JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **200,** 259–271 (1996)
ARTICLE NO. 0203

Iterative Solution of Nonlinear Equations of the Φ-Strongly Accretive Type

M. O. Osilike

Department of Mathematics, University of Nigeria, Nsukka, Nigeria

Let $q > 1$, and let E be a real q-uniformly smooth Banach space. Let $T: E \to E$ be a Lipschitz ϕ-strongly accretive operator and suppose the equation $Tx = f$, $f \in E$, has a solution. It is proved that under suitable conditions on the real sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$, the iteration process

$$x_0 \in E$$

$$y_n = (1 - \beta_n)x_n + \beta_n\big(f + (I - T)x_n\big), \qquad n \geq 0$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n\big(f + (I - T)y_n\big), \qquad n \geq 0$$

converges strongly to the unique solution of the equation $Tx = f$. A related result deals with the approximation of fixed points of ϕ-hemicontractive operators—a class of operators which is much more general than the important class of strongly pseudocontractive operators. © 1996 Academic Press, Inc.

INTRODUCTION

Let E be a real Banach space and let $q > 1$. We denote by J_q the generalized duality mapping from E into 2^{E^*} given by

$$J_q(x) = \left\{f \in E^*: \langle x, f^* \rangle = \|x\|^q \text{ and } \|f^*\| = \|x\|^{q-1}\right\},$$

where E^* denotes the dual of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. In particular, $J = J_2$ is called the normalized duality mapping and $J_q(x) = \|x\|^{q-2}J(x)$ if $x \neq 0$. E is uniformly smooth if and only if J_q is single-valued and uniformly continuous on any bounded subset of E. In the following we shall denote the single-valued generalized duality mapping by j_q.

An operator T with domain $D(T)$ and range $R(T)$ in E is called *strongly accretive* if for all $x, y \in D(T)$, there exist $j(x - y) \in J(x - y)$ and

0022-247X/96 $18.00

a constant $k > 0$ such that

$$\langle Tx - Ty, j(x-y)\rangle \geq k\|x-y\|^2. \quad (1)$$

Without loss of generality we may assume $k \in (0,1)$. T is called *accretive* if for all $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \geq 0. \quad (2)$$

T is called *ϕ-strongly accretive* (see for example [9]) if for all $x, y \in D(T)$ there exist $j(x-y) \in J(x-y)$ and a strictly increasing function $\phi: [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that

$$\langle Tx - Ty, j(x-y)\rangle \geq \phi(\|x-y\|)\|x-y\|. \quad (3)$$

Every strongly accretive operator is ϕ-strongly accretive with $\phi: [0,\infty) \to [0,\infty)$ defined by $\phi(s) = ks$. The following example shows that the class of strongly accretive operators is a proper subset of the class of ϕ-strongly accretive operators.

EXAMPLE. Let $E = \Re$ (the reals with the usual norm) and let $K = [0,\infty)$. Define $T: K \to K$ by

$$Tx = x - \frac{x}{1+x}.$$

It is easy to verify that T is ϕ-strongly accretive with $\phi: [0,\infty) \to [0,\infty)$ defined by $\phi(s) = s^2/(1+s)$. However, given any $k \in (0,1)$, if we choose $x \in K$ such that $0 < x < k/(1-k)$ and $y = 0$ then

$$\langle Tx - Ty, (x-y)\rangle < k|x-y|^2.$$

Hence T is not strongly accretive.

Closely related to the class of accretive operators is the class of *pseudocontractive* operators. An operator T with domain $D(T)$ and range $R(T)$ in E is called *strongly pseudocontractive* if for all $x, y \in D(T)$, there exist $j(x-y) \in J(x-y)$ and a constant $t > 1$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \frac{1}{t}\|x - \|^2. \quad (4)$$

If $t = 1$ in (4), then T is called *pseudocontractive*.

We call T *ϕ-strongly pseudocontractive* if for all $x, y \in D(T)$ there exist $j(x-y) \in J(x-y)$ and a strictly increasing function $\phi: [0,\infty) \to [0,\infty)$

with $\phi(0) = 0$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \phi(\|x - y\|)\|x - y\|. \tag{5}$$

T is called *ϕ-hemicontractive* if $F(T) = \{x \in D(T): Tx = x\} \neq \emptyset$ and for all $x \in D(T)$, $x^* \in F(T)$, there exist $j(x - x^*) \in J(x - x^*)$ and a strictly increasing function $\phi: [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle Tx - x^*, j(x - x^*) \rangle \leq \|x - x^*\|^2 - \phi(\|x - x^*\|)\|x - x^*\|. \tag{6}$$

Every strongly pseudocontractive operator is ϕ-strongly pseudocontractive, and every ϕ-strongly pseudocontractive operator with a nonempty fixed point set is ϕ-hemicontractive. If E, K, and T are as defined in the example above, and I denotes the identity operator, then the operator $(I - T): K \to K$ is ϕ-strongly pseudocontractive but not strongly pseudocontractive. For an example of an operator which is ϕ-hemicontractive but not ϕ-strongly pseudocontractive, the reader may consult [11].

It follows from inequalities (1)–(5) that T is pseudocontractive (respectively, strongly pseudocontractive, ϕ-strongly pseudocontractive) if and only if $(I - T)$ is accretive (respectively, strongly accretive, ϕ-strongly accretive), so that the mapping theory for accretive operators (respectively, strongly accretive operators, ϕ-strongly accretive operators) is intimately connected with the fixed point theory of pseudocontractive operators (respectively, strongly pseudocontractive operators, ϕ-strongly pseudocontractive operators).

The accretive operators were introduced independently in 1967 by Browder [3] and Kato [17]. Interest in accretive operators stems mainly from their firm connection with the existence theory for nonlinear evolution equations in Banach spaces. It is well known that many physically significant problems can be modeled in terms of an initial value problem of the form

$$\frac{du}{dt} = -Tu, \qquad u(0) = u_0, \tag{7}$$

where T is strongly accretive, accretive, or ϕ-strongly accretive in an appropriate Banach space. Typical examples of how such evolution equations arise are found in models involving the heat, the wave, or the Schrödinger equation (see for example [31]).

An early fundamental result in the theory of accretive operators due to Browder [3] states that the initial value problem (7) is solvable if T is locally Lipschitzian and accretive on E—a result which was subsequently generalized by Martin [19] to the continuous accretive operators. As a consequence of the result of Martin [19] (see also Morales [20]), if

$T: E \to E$ is strongly accretive and continuous, then T is surjective, so that the equation

$$Tx = f \tag{8}$$

has a solution for any given $f \in E$. It is easy to verify from inequality (1) that if T is strongly accretive in any Banach space E and Eq. (8) has a solution, then the solution is unique.

If $T: E \to E$ is strongly accretive and (8) has a solution, methods for approximating the solution have been investigated by several researchers (see for example Chidume [7], Chidume and the author [10], Deng [13, 14], and Tan and Xu [28]). In [7] Chidume proved that if $E = L_p$ (or l_P), $p \geq 2$, and $T: E \to E$ is a Lipschitz strongly accretive operator, then an iteration process of the type introduced by Mann [18] can be used to approximate the solution of (8). As a consequence of this result he proved that if C is a nonempty closed convex and bounded subset of E and $T: C \to C$ is a Lipschitz strongly pseudocontractive operator, then the Mann iteration process converges strongly to the unique fixed point of T. These results of Chidume have been generalized and extended in several ways by several researchers (see for example [10, 13, 14, 28]). In [13] Deng extended the results to the Ishikawa iteration method. Recently, Tan and Xu [28] extended the results of both Chidume [7] and Deng [13] to the more general real Banach spaces which are q-uniformly smooth, $1 < q < 2$ (see the definition below). These results of Tan and Xu have recently been extended to all real q-uniformly smooth Banach spaces, $q > 1$ (see Chidume and the author [10] and Deng [14]).

Let $q > 1$, and let E be a real q-uniformly smooth Banach space. It is our purpose in this paper to prove that if $T: E \to E$ is Lipschitz and ϕ-strongly accretive and Eq. (8) has a solution, then the Mann and Ishikawa iteration methods converge strongly to the unique solution. Furthermore, we prove that if $T: E \to E$ is Lipschitz and ϕ-hemicontractive, then the Mann and Ishikawa iteration methods converge strongly to the unique fixed point of T. Our results will thus extend the results of Chidume [7], Chidume and the author [10], Deng [13], and Tan and Xu [28], and Theorems 1 and 2 of Deng [14] to the more general classes of operators considered here.

PRELIMINARIES

We start by defining the two fixed point iteration methods which will be needed in the following.

(a) *The Ishikawa Iteration Process* (see for example [16, 25]) is defined as follows: For K a convex subset of a Banach space E, and T a mapping of

K into itself, the sequence $\{x_n\}_{n=0}^{\infty}$ is defined by

$$x_0 \in K$$
$$y_n = (1 - \beta_n) x_n + \beta_n T x_n, \qquad n \geq 0$$
$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T y_n, \qquad n \geq 0,$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are real sequences satisfying (i) $0 \leq \alpha_n \leq \beta_n < 1$, (ii) $\lim_{n \to \infty} \beta_n = 0$, and (iii) $\sum_{n=0}^{\infty} \alpha_n \beta_n = \infty$.

(b) *The Mann Iteration Process* (see for example [18, 25]) is defined as follows: With K and T as in (a) the sequence $\{x_n\}_{n=0}^{\infty}$ is defined by

$$x_0 \in K$$
$$x_{n+1} = (1 - c_n) x_n + c_n T x_n, \qquad n \geq 0$$

where (i) $0 \leq c_n < 1$, (ii) $\lim_{n \to \infty} c_n = 0$, and (iii) $\sum_{n=0}^{\infty} c_n = \infty$. In some applications, condition (iii) is replaced by $\sum_{n=0}^{\infty} c_n(1 - c_n) = \infty$. This change will be reflected in the following.

The iteration processes (a) and (b) have been employed by various researchers to approximate solutions of several nonlinear operator equations in Banach spaces (see for example [6–11, 13, 14, 16, 18, 25, 26–28, 30]). Moreover, it is well known that the two processes may exhibit different behaviour for different classes of nonlinear mappings (see for example [25] for a detailed comparison of the two processes).

Let E be a Banach space. The *modulus of smoothness* of E is the function

$$\rho_E \colon [0, \infty) \to [0, \infty)$$

defined by

$$\rho_E(t) = \sup\left\{\tfrac{1}{2}(\|x + y\| + \|x - y\|) - 1 \colon \|x\| \leq 1, \|y\| \leq t\right\}.$$

E is uniformly smooth if $\lim_{t \to \infty}(\rho_E(t)/t) = 0$.

Let $q > 1$; E is said to be *q-uniformly smooth* (*or to have a modulus of smoothness of power type* $q > 1$) if there exists a constant $c > 0$ such that $\rho_E(t) \leq ct^q$. Hilbert spaces, L_p (or l_p) spaces, and the Sobolev spaces W_m^p are q-uniformly smooth. Hilbert spaces are 2-uniformly smooth while

$$L_p \ (\text{or } l_p) \text{ or } W_m^p \text{ is } \begin{cases} p\text{-uniformly smooth} & \text{if } 1 < p \leq 2 \\ 2\text{-uniformly smooth} & \text{if } p \geq 2 \end{cases}$$

In the following we shall need

THEOREM HKS [29, Corollary 1′, p. 1130]. *Let $q > 1$ and E be a real Banach space. Then E is q-uniformly smooth if and only if there exists a* *constant* $d_q > 0$ *such that*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + d_q\|y\|^q \tag{9}$$

for all $x, y \in E$.

LEMMA TX (Tan and Xu [27, p. 303]). *Suppose that* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *are two sequences of nonnegative numbers such that*

$$a_{n+1} \leq a_n + b_n, \qquad \forall n \geq 1,$$

If $\sum_{n=1}^{\infty} b_n$ *converges, then* $\lim_{n\to\infty} a_n$ *exits*.

MAIN RESULTS

THEOREM 1. *Let* $q > 1$, *and let* E *be a real* q*-uniformly smooth Banach space. Let* $T: E \to E$ *be a Lipschitz* ϕ*-strongly accretive operator. Suppose the equation* $Tx = f$ *has a solution for each* $f \in E$. *Let* $\{\alpha_n\}_{n=0}^{\infty}$ *and* $\{\beta_n\}_{n=0}^{\infty}$ *be real sequences satisfying*

(i) $0 < \alpha_n < 1$, $n \geq 0$,

(ii) $0 \leq \beta_n \leq \alpha_n^{q-1}$, $n \geq 0$

(iii) $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n)^{q-1} = \infty$,

(iv) $\sum_{n=0}^{\infty} \alpha_n^q < \infty$.

Then the sequence $\{x_n\}_{n=0}^{\infty}$ *generated from any* $x_0 \in E$ *by*

$$y_n = (1 - \beta_n)x_n + \beta_n(f + (I - T)x_n), \qquad n \geq 0, \tag{10}$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(f + (I - T)y_n), \qquad n \geq 0, \tag{11}$$

converges strongly to the solution of the equation $Tx = f$.

Proof. It follows from inequality (3) that if $Tx = f$ has a solution then the solution is unique. Let x^* denote the unique solution and L the Lipschitz constant of T. Define $S: E \to E$ by

$$Sx = f + (I - T)x.$$

Then x^* is a fixed point of S and S is Lipschitz with Lipschitz constant $L_* = 1 + L$. Furthermore,

$$\langle Sx - Sy, j(x - y)\rangle \leq \|x - y\|^2 - \phi(\|x - y\|)\|x - y\|. \tag{12}$$

Using the inequalities (9)–(12) we obtain

$$\begin{aligned}\|x_{n+1} - x^*\|^q &= \|(1-\alpha_n)(x_n - x^*) + \alpha_n(Sy_n - x^*)\|^q \\ &\le (1-\alpha_n)^q\|x_n - x^*\|^q + q\alpha_n(1-\alpha_n)^{q-1} \\ &\quad \times \langle Sy_n - x^*, j_q(x_n - x^*)\rangle + \alpha_n^q\, d_q\|Sy_n - x^*\|^q \\ &\le (1-\alpha_n)^q\|x_n - x^*\|^q + q\alpha_n(1-\alpha_n)^{q-1} \\ &\quad \times \langle Sy_n - x^*, j_q(x_n - x^*)\rangle + \alpha_n^q\, d_q L_*^q\|y_n - x^*\|^q. \qquad (13)\end{aligned}$$

Observe that

$$\begin{aligned}\langle Sy_n - x^*, j_q(x_n - x^*)\rangle &= \langle Sy_n - Sx_n, j_q(x_n - x^*)\rangle \\ &\quad + \langle Sx_n - x^*, j_q(x_n - x^*)\rangle \\ &\le \|Sy_n - Sx_n\|\,\|x_n - x^*\|^{q-1} \\ &\quad + \|x_n - x^*\|^q - \phi(\|x_n - x^*\|)\|x_n - x^*\|^{q-1} \\ &\le [\beta_n L_*(1 + L_*) + 1]\|x_n - x^*\|^q \\ &\quad - \phi(\|x_n - x^*\|)\|x_n - x^*\|^{q-1}. \qquad (14)\end{aligned}$$

Furthermore,

$$\begin{aligned}\|y_n - x^*\|^q &= \|(1-\beta_n)(x_n - x^*) + \beta_n(Sx_n - x^*)\|^q \\ &\le (1-\beta_n)^q\|x_n - x^*\|^q + q\beta_n(1-\beta_n)^{q-1} \\ &\quad \times \langle Sx_n - x^*, j(x_n - x^*)\rangle + \beta_n^q\, d_q\|Sx_n - x^*\|^q \\ &\le \left[(1-\beta_n)^q + q\beta_n(1-\beta_n)^{q-1} + \beta_n^q\, d_q L_*^q\right]\|x_n - x^*\|^q \\ &\quad - q\beta_n(1-\beta_n)^{q-1}\phi(\|x_n - x^*\|)\|x_n - x^*\|^{q-1} \\ &\le \left[(1-\beta_n)^q + q\beta_n(1-\beta_n)^{q-1} + \beta_n^q\, d_q L_*^q\right]\|x_n - x^*\|^q. \qquad (15)\end{aligned}$$

Using (14) and (15) in (13) we obtain

$$\begin{aligned}\|x_{n+1} - x^*\|^q \le \Big[&(1-\alpha_n)^q + q\alpha_n(1-\alpha_n)^{q-1} \\ &+ q\alpha_n\beta_n(1-\alpha_n)^{q-1}L_*(1 + L_*) \\ &+ \alpha_n^q\, d_q L_*^q\Big((1-\beta_n)^q + q\beta_n(1-\beta_n)^{q-1} \\ &\qquad + \beta_n^q\, d_q L_*^q\Big)\Big]\|x_n - x^*\|^q \\ &- q\alpha_n(1-\alpha_n)^{q-1}\phi(\|x_n - x^*\|)\|x_n - x^*\|^{q-1}.\end{aligned}$$

Since $q - 1 > 0$, condition (ii) implies

$$\|x_{n+1} - x^*\|^q \leq \Big[(1-\alpha_n)^q + q\alpha_n(1-\alpha_n)^{q-1} + q\alpha_n^q L_*(1+L_*) + \alpha_n^q d_q L_*{}^q\Big((1-\beta_n)^q + q\beta_n(1-\beta_n)^{q-1} + d_q L_*^q\Big)\Big]\|x_n - x^*\|^q - q\alpha_n(1-\alpha_n)^{q-1}\phi\big(\|x_n - x^*\|\big)\|x_n - x^*\|^{q-1}. \quad (16)$$

Consider the function $f\colon [0,\infty) \to [0,\infty)$ defined for each $x \in [0,\infty)$ by $f(x) = (1+x)^q$. Then there exists $c \in (0,x)$ such that

$$f(x) = f(0) + f'(0)x + \frac{f''(c)}{2}x^2.$$

Since $f''(c) \geq 0$, we obtain

$$f(0) + f'(0)x \leq f(x). \quad (17)$$

Setting $x = \alpha_n/(1-\alpha_n)$ in (17) we obtain

$$(1-\alpha_n)^q + q\alpha_n(1-\alpha_n)^{q-1} \leq 1. \quad (18)$$

Similarly setting $x = \beta_n/(1-\beta_n)$ in (17) we obtain

$$(1-\beta_n)^q + q\beta_n(1-\beta_n)^{q-1} \leq 1. \quad (19)$$

Using (18) and (19) in (16) we obtain

$$\begin{aligned}\|x_{n+1} - x^*\|^q &\leq \Big[1 + q\alpha_n^q L_*(1+L_*) + \alpha_n^q d_q L_*^q\big(1 + d_q L_*^q\big)\Big]\|x_n - x^*\|^q \\ &\quad - q\alpha_n(1-\alpha_n)^{q-1}\phi\big(\|x_n - x^*\|\big)\|x_n - x^*\|^{q-1} \\ &= \big[1 + M\alpha_n^q\big]\|x_n - x^*\|^q \\ &\quad - q\alpha_n(1-\alpha_n)^{q-1}\phi\big(\|x_n - x^*\|\big)\|x_n - x^*\|^{q-1} \quad (20)\end{aligned}$$

where $M = qL_*(1+L_*) + d_q L_*^q(1 + d_q L_*^q)$. It follows from inequality (20) and condition (iv) that $\{\|x_n - x^*\|\}_{n=0}^{\infty}$ is bounded. Suppose $\|x_n - x^*\|^q \leq K$, for all $n \geq 0$. Then (20) implies that

$$\|x_{n+1} - x^*\|^q \leq \|x_n - x^*\|^q + MK\alpha_n^q - q\alpha_n(1-\alpha_n)^{q-1}\phi\big(\|x_n - x^*\|\big)\|x_n - x^*\|^{q-1}. \quad (21)$$

Thus $\|x_{n+1} - x^*\|^q \le \|x_n - x^*\|^q + MK\alpha_n^q$, and condition (iv) and an application of Lemma TX imply that $\lim \|x_n - x^*\|$ exists. Suppose $\lim \|x_n - x^*\| = \delta \ge 0$. We show that $\delta = 0$. Suppose $\delta > 0$. Let $N > 0$ be an integer such that $\|x_n - x^*\| \ge \delta/2$ for all $n \ge N$. Then

$$\liminf \phi(\|x_n - x^*\|) \ge \phi\left(\frac{\delta}{2}\right) > 0. \tag{22}$$

From inequality (21) we obtain

$$\|x_{N+1} - x^*\|^q \le \|x_0 - x^*\|^q - q \sum_{n=0}^{N} \alpha_n (1 - \alpha_n)^{q-1} \times \phi(\|x_n - x^*\|)\|x_n - x^*\|^{q-1} + MK \sum_{n=0}^{N} \alpha_n^q.$$

As $N \to \infty$, using condition (iv) we obtain

$$\sum_{n=0}^{\infty} \alpha_n (1 - \alpha_n)^{q-1} \phi(\|x_n - x^*\|)\|x_n - x^*\|^{q-1} < \infty.$$

Condition (iii) and the assumption that $\|x_n - x^*\| \to \delta > 0$ imply that

$$\liminf \phi(\|x_n - x^*\|) = 0,$$

contradicting (22). Thus, $\delta = 0$, completing the proof of Theorem 1.

COROLLARY 1. *Let $q > 1$, and let E be a real q-uniformly smooth Banach space. Let $T: E \to E$ be a Lipschitz strongly accretive operator. Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be real sequences satisfying conditions* (i)–(iv). *Then the sequence $\{x_n\}_{n=0}^{\infty}$ generated from any $x_0 \in E$ by* (10) *and* (11) *converges strongly to the solution of the equation $Tx = f$.*

Proof. The existence of a solution follows from Martin [19] (see also Morales [20]) and the result follows from Theorem 1.

COROLLARY 2. *Let E and T be as in Theorem* 1. *Suppose the equation $Tx = f$ has a solution for each $f \in E$. Let $\{\alpha_n\}_{n=0}^{\infty}$ be a real sequence satisfying conditions* (i), (iii), *and* (iv). *Then the sequence $\{x_n\}_{n=0}^{\infty}$ generated from any $x_0 \in E$ by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(f + (I - T)x_n), \qquad n \ge 0$$

converges strongly to the solution of the equation $Tx = f$.

Proof. Obvious from Theorem 1.

Since

$$L_p \text{ (or } l_p\text{) is } \begin{cases} p\text{-uniformly smooth} & \text{if } 1 < p \leq 2 \\ 2\text{-uniformly smooth} & \text{if } p \geq 2; \end{cases}$$

it follows that the following corollaries are immediate consequences of Theorem 1.

COROLLARY 3. *Let $E = L_p$ (or l_p), $p \geq 2$, and $T: E \to E$ be a Lipschitz ϕ-strongly accretive operator. Suppose the equation $Tx = f$ has a solution for each $f \in E$. Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be real sequences satisfying*

(i) $0 < \alpha_n < 1$, $n \geq 0$,

(ii) $0 \leq \beta_n \leq \alpha_n$, $n \geq 0$,

(iii) $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$,

(iv) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$.

Then the sequence $\{x_n\}_{n=0}^{\infty}$ generated from any $x_0 \in E$ by (10) *and* (11) *converges strongly to the solution of the equation $Tx = f$.*

COROLLARY 4. *Let $E = L_p$ (or l_p), $1 < p \leq 2$, and $T: E \to E$ be a Lipschitz ϕ-strongly accretive operator. Suppose the equation $Tx = f$ has a solution for each $f \in E$. Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be real sequences satisfying*

(i) $0 < \alpha_n < 1$, $n \geq 0$,

(ii) $0 \leq \beta_n \leq \alpha_n^{p-1}$, $n \geq 0$,

(iii) $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n)^{p-1} = \infty$,

(iv) $\sum_{n=0}^{\infty} \alpha_n^p < \infty$.

Then the sequence $\{x_n\}_{n=0}^{\infty}$ generated from any $x_0 \in E$ by (10) *and* (11) *converges strongly to the solution of the equation $Tx = f$.*

THEOREM 2. *Let $q > 1$, and let E be a real q-uniformly smooth Banach space. Let K be a nonempty closed convex subset of E and $T: K \to K$ be a Lipschitz ϕ-hemicontractive operator. Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be real sequences satisfying*

(i) $0 < \alpha_n < 1$, $n \geq 0$,

(ii) $0 \leq \beta_n \leq \alpha_n^{q-1}$, $n \geq 0$,

(iii) $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n)^{q-1} = \infty$,

(iv) $\sum_{n=0}^{\infty} \alpha_n^q < \infty$.

Then the sequence $\{x_n\}_{n=0}^{\infty}$ *generated from any* $x_0 \in K$ *by*

$$y_n = (1 - \beta_n) x_n + \beta_n T x_n, \qquad n \geq 0 \tag{23}$$

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T y_n, \qquad n \geq 0 \tag{24}$$

converges strongly to the fixed point of T.

Proof. Inequality (6) implies that $F(T)$ is singleton. Let x^* denote the fixed point of T and L the Lipschitz constant of T. Then using (9), (23), and (24) we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^q &= \|(1 - \alpha_n)(x_n - x^*) + \alpha_n(Ty_n - x^*)\|^q \\
&\leq (1 - \alpha_n)^q \|x_n - x^*\|^q + q\alpha_n(1 - \alpha_n)^{q-1} \\
&\quad \times \langle Ty_n - x^*, j_q(x_n - x^*)\rangle + \alpha_n^q d_q \|Ty_n - x^*\|^q \\
&\leq \left[(1 - \alpha_n)^q + q\alpha_n(1 - \alpha_n)^{q-1}\right] \|x_n - x^*\|^q \\
&\quad + q\alpha_n(1 - \alpha_n)^{q-1} \langle Ty_n - Tx_n, j_q(x_n - x^*)\rangle \\
&\quad - q\alpha_n(1 - \alpha_n)^{q-1} \phi(\|x_n - x^*\|) \|x_n - x^*\|^{q-1} \\
&\quad + \alpha_n^q d_q L^q \|y_n - x^*\|^q \\
&\leq \Big[(1 - \alpha_n)^q + q\alpha_n(1 - \alpha_n)^{q-1} + q\alpha_n \beta_n (1 - \alpha_n)^{q-1} \\
&\quad \times L(1 + L) + \alpha_n^q d_q L^q \Big((1 - \beta)^q + q\beta_n(1 - \beta_n)^{q-1} \\
&\qquad + \beta_n^q d_q L^q\Big)\Big] \|x_n - x^*\|^q \\
&\quad - q\alpha_n(1 - \alpha_n)^{q-1} \phi(\|x_n - x^*\|) \|x_n - x^*\|^{q-1} \\
&\leq \left[1 + \left(qL(1 + L) + d_q L^q (1 + d_q L^q)\right) \alpha_n^q\right] \|x_n - x^*\|^q \\
&\quad - q\alpha_n(1 - \alpha_n)^{q-1} \phi(\|x_n - x^*\|) \|x_n - x^*\|^{q-1}.
\end{aligned}$$

Thus

$$\begin{aligned}
\|x_{n+1} - x^*\|^q \leq [1 + M\alpha_n^q] \|x_n - x^*\|^q - q\alpha_n(1 - \alpha_n)^{q-1} \\
\times \phi(\|x_n - x^*\|) \|x_n - x^*\|^{q-1},
\end{aligned}$$

where $M = qL(1 + L) + d_q L^q(1 + d_q L^q)$. The rest of the argument now follows as in the Proof of Theorem 1 to yield that $x_n \to x^*$ as $n \to \infty$.

COROLLARY 5. *Let* $q > 1$, *and let* E *be a real* q-*uniformly smooth Banach space. Let* K *be a nonempty closed convex subset of* E, *and* $T: K \to K$ *a*

Lipschitz strongly pseudocontractive mapping. Let $\{\alpha_n\}_{n=0}^{\infty}$ *and* $\{\beta_n\}_{n=0}^{\infty}$ *be real sequences satisfying conditions* (i)–(iv). *Then the sequence* $\{x_n\}_{n=0}^{\infty}$ *generated from any* $x_0 \in K$ *by* (23) *and* (24) *converges strongly to the fixed point of* T.

Proof. The existence of a fixed point follows from Deimling [12]. Since $F(T) \neq \emptyset$, T is ϕ-hemicontractive and hence the result follows from Theorem 2.

COROLLARY 6. *Let* E, K, *and* T *be as in Theorem* 2 *and let* $\{\alpha_n\}_{n=0}^{\infty}$ *be a real sequence satisfying* (i), (iii), *and* (iv). *Then the Mann iteration process*

$$x_0 \in K$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n$$

converges strongly to the fixed point of T.

Remark. Theorems 1 and 2 of our results extend the results of Chidume and the author [10] and Tan and Xu [28], Theorems 1 and 2 of Deng [14], and several other results from the class of strongly accretive operators and the class of strongly pseudocontractive operators to the more general classes of operators considered here. Furthermore, Corollaries 1 and 5 which are varied for all q-uniformly smooth Banach spaces, $q > 1$, are improvements of the results of Tan and Xu which are varied for only q-uniformly smooth Banach spaces, $1 < q \leq 2$.

Prototypes of our sequences are

$$\alpha_n = \frac{1}{1+n}, \quad n \geq 0, \qquad \beta_n = \frac{1}{(1+n)^{q-1}}, \quad n \geq 0.$$

REFERENCES

1. M. Bethke, Approximation von fixpunkten streng pseudo kontrtiver operatoren (Approximation of fixed points of strongly pseudocontractive operators), *Math. Naturwiss Fak.* **27**, No. 2 (1989), 263–270.
2. J. Bogin, On strict pseudocontrations and fixed point theorems, Technion Preprint MT 219, Haifa, Israel, 1974.
3. F. E. Browder, Nonlinear mappings of nonexpansive and accretive type in Banach spaces, *Bull. Amer. Math. Soc.* (1967), 875–882.
4. F. E. Browder, Nonlinear monotone and accretive operators in Banach spaces, *Proc. Natl. Acad. Sci. USA* **61** (1968), 388–393.
5. F. E. Browder and W. V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert spaces, *J. Math. Anal. Appl.* **20** (1967), 197–228.
6. C. E. Chidume, Iterative approximation of fixed points of Lipschitz strictly pseudocontractive mappings, *Proc. Amer. Math. Soc.* **99**, No. 2 (1987), 283–288.

7. C. E. Chidume, An iterative process for nonlinear Lipschitzian strongly accretive mappings in L_p spaces, *J. Math. Anal. Appl.* **151** (1990), 453–461.
8. C. E. Chidume, Approximation of fixed points of strongly pseudocontractive mappings, *Proc. Amer. Math. Soc.* **120**, No. 2 (1994), 545–551.
9. C. E. Chidume, Steepest descent approximations for accretive operator equations, *Nonlinear Anal.* **26**, (1996), 299–311.
10. C. E. Chidume and M. O. Osilike, Ishikawa iteration process for nonlinear Lipschitz strongly accretive mappings, *J. Math. Anal. Appl.* **192** (1995), 727–741.
11. C. E. Chidume and M. O. Osilike, Fixed point iterations for strictly hemi-contractive maps in uniformly smooth Banach spaces, *Numerical Funct. Anal. Optim.* **15**, (1994), 779–790.
12. K. Deimling, Zeros of accretive operators, *Manuscript Math.* **13** (1974), 365–374.
13. L. Deng, On Chidume's open questions, *J. Math. Anal. Appl.* **174**, No. 2 (1993), 441–449.
14. L. Deng, An iterative process for nonlinear Lipschitz and strongly accretive mappings in uniformly convex and uniformly smooth Banach spaces, *Acta Appl. Math.* **32** (1993), 183–196.
15. J. A. Gatica and W. A. Kirk, Fixed point theorems for Lipschitz pseudocontractive mappings, *Proc. Amer. Math. Soc.* **36** (1972), 111–115.
16. S. Ishikawa, Fixed point by a new iteration method, *Proc. Amer. Math. Soc.* **44**, No. 1 (1974), 147–150.
17. T. Kato, Nonlinear semigroups and evolution equations, *J. Math. Soc. Japan* **19** (1964), 508–520.
18. W. R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.* **4** (1953), 506–510.
19. R. H. Martin Jr., A global existence theorem for autonomous differential equations in Banach spaces, *Proc. Amer. Math. Soc.* **26** (1970), 307–314.
20. C. Morales, Surjectivity theorems for multi-valued mappings of accretive type, *Comment. Math. Univ. Carolin.* **26**, No. 2 (1985), 397–413.
21. G. Nevanlinna and S. Reich, Strong convergence of contraction semigroups and of iterative methods for accretive operators in Banach spaces, *Israel J. Math.* **32** (1979), 44–58.
22. S. Reich, An iterative procedure for constructing zeros of accretive sets in Banach spaces, *Nonlinear Anal.* **2** (1978), 85–92.
23. S. Reich, Constructing zeros of accretive and monotone operators, *in* "Applied Nonlinear Analysis," (V. Lakshmikamthan, Ed.), pp. 335–345, Academic Press, New York, 1979.
24. S. Reich, Strong convergence theorems for resolvents of accretive operators in Banach spaces, *J. Math. Anal. Appl.* **85** (1980), 287–292.
25. B. E. Rhoades, Comments on two fixed point iteration methods, *J. Math. Anal. Appl.* **56** (1976), 741–750.
26. J. Schu, On a theorem of C. E. Chidume concerning the iterative approximation of fixed points, *Math. Nachr.* **153** (1991), 313–319.
27. K.-K. Tan and H.-K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, *J. Math. Anal. Appl.* **178** (1993), 301–308.
28. K.-K. Tan and H.-K. Xu, Iterative solutions to nonlinear equations of strongly accretive operators in Banach spaces, *J. Math. Anal. Appl.* **178** (1993), 9–21.
29. H.-K. Xu, Inequalities in Banach spaces with Applications, *Nonlinear Anal.* **16**, No. 12 (1991), 1127–1138.
30. Z. Xu and G. F. Roach, A necessary and sufficient condition for convergence of steepest descent approximation to accretive operator equations, *J. Math. Anal. Appl.* **167** (1992), 340–354.
31. E. Zeidler, "Nonlinear Functional Analysis and Its Applications," Part II, "Monotone Operators," Springer-Verlag, New York/Berlin, 1985.