derivative exists almost everywhere. Take a point $x \in [0, 1]$ with $f'(x) \in \mathbb{R}$ and assume $f'(x) \neq 0$. With $x_n, y_n$ as defined above, we have $(f(y_n) - f(x_n))/(y_n - x_n) \to f'(x)$, and it follows that

$$q_n := \frac{(f(y_{n+1}) - f(x_{n+1}))/ (y_{n+1} - x_{n+1})}{(f(y_n) - f(x_n))/ (y_n - x_n)}$$

$$= \frac{y_n - x_n}{y_{n+1} - x_{n+1}} \frac{f(y_{n+1}) - f(x_{n+1})}{f(y_n) - f(x_n)} \to 1.$$  

We will now show that this is impossible for $x \not\in \bigcup_{N \in \mathbb{N}} S_N$, i.e., almost everywhere.

If $\xi_{n+1} = 0$ (or equivalently $x_n = x_{n+1}$), there exists, by Condition (3), a $t_n \in T$ such that $f(y_{n+1}) = t_n f(x_n) + (1 - t_n) f(y_n)$, which implies

$$\frac{f(y_{n+1}) - f(x_{n+1})}{f(y_n) - f(x_n)} = t_n \frac{f(y_n) + (1 - t_n) f(y_n) - f(x_n)}{f(y_n) - f(x_n)} = 1 - t_n.$$  

We also have

$$\frac{y_n - x_n}{y_{n+1} - x_{n+1}} = \frac{b_{i_{n+1}, n + 1} \cdot b_{i_{n+1}, n} + 1}{b_{i_{n+1}, n} \cdot b_{i_{n+1}, n} + 1} = \frac{b_{i_{n+1}, n} \cdot b_{i_{n+1}, n}}{b_{i_{n+1}, n} \cdot b_{i_{n+1}, n}}$$

$$= 1 + \frac{b_{i_{n+1}, n}}{b_{i_{n+1}, n}}.$$  

If $\xi_{n+1} = 1$, we similarly get

$$\frac{f(y_{n+1}) - f(x_{n+1})}{f(y_n) - f(x_n)} = t_n \quad \text{and} \quad \frac{y_n - x_n}{y_{n+1} - x_{n+1}} = 1 + \frac{b_{i_{n+1}, n}}{b_{i_{n+1}, n}}.$$  

Define

$$c_n := \begin{cases} \frac{b_{i_{n+1}, n}}{b_{i_{n+1}, n}} & \text{if } \xi_{n+1} = 0, \\ \frac{b_{i_{n+1}, n}}{b_{i_{n+1}, n}} & \text{if } \xi_{n+1} = 1. \end{cases}$$  

If there are two consecutive 0’s in the Farey expansion of $x$, $\xi_{n+1} = \xi_{n+2} = 0$, we have

$$c_{n+1} = \frac{b_{i_{n+1}, n + 1} + 1}{b_{i_{n+1}, n} \cdot b_{i_{n+1}, n}} = \frac{1}{1 + 1/c_n}.$$  

Similarly, if $\xi_{n+1} = \xi_{n+2} = 1$, we again have

$$c_{n+1} = \frac{b_{i_{n+1}, n + 1} + 1}{b_{i_{n+1}, n} \cdot b_{i_{n+1}, n}} = \frac{1}{1 + 1/c_n}.$$
Thus, if there are two consecutive 0’s or 1’s in the Farey expansion of \(x\), then \(c_n \leq 1/m\) implies \(c_{n+1} \leq 1/(m+1)\).

Assume now that \(T\) is bounded away from 0. Also, since \(x\) is not in any \(S_N\), we can find arbitrarily many consecutive 0’s in the Farey expansion of \(x\). This means that there is a subsequence \(n_k\) such that \(\xi_{n_k+1} = 0\) and \(c_{n_k} \to 0\) as \(k \to \infty\). Therefore,

\[
q_{n_k} = (1 + c_{n_k}) \cdot (1 - t_{n_k}) \\
\leq (1 + c_{n_k}) \cdot (1 - \text{dist}(T,0)) \to 1 - \text{dist}(T,0) < 1,
\]

which contradicts \(q_n \to 1\). If \(T\) is bounded away from 1, then we get a similar contradiction from the fact that we can also find arbitrarily many consecutive 1’s in the Farey expansion of \(x\).

**Corollary.** (1) The solution \(f_t\) of (F) is singular for every \(t\).

(2) The solution \(r_t\) of (R) is singular for every \(t\).

**Proof.** (1) We will now prove that we can choose \(T = (t,1-t)\) in Theorem 2. In fact, define

\[
t_{i,n} := \begin{cases} 1 - t & \text{if } i \text{ is even,} \\ t & \text{if } i \text{ is odd,} \end{cases} \text{ for } i = 0, \ldots, 2^n - 1.
\]

Then \(f_t\) satisfies

\[
f_t \left( \frac{a_{2i+1,n+1}}{b_{2i+1,n+1}} \right) = t_{i,n} f_t \left( \frac{a_{i,n}}{b_{i,n}} \right) + (1 - t_{i,n}) f_t \left( \frac{a_{i+1,n}}{b_{i+1,n}} \right),
\]

for \(i = 0, \ldots, 2^n - 1\).

This is exactly Condition (3) and can be proved by induction: For \(n = 0\), we have \(f_t(a_{1,1}/b_{1,1}) = f_t(1/2) = t = (1-t)f_t(0) + tf_t(1)\) (and \(t_{0,0} = 1-t\)); and if the formula is already proved for \(n\), then we get for \(i = 0, \ldots, 2^n - 1\),

\[
f_t \left( \frac{a_{2i+1,n+2}}{b_{2i+1,n+2}} \right) = f_t \left( \frac{a_{2i+1,n+1}}{a_{2i+1,n+1} + b_{2i+1,n+1}} \right) = f_t \left( \frac{a_{2i+1,n+1}}{b_{2i+1,n+1}} \right) \\
= t_{i,n} f_t \left( \frac{a_{i,n}}{b_{i,n}} \right) + (1 - t_{i,n}) f_t \left( \frac{a_{i+1,n}}{b_{i+1,n}} \right) \\
= t_{i,n} f_t \left( \frac{a_{i,n} + b_{i,n}}{a_{i,n}} \right) + (1 - t_{i,n}) f_t \left( \frac{a_{i+1,n}}{a_{i+1,n} + b_{i+1,n}} \right) \\
= t_{i,n} f_t \left( \frac{a_{i,n+1} + 1}{b_{i,n+1}} \right) + (1 - t_{i,n}) f_t \left( \frac{a_{i+1,n+1}}{b_{i+1,n+1}} \right),
\]

for \(i = 0, \ldots, 2^n - 1\).
Finally, the definition of the $t_i$ implies $t_i, n = t_i, n + 1$ for $i = 0, \ldots, 2^n - 1$, and $t_{2^n - 1 - i, n} = 1 - t_{i, n + 1}$ for $i = 2^n, \ldots, 2^{n+1} - 1$.

(2) It is not difficult to prove (again by induction) that $r_i$ satisfies

$$r_i \left( \frac{a_{2i+1, n+1}}{b_{2i+1, n+1}} \right) = (1 - t) r_i \left( \frac{a_i, n}{b_i, n} \right) + t r_i \left( \frac{a_{i+1, n}}{b_{i+1, n}} \right),$$

so that we can choose $T = (1 - t)$ in Theorem 2.

Theorem 2 gives us a condition (Formula (3)) for a continuous function to be singular. But it is also possible to use that formula to construct a function $f: \mathbb{Q} \cap [0, 1] \to \mathbb{R}$ as follows. Fix a set $T \subset (0, 1)$ and let $f(0) := 0$ and $f(1) := 1$ (without loss of generality). Then define $f$ recursively on the sets $G_1, G_2, \ldots$ with the use of Formula (3) and arbitrary $t_i, n \in T$. In this way, $f$ is defined on the rationals. If $f$ were continuously extendable to the whole interval $[0, 1]$ then the extension would be a singular function (by Theorem 2). Under which conditions on $T$ is this possible?

**Theorem 3.** Let $f: \mathbb{Q} \cap [0, 1] \to \mathbb{R}$ be the function described above. Then $f$ is continuously extendable if $T$ is bounded away from 0 and from 1.

**Proof.** We start by showing that

$$M_n := \max_{i = 0, \ldots, 2^n - 1} \left| f \left( \frac{a_i, n}{b_i, n} \right) - f \left( \frac{a_{i+1, n}}{b_{i+1, n}} \right) \right| \to 0 \quad (n \to \infty).$$

This follows from Formula (3). Let $d := \min(\text{dist}(T, 0), \text{dist}(T, 1)) > 0$. Then

$$\left| f \left( \frac{a_{2i, n+1}}{b_{2i, n+1}} \right) - f \left( \frac{a_{2i+1, n+1}}{b_{2i+1, n+1}} \right) \right| = \left| f \left( \frac{a_i, n}{b_i, n} \right) - t_i, n f \left( \frac{a_i, n}{b_i, n} \right) - (1 - t_i, n) f \left( \frac{a_{i+1, n}}{b_{i+1, n}} \right) \right|$$

$$= (1 - t_i, n) \cdot \left| f \left( \frac{a_i, n}{b_i, n} \right) - f \left( \frac{a_{i+1, n}}{b_{i+1, n}} \right) \right| \leq (1 - d) \cdot M_n.$$
and, similarly,
\[
\left| f\left(\frac{a_{2i+1,n+1}}{b_{2i+1,n+1}}\right) - f\left(\frac{a_{2i+2,n+1}}{b_{2i+2,n+1}}\right) \right| \leq (1 - d) \cdot M_n.
\]

Therefore \(M_{n+1} \leq (1 - d) \cdot M_n\), and the assertion follows.

To prove extendability of \(f\), we have to show that for every \(\varepsilon > 0\) there exists a \(\delta > 0\) such that \(|f(a_{j_1,k_1}/b_{j_1,k_1}) - f(a_{j_2,k_2}/b_{j_2,k_2})| < \varepsilon\) whenever \(|a_{j_1,k_1}/b_{j_1,k_1} - a_{j_2,k_2}/b_{j_2,k_2}| < \delta\).

First of all, we remark that monotonicity of \(f\) on \(G_n\) can be shown exactly as in the proof of Theorem 2.

Now, if \(\varepsilon > 0\) is given, we can choose \(n\) such that \(M_n < \varepsilon/2\). Let
\[
\delta < \min_{i=0,\ldots,2^n-1} \{a_{i+1,n}/b_{i+1,n} - a_{i,n}/b_{i,n}\}.
\]

Then \(|a_{j_1,k_1}/b_{j_1,k_1} - a_{j_2,k_2}/b_{j_2,k_2}| < \delta\) implies that there exists an \(i\) such that
\[
a_{i-1,n}/b_{i-1,n} \leq a_{j_1,k_1}/b_{j_1,k_1}, a_{j_2,k_2}/b_{j_2,k_2} \leq a_{i+1,n}/b_{i+1,n}.
\]

Because of the monotonicity we have
\[
\left| f\left(\frac{a_{j_1,k_1}}{b_{j_1,k_1}}\right) - f\left(\frac{a_{j_2,k_2}}{b_{j_2,k_2}}\right) \right|
\leq f\left(\frac{a_{i+1,n}}{b_{i+1,n}}\right) - f\left(\frac{a_{i,n}}{b_{i,n}}\right)
\leq 2 \cdot M_n \leq \varepsilon.
\]

4. SOME REMARKS

What is the relationship between the systems (F) and (R)? It is quite easy to see that for \(t \neq 1/2\) the unique bounded solution \(f_t\) of (F) is different from the solution \(r_t\) of (R). In fact, from the functional equations we get \(f_t(0) = r_t(0) = 0, f_t(1) = r_t(1) = 1, f_t(1/2) = r_t(1/2) = t\), but \(f_t(2/3) = 1 - r_t(1 - t) \neq r_t(2 - t) = r_t(2/3)\) if \(t \neq 1/2\).

The case \(t = 1/2\) is slightly less obvious. As we have seen, the Minkowski function is then the only bounded solution of either system. Does this mean that the two systems are equivalent, i.e., that any solution (bounded or not) of (F) also solves (R) and vice versa? The answer to the first part of this question is positive: Solutions to (F) are also solutions to (R) as can be seen from the following.
Observation. Any solution $f$ of $(F)$ is symmetric with respect to the point $(1/2, 1/2)$; more precisely, it satisfies the functional equation

$$f(x) + f(1-x) = 1 \quad \text{for all } x \in [0,1]. \quad (S)$$

Proof. Without loss of generality we can assume that $x \in [0,1/2]$. Then there exists a $y \in [0,1]$ such that $x = y/(y+1)$. It follows that $1-x = 1/(y+1)$. Therefore,

$$f(x) + f(1-x) = f\left(\frac{y}{y+1}\right) + f\left(\frac{1}{y+1}\right) = \frac{1}{2}f(y) + 1 - \frac{1}{2}f(y) = 1.$$

With the use of $(S)$ we can now show that any solution $f$ of $(F)$ must also satisfy $(R)$:

$$f\left(\frac{1}{2} - x\right) = f\left(\frac{1}{(1-x) + 1}\right) = 1 - \frac{1}{2}f(1-x)$$

$$= 1 - \frac{1}{2}(1 - f(x)) = \frac{1 + f(x)}{2}.$$

However, the converse is not true. It can be shown by constructing the general solution of both $(F)$ and $(R)$ that there is an infinite family of functions which solve $(R)$ but not $(F)$.

In [5], G. de Rham, besides considering the Minkowski function, also introduces another class of singular functions. He defines the function $m_t$ as the only continuous solution of the system

$$m\left(\frac{x}{2}\right) = tm(x), \quad m\left(\frac{x + 1}{2}\right) = t + (1-t)m(x) \quad \text{for all } x \in [0,1],$$

$$(R')$$

where $t \in (0,1)$. He then proves that for $t \neq 1/2$, $m_t$ is singular. The functions $m_t$ and $r_t$ are related by the Minkowski function $?$: $m_t(?(x)) = r_t(x)$. This can be seen from the functional equations; the function $m_t$ satisfies

$$m_t\left(\frac{x}{x + 1}\right) = m_t\left(\frac{1}{2}?(x)\right) = tm_t(?(x))$$

and

$$m_t\left(\frac{1}{2} - x\right) = m_t\left(\frac{1 + ?(x)}{2}\right) = t + (1-t)m_t(?(x)).$$
Since \( r \) is the only bounded function satisfying these two functional equations (and since \( m, e \)? is bounded), it follows that \( r = m, e \). Of course, this relationship does not mean that one could deduce the singularity of \( r \) directly from that of \( m \).

We have seen that the “Farey expansion” behaves in some ways like the usual dyadic expansion. So is there a meaningful generalization of the Farey expansion which corresponds to the general \( b \)-adic expansion? In the case \( b = 3 \), for example, one could do the following. Define the \( n \)th ternary Farey sequences \( G_n^{(3)} \) inductively: \( G_0^{(3)} = [0/1, 1/1] \), and if \( a/b, c/d \) are two consecutive elements of \( G_n^{(3)} \), then \( a/b, (2a + c)/(2b + d), (a + 2c)/(b + 2d), c/d \) are consecutive elements of \( G_{n+1}^{(3)} \). (Check that these fractions are arranged in increasing order!) If we write \( G_n^{(3)} = [a_{0,n}/b_{0,n}, \ldots, a_{3^n,n}/b_{3^n,n}] \), then a “ternary Farey expansion” \( x = (0, \xi_1, \xi_2, \xi_3, \ldots)_{3,3} \) (with \( \xi_n \in \{0,1,2\} \)) could be interpreted by setting \( x_n = a_{i_n,n}/b_{i_n,n} \) where \( i_n = \sum_{k=1}^{n} \xi_k 3^{n-k} \), and then taking \( x := \lim_{n \to \infty} x_n \). This generalization would make sense from a real-variable point of view, because the \( b \)-adic Farey fractions can likely be used to construct a larger class of singular functions, much in the same way as in the dyadic case. (This would require a generalization of Lemma 3—another reason for giving its full proof here.) However, the number theoretic value of this generalization is limited, because the \( b \)-adic Farey fractions do not have properties (a)–(c), as given in Lemma 2, which are crucial in Diophantine approximation.

REFERENCES