Analytic properties of double zeta-functions

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Abstract

We shall derive a new expression for the double zeta-function of Euler–Zagier type
\[
\zeta_2(s_1, s_2) = \sum_{1 \leq n_1 < n_2} n_1^{-s_1} n_2^{-s_2}
\]
in the region \(0 < \text{Re}\ s_j < 1\) \((j = 1, 2)\), and give some applications relating to the lower bounds, and an approximate functional equation for this zeta-function.

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1. Introduction

Let \(s_j = \sigma_j + it_j\) \((j = 1, 2)\) be complex variables. The double zeta-function of Euler–Zagier type is defined by

\[
\zeta_2(s_1, s_2) = \sum_{1 \leq n_1 < n_2} \frac{1}{n_1^{s_1} n_2^{s_2}}.
\]

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The series (1.1) is convergent absolutely for $\sigma_2 > 1$ and $\sigma_1 + \sigma_2 > 2$. The values of $\zeta_2(s_1, s_2)$ for positive integers were first studied by Euler in 1775, who observed the connection between $\zeta_2(k_1, k_2)$ and the Riemann zeta-values for positive integers greater than 1. Recently, generalizations of Euler’s formula and other relations among the double and more general multiple zeta-values have been studied extensively.

It is known that the double zeta-function and more general $r$-ple multiple zeta-function can be continued analytically to the whole space $\mathbb{C}^r$; cf. Akiyama et al. [1], Zhao [14] and Matsumoto [11]. See also Akiyama and Tanigawa [2]. Recently, Kiuchi and Tanigawa [10] obtained some results on the order of magnitude of the double zeta-function (1.1) in the region $0 \leq \sigma_j < 1$ ($j = 1, 2$).

In this paper, we restrict ourselves to the double zeta-functions and give a new formula for (1.1) and its applications. To state our first theorem, let $\sigma_\alpha(n)$ be the arithmetical function defined by

$$\sigma_\alpha(n) = \sum_{d \mid n} d^\alpha,$$

where $\alpha$ is a complex number. This is the Dirichlet convolution of 1 with $n^\alpha$, and hence

$$\sum_{n=1}^{\infty} \frac{\sigma_\alpha(n)}{n^s} = \zeta(s)\zeta(s - \alpha)$$

(1.2)

for $\Re s > \max(1, \Re \alpha + 1)$. Then our first theorem is:

**Theorem 1.** For $0 < \sigma_1 < 1$ and $0 < \sigma_2 < 1$, we have

$$\zeta_2(s_1, s_2) = \frac{\zeta(s_1 + s_2 - 1)}{s_2 - 1} - \frac{1}{2} \zeta(s_1 + s_2)$$

$$+ \frac{s_2}{\pi} \sum_{n=1}^{\infty} \frac{\sigma_{1-s_1-s_2}(n)}{n} \int_{1}^{\infty} \sin(2\pi nx)x^{-s_2-1}dx. \quad (1.3)$$

Let $J(s_1, s_2)$ denote the last term on the right-hand side of (1.3), namely

$$J(s_1, s_2) = \frac{s_2}{\pi} \sum_{n=1}^{\infty} \frac{\sigma_{1-s_1-s_2}(n)}{n} \int_{1}^{\infty} \sin(2\pi nx)x^{-s_2-1}dx. \quad (1.4)$$

This is the function that we shall consider in the sequel. Our second theorem gives the truncated expression for the function $J(s_1, s_2)$.

**Theorem 2.** Suppose that $0 < \sigma_1 < 1$ and $0 < \sigma_2 < 1$, and let $\epsilon$ be any positive real number. Then we have

$$J(s_1, s_2) = \chi(s_2) \sum_{n \leq |t_2|^{1/2}} \frac{\sigma_{1-s_1-s_2}(n)}{n^{1-s_2}} + O(|t_2|^{\delta+\epsilon}), \quad (1.5)$$

where $\chi(s_2) = 2(2\pi)^{s_2-1} \sin(\pi s_2) \Gamma(1-s_2)$ and $\delta = \max(0, 1 - \sigma_1 - \sigma_2)$. The $O$-constant is independent of $t_1$.

Under the above conditions, we have

$$|J(s_1, s_2)| \ll |t_2|^{1/2+\delta+\epsilon} \quad (1.6)$$
by the trivial estimate of the sum in the right-hand side of (1.5). We should note that the implied constant in (1.6) is independent of $t_1$.

We shall give two applications of our theorems. In [10], the first two authors derived an upper bound of $\zeta_2(s_1, s_2)$ in the strip $0 \leq \sigma_j < 1$ ($j = 1, 2$). As an application of (1.5) or its consequence (1.6), we can deduce a lower bound of $\zeta_2(s_1, s_2)$. *That the implied constant is independent of $t_1$ in (1.6) plays an important role.*

**Corollary 1.** Suppose that $\sigma_1 > 0$, $\sigma_2 > 0$, $\sigma_1 + \sigma_2 \leq 1$ and
\[
|t_2| \ll |t_1|^{\frac{3-2(\sigma_1+\sigma_2)}{2-4(\sigma_1+\sigma_2)} - \epsilon}.
\]
Then we have
\[
|\zeta_2(s_1, s_2)| \asymp \frac{|t_1|^{\frac{3}{2}-(\sigma_1+\sigma_2)}}{|t_2|} \quad \text{if } \sigma_1 + \sigma_2 < 1,
\]
and
\[
\zeta_2(s_1, s_2) = \Omega \left( \frac{|t_1|^{\frac{1}{2}} \log \log |t_1|}{|t_2|} \right) \quad \text{if } \sigma_1 + \sigma_2 = 1.
\]

For example, if we take $\sigma_1 = \frac{1}{2}$, $\sigma_2 = \frac{1}{4}$ or $\sigma_1 = \frac{1}{4}$, $\sigma_2 = \frac{1}{2}$ in Corollary 1, then we find
\[
|\zeta_2(s_1, s_2)| \asymp \frac{|t_1|^\frac{3}{4}}{|t_2|}.
\]

Also if we take $\sigma_1 = \sigma_2 = \frac{1}{2}$, then we get
\[
\zeta_2 \left( \frac{1}{2} + it_1, \frac{1}{2} + it_2 \right) = \Omega \left( \frac{|t_1|^{\frac{1}{2}} \log \log |t_1|}{|t_2|} \right).
\]

See Section 6 for other remarks.

As a second application of Theorem 2, we shall show a certain relation between $\zeta_2(s_1, s_2)$ and $\zeta_2(1-s_2, 1-s_1)$ which can be regarded as an “approximate functional equation” of the double zeta-function. More precisely, we have:

**Corollary 2.** Suppose that $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ and
\[
||t_1| - |t_2|| \ll |t_1|^\frac{1}{2}.
\]
Then we have
\[
\frac{J(s_1, s_2)}{\chi(s_2)} - \frac{J(1-s_2, 1-s_1)}{\chi(1-s_1)} \ll \begin{cases} 
|t_1|^{|\sigma_2-\frac{1}{2}+\epsilon} & \text{if } \sigma_1 + \sigma_2 \geq 1 \\
|t_1|^{|\frac{1}{2}-\sigma_1+\epsilon} & \text{if } \sigma_1 + \sigma_2 < 1 \\
|t_1|^\frac{1}{2} & \text{if } \sigma_1 = \sigma_2 = \frac{1}{2}.
\end{cases}
\]

See Remark 4 for the comparison with the functional equation given by Matsumoto [12].

In Section 2, we shall recall the work of Atkinson on the double zeta-function $\zeta_2(s_1, s_2)$. In Section 3, we shall study a function $f_s(Y)$ which will be used in the proof of our theorems. We shall prove the theorems and corollaries in Sections 4–6.
2. Atkinson’s work on the double zeta-function

As is mentioned in the introduction, the double zeta-function $\zeta_2(s_1, s_2)$ for the integers $s_1 \geq 1$ and $s_2 \geq 2$ was introduced by Euler, whereas its properties as a function of complex variables $s_1$ and $s_2$ were first studied by Atkinson [4]. He showed the analytic continuation of $\zeta_2(s_1, s_2)$ on a certain range beyond the domain of absolute convergence by using the Poisson summation formula and applied it in his research on the mean value formula of the Riemann zeta-function $\zeta(s)$. In fact he showed that

$$\zeta_2(s_1, s_2) = \frac{\zeta(s_1 + s_2 - 1)}{s_2 - 1} + \frac{1}{2} \zeta(s_1 + s_2)$$

is regular for $\sigma_1 + \sigma_2 > 0$, and furthermore putting

$$g(s_1, s_2) = \zeta_2(s_1, s_2) - \frac{\Gamma(s_1 + s_2 - 1)\Gamma(1 - s_1)}{\Gamma(s_2)}\zeta(s_1 + s_2 - 1),$$

he derived the formula

$$g(s_1, s_2) = 2 \sum_{n=1}^{\infty} \sigma_{1-s_1-s_2}(n) \int_0^{\infty} y^{-s_1} (1 + y)^{-s_2} \cos(2\pi ny)dy.$$  

(2.3)

Equation (2.3) holds true for $\sigma_1 < 0, \sigma_2 > 1$ and $\sigma_1 + \sigma_2 > 0$. Atkinson put $s_2 = 1 - s_1$ in (2.3) and applied the Voronoi formula to get a certain expression for $g(s_1, 1 - s_1)$ which holds in the region containing the critical line $\text{Re} s_1 = 1/2$. From this, he obtained the series expression $E(T) = \int_0^T |\zeta(1/2 + it)|^2 dt - T \log(T/2\pi) - (2\gamma - 1)T$, called Atkinson’s formula, which is one of the most important formulas in the theory of the Riemann zeta-function. See also Chapter 15 of Ivić [9].

The integral in the right-hand side of (2.3) can be expressed using the confluent hypergeometric function $\Psi(a, c; x)$, whose integral representation is given by

$$\Psi(a, c; x) = \frac{1}{\Gamma(a)} \int_0^{\infty} e^{\phi} x^{-a-1}(1 + y)^{c-a-1}dy.$$  

(2.4)

for $\text{Re} a > 0, -\pi < \phi < \pi$ and $-\frac{1}{2} \pi < \phi + \arg x < \frac{1}{2} \pi$ (Erdélyi et al. [5, 6.5 (2)]). Hence the formula (2.3) can be written as

$$g(s_1, s_2) = \Gamma(1 - s_1) \sum_{n=1}^{\infty} \sigma_{1-s_1-s_2}(n) \left( \Psi(1 - s_1, 2 - s_1 - s_2; 2\pi in) \right.$$

$$+ \Psi(1 - s_1, 2 - s_1 - s_2; -2\pi in) \right).$$  

(2.5)

It is appropriate to give here a certain interpretation for the formula (2.5) from our point of view. In Section 4, we shall derive the formula for the double zeta-function

$$\zeta_2(s_1, s_2) = \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(s_2 + z)\Gamma(-z)}{\Gamma(s_2)} \zeta(s_1 + s_2 + z)\zeta(-z)dz$$

which holds true for $-\sigma_2 < c < -1$ and $\sigma_1 + \sigma_2 + c > 1$. Our Theorem 1 will be shown by applying the functional equation for the Riemann zeta-function to $\zeta(-z)$, while (2.3) or equivalently (2.5) can be shown by applying the functional equation to $\zeta(s_1 + s_2 + z)$, which we shall see in the remaining part of this section.
We assume that \( \sigma_1 < 0 \), and hence \(-\sigma_2 < -(\sigma_1 + \sigma_2)\). Take \( c_1 \) such that

\[-\sigma_2 < c_1 < -(\sigma_1 + \sigma_2)\]

and move the line of integration to the left from \((c)\) to \((c_1)\). The pole encountered is \( z = 1 - s_1 - s_2 \) and its residue is

\[
\frac{\Gamma(1 - s_1) \Gamma(s_1 + s_2 - 1)}{\Gamma(s_2)} \zeta(s_1 + s_2 - 1).
\]

Hence comparing with the definition (2.2) of the function \( g(s_1, s_2) \), we have

\[
g(s_1, s_2) = \frac{1}{2\pi i} \int_{(c_1)} \frac{\Gamma(s_2 + z) \Gamma(-z)}{\Gamma(s_2)} \zeta(s_1 + s_2 + z) \zeta(-z) dz.
\]

(2.6)

We apply the functional equation for the Riemann zeta-function to \( \zeta(s_1 + s_2 + z) \). Then we have

\[
g(s_1, s_2) = \frac{(2\pi)^{s_1+s_2}}{\pi} \frac{1}{2\pi i} \int_{(c_1)} \frac{\Gamma(s_2 + z) \Gamma(-z)}{\Gamma(s_2)} \Gamma(1 - s_1 - s_2 - z)
\]

\[
\times \sin \frac{\pi (s_1 + s_2 + z)}{2} \zeta(-z) \zeta(-z - s_1 - s_2 + 1)(2\pi)^z dz.
\]

Since \( c_1 < -(\sigma_1 + \sigma_2) < c < -1 \), we can expand the product of the zeta-function \( \zeta(-z) \zeta(-z - s_1 - s_2 + 1) \) into the Dirichlet series \( \sum_{n=1}^{\infty} \frac{\sigma_{s_1+s_2-1}(n)}{n^z} \) by (1.2). Interchanging the integration and summation we get

\[
g(s_1, s_2) = \frac{(2\pi)^{s_1+s_2}}{\pi} \sum_{n=1}^{\infty} \sigma_{s_1+s_2-1}(n) F_{s_1,s_2}(n),
\]

(2.7)

where

\[
F_{s_1,s_2}(n) = \frac{1}{2\pi i} \int_{(c_1)} \frac{\Gamma(s_2 + z) \Gamma(-z)}{\Gamma(s_2)} \Gamma(1 - s_1 - s_2 - z)
\]

\[
\times \sin \frac{\pi (s_1 + s_2 + z)}{2} (2\pi n)^z dz.
\]

The function \( F_{s_1,s_2}(n) \) can be expressed by means of the confluent hypergeometric function \( \psi(a, c; x) \) as follows:

\[
F_{s_1,s_2}(n) = \frac{1}{2i} \frac{1}{2\pi i} \int_{(c_1)} \frac{\Gamma(s_2 + z) \Gamma(-z)}{\Gamma(s_2)} \Gamma(1 - s_1 - s_2 - z)
\]

\[
\times \left( e^{\frac{\pi i}{2}(s_1+s_2+z)} - e^{-\frac{\pi i}{2}(s_1+s_2+z)} \right) (2\pi n)^z dz
\]

\[
= \frac{1}{2i} \left\{ e^{\frac{\pi i}{2}(s_1+s_2)} \frac{1}{2\pi i} \int_{(c_1)} \frac{\Gamma(s_2 + z) \Gamma(-z)}{\Gamma(s_2)} \Gamma(1 - s_1 - s_2 - z) (2\pi in)^z dz
\]

\[
- e^{-\frac{\pi i}{2}(s_1+s_2)} \frac{1}{2\pi i} \int_{(c_1)} \frac{\Gamma(s_2 + z) \Gamma(-z)}{\Gamma(s_2)} \Gamma(1 - s_1 - s_2 - z) (-2\pi in)^z dz \right\}
\]

\[
= \Gamma(1 - s_1) \left\{ e^{\frac{\pi i}{2}(s_1+s_2)} \psi(s_2, s_1 + s_2; 2\pi in)
\]

\[- e^{-\frac{\pi i}{2}(s_1+s_2)} \psi(s_2, s_1 + s_2; -2\pi in) \right\},
\]

(2.8)
where we used the Mellin–Barnes integral expression for the function $\Psi(a, c; x)$:

$$\Psi(a, c; x) = \frac{1}{2\pi i} \int_{(\gamma)} \frac{\Gamma(a+s) \Gamma(-s) \Gamma(1-c-s)}{\Gamma(a) \Gamma(a-c+1)} x^s ds$$

for $-\text{Re} \ a < \gamma < \min(0, 1 - \text{Re} \ c), -3\pi/2 < \arg x < 3\pi/2$ (Erdélyi et al. [5, 6.5 (5)]).

The well-known transformation formula

$$\Psi(a, c; x) = x^{1-c} \Psi(a - c + 1, 2 - c; x)$$ (2.9)

implies that

$$\Psi(s_2, s_1 + s_2; \pm 2\pi in) = \pm 2\pi i (2\pi)^{-(s_1+s_2)} e^{\mp \frac{\pi i}{2}(s_1+s_2)} n^{1-(s_1+s_2)} \times \Psi(1-s_1, 2-s_1-s_2; \pm 2\pi in).$$ (2.10)

From (2.7), (2.8), (2.10) and the trivial relation $\sigma_\alpha(n) = n^\alpha \sigma_{-\alpha}(n)$, we get immediately that

$$g(s_1, s_2) = \Gamma(1-s_1) \sum_{n=1}^\infty \sigma_{1-s_1-s_2}(n) \left( \Psi(1-s_1, 2-s_1-s_2; 2\pi in) + \Psi(1-s_1, 2-s_1-s_2; -2\pi in) \right),$$

which is the formula (2.5) for $g(s_1, s_2)$.

3. The function $f_s(Y)$

Suppose that $0 < \eta < 1$ and let $s$ be a complex variables with $\text{Re} \ s + \eta > 0$. Let $f_s(Y)$ be a function defined by

$$f_s(Y) = \frac{1}{2\pi i} \int_{(\eta)} \frac{\Gamma(s+z)}{\cos \frac{\pi z}{2}} Y^{-z} dz$$ (3.1)

where $Y > 0$ and $(c)$ indicates that the line of integration is the vertical line $z = c + iy$, $-\infty < y < \infty$.

We make use of the other type of confluent hypergeometric function $\Phi(a, c; x)$ (sometimes this is written as $1F_1(a, c; x)$) which we recall now. It is defined by

$$\Phi(a, \gamma; z) = \sum_{n=0}^\infty \frac{\alpha(\alpha+1) \ldots (\alpha+n-1)}{\gamma(\gamma+1) \ldots (\gamma+n-1)} \frac{z^n}{n!}.$$ (3.2)

The integral representation of $\Phi(a, c; x)$ is given by

$$\Phi(a, c; x) = \frac{\Gamma(c)}{\Gamma(a) \Gamma(c-a)} \int_0^1 e^{yu} u^{a-1} (1-u)^{c-a-1} du,$$ (3.3)

for $\text{Re} \ c > \text{Re} \ a > 0$ (cf. Erdélyi et al. [5, p. 248 (1), p. 255 (1)])

We have the following expression for the function $f_s(Y)$ obtained by means of $\Phi(a, c; x)$.

Lemma 1. Let the notation be as above. Then we have

$$f_s(Y) = Y^s \left( \frac{\cos Y}{\cos \frac{\pi s}{2}} + \frac{\sin Y}{\sin \frac{\pi s}{2}} \right) + \frac{2}{\pi} \Gamma(s) g_s(Y),$$ (3.4)
where
\[ g_s(Y) = \frac{i}{2} \left\{ \Phi(1, 1 - s; iY) - \Phi(1, 1 - s; -iY) \right\}. \] (3.5)

**Proof.** We shall move the line of integration to the left. By Stirling’s formula for the gamma-function, we see easily that
\[
\lim_{N \to \infty} \int_{(-N+\frac{1}{2})} \left| \frac{\Gamma(s+z)}{\cos \frac{\pi z}{2}} \right| N^{-\frac{1}{2}} |d\zeta| = 0.
\]
Therefore by Cauchy’s theorem, \( f_s(Y) \) coincides with the sum of residues in the left half-plane \( \text{Re} \, z < \eta < 1 \). The sum of the residues occurring from the poles of \( \Gamma(s+z) \) is equal to
\[
\sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \frac{Y^{l+s}}{\cos \frac{\pi}{2}(s+l)} = Y^s \left( \frac{\cos Y}{\cos \frac{\pi}{2} s} + \frac{\sin Y}{\sin \frac{\pi}{2} s} \right).
\]
On the other hand, the sum of the residues occurring from the poles of \( 1/\cos \frac{\pi z}{2} \) is given by
\[
\sum_{l=1}^{\infty} \frac{2}{\pi} (-1)^{l+1} \frac{Y^{2l-1}}{(s-1)(s-2)\ldots(s-2l+1)} = \frac{1}{2} \left\{ \Phi(1, 1 - s; iY) - \Phi(1, 1 - s; -iY) \right\},
\]
which proves the assertion of Lemma 1. \( \square \)

**Lemma 2.** Let \( g_s(Y) \) be the function defined by (3.5). Then, for an integer \( n \), we have
\[
g_s(2\pi n) = -s \int_0^1 \sin(2\pi n x)x^{-s-1} \, dx, \quad (\text{Re} \, s < 1).
\] (3.6)

**Proof.** First we assume that \( \text{Re} \, s < 0 \). From (3.3) and (3.5), we have
\[
g_s(Y) = s \int_0^1 \sin(Y(1-x))x^{-s-1} \, dx.
\]
Hence for \( Y = 2\pi n \), we get the expression (3.6). We note that the integral converges for \( \text{Re} \, s < 1 \). \( \square \)

4. **Proof of Theorem 1**

We make use of the Mellin–Barnes integral expression of the double zeta-function which we recall for the sake of completeness. The well-known Mellin–Barnes formula which we need is
\[
\Gamma(s)(1+\lambda)^{-s} = \frac{1}{2\pi i} \int_{(c)} \Gamma(s+z)\Gamma(-z)\lambda^z \, dz
\] (4.1)
where \( \lambda > 0 \), \( \text{Re} \, s > 0 \) and \( -\text{Re} \, s < c < 0 \) (see e.g. [3, p.85]).
First assume that \( \sigma_2 > 1 \) and \( \sigma_1 + \sigma_2 > 2 \). Taking \( -\sigma_2 < c < -1 \), \( \sigma_1 + \sigma_2 + c > 1 \) and setting \( \lambda = n/m \) in (4.1), we have

\[
\zeta_2(s_1, s_2) = \lim_{c \to -\sigma_2} \frac{1}{m^{s_1 + s_2} \Gamma(s_2)} \left( \frac{n}{m} \right)^z \Gamma(s_2 + z) \Gamma(-z) \zeta(s_1 + s_2 + z) \zeta(-z) \text{d}z.
\]

(cf. Matsumoto [11]). Now, we shift the path of integration to \( \Re z = \eta \) (\( 0 < \eta < 1 \)). The poles encountered are \( z = -1 \) and \( z = 0 \) and the residues at these poles are given by

\[
-\frac{\zeta(s_1 + s_2 - 1)}{s_2 - 1} \quad \text{and} \quad \frac{1}{2} \zeta(s_1 + s_2),
\]

respectively. Hence we have

\[
\zeta_2(s_1, s_2) = \frac{\zeta(s_1 + s_2 - 1)}{s_2 - 1} - \frac{1}{2} \zeta(s_1 + s_2) + J.
\]

where

\[
J = \frac{1}{2\pi i} \int_{(\eta)} \frac{\Gamma(s_2 + z) \Gamma(-z) \zeta(s_1 + s_2 + z) \zeta(-z)}{\Gamma(s_2)} \text{d}z.
\]

Therefore our task is to show that the above function \( J \) coincides with the one defined by (1.4).

The formula (4.4) shows that \( J \) can be continued to the range \( \sigma_2 + \eta > 0 \), and \( \sigma_1 + \sigma_2 + \eta > 1 \).

So we first assume that

\[-\eta < \sigma_2 < 0, \quad \text{and} \quad \sigma_1 + \sigma_2 > 1.\]

Substituting the functional equation of the Riemann zeta-function ([9, (1.23)])

\[
\Gamma(-z) \zeta(-z) = \frac{1}{2} (2\pi)^{-z} \frac{\zeta(1 + z)}{\cos \frac{\pi z}{2}}
\]

into (4.4), and expanding the product of the Riemann zeta-functions into Dirichlet series by using (1.2), we get

\[
J = \frac{1}{2\Gamma(s_2)} \cdot \frac{1}{2\pi i} \int_{(\eta)} \frac{(2\pi)^{-z} \Gamma(s_2 + z)}{\cos \frac{\pi z}{2}} \frac{\zeta(1 + z) \zeta(s_1 + s_2 + z)}{\zeta(s_2)} \text{d}z
\]

\[
= \frac{1}{2\Gamma(s_2)} \sum_{n=1}^{\infty} \sigma_{1-s_1-s_2}(n) \frac{1}{n} \frac{\Gamma(s_2 + z)}{\cos \left( \frac{\pi z}{2} \right)} (2\pi n)^{-z} \text{d}z
\]

\[
= \frac{1}{2\Gamma(s_2)} \sum_{n=1}^{\infty} \sigma_{1-s_1-s_2}(n) f_s(2\pi n),
\]

(4.5)
where \( f_s(Y) \) is the function defined by (3.1). Since
\[
f_s(2\pi n) = \frac{(2\pi n)^s}{\cos \frac{s\pi}{2}} - \frac{2}{\pi} \Gamma(s + 1) \int_0^1 \sin(2\pi nx)x^{-s-1}dx,
\]
by (3.4) and (3.6), we get
\[
J = \frac{(2\pi)^s_2}{2\Gamma(s_2)\cos \frac{s_2\pi}{2}} \sum_{n=1}^{\infty} \frac{\sigma_1 - s_1 - s_2(n)}{n^{1-s_2}}
\]
\[
- \frac{s_2}{\pi} \sum_{n=1}^{\infty} \frac{\sigma_1 - s_1 - s_2(n)}{n} \int_0^1 \sin(2\pi nx)x^{-s_2-1}dx.
\]
We apply here the well-known formula for the gamma-function (see e.g. [6, 3.761.4]):
\[
\int_0^1 \sin(2\pi nx)x^{-s_2-1}dx = -(2\pi n)^{s_2} \Gamma(-s_2) \sin \frac{\pi s_2}{2} - \int_1^{\infty} \sin(2\pi nx)x^{-s_2-1}dx.
\]
Hence we get
\[
J = (2\pi)^{s_2} \left\{ \frac{1}{2\Gamma(s_2)\cos \frac{s_2\pi}{2}} + \frac{s_2}{\pi} \Gamma(-s_2) \sin \frac{\pi s_2}{2} \right\} \sum_{n=1}^{\infty} \frac{\sigma_1 - s_1 - s_2(n)}{n^{1-s_2}}
\]
\[
+ \frac{s_2}{\pi} \sum_{n=1}^{\infty} \frac{\sigma_1 - s_1 - s_2(n)}{n} \int_1^{\infty} \sin(2\pi nx)x^{-s_2-1}dx.
\]
From the properties of the gamma-function \( \Gamma(s + 1) = s \Gamma(s) \) and \( \Gamma(s) \Gamma(1 - s) = \pi / \sin(\pi s) \), it follows that the first term in the above expression vanishes. Hence we have
\[
J = \frac{s_2}{\pi} \sum_{n=1}^{\infty} \frac{\sigma_1 - s_1 - s_2(n)}{n} \int_1^{\infty} \sin(2\pi nx)x^{-s_2-1}dx.
\]
We should note that the above procedure is justified by the estimate
\[
\int_1^{\infty} \sin(2\pi nx)x^{-s_2-1}dx \ll \frac{1}{n}
\]
for \( \sigma_2 > -1 \) and \( n \gg |t_2| \), which also shows that the series in (4.7) is absolutely convergent in the region \( \sigma_2 > -1 \) and \( \sigma_1 + \sigma_2 > 0 \).

This completes the proof of Theorem 1.

Remark 1. By using the formulas (3.5), (3.6) and (4.6), \( J(s_1, s_2) \) can be expressed as
\[
J(s_1, s_2) = \zeta(s_1)\zeta(s_2)
\]
\[
+ \frac{\pi i}{2} \sum_{n=1}^{\infty} \frac{\sigma_1 - s_1 - s_2(n)}{n} \left\{ \Phi(1, 1 - s_2; 2\pi in) - \Phi(1, 1 - s_2; -2\pi in) \right\}.
\]
This is analogous to the formula (2.5) of \( g(s_1, s_2) \).
5. Proof of Theorem 2

We assume that $0 < \sigma_2 < 1$ in what follows. Suppose that

$$I(\xi, s) = \int_\xi^\infty x^{-s} \cos x \, dx \quad \text{for} \ Re \ s > 0.$$  \hspace{1cm} (5.1)

From integration by parts, we see easily that

$$s_2 \int_1^\infty x^{-s_2-1} \sin(2\pi n x) \, dx = (2\pi n)^{s_2} I(2\pi n, s_2),$$

and hence we have

$$J = 2(2\pi)^{s_2-1} \sum_{n=1}^\infty \frac{\sigma_{1-s_1-s_2}(n)}{n^{1-s_2}} I(2\pi n, s_2).$$  \hspace{1cm} (5.2)

The function $I(\xi, s)$ was considered by Hardy and Littlewood in [7] and in [8] in more detail. For our purpose it is enough to quote from [7].

Lemma 3 (Hardy and Littlewood [7, Lemma 12]). Suppose that $\sigma$ is fixed, $0 < \sigma < 1$, and $A_0$, $A_1$ are fixed constants such that $0 < A_0 < 1$ and $1 < A_1$ respectively. Then for $s = \sigma + it$, we have

$$I(\xi, s) = \Gamma(1-s) \sin \frac{\pi s}{2} + O \left( \frac{\xi^{1-\sigma}}{|t|} \right) \quad \text{for} \ \xi < A_0|t| < |t|, \hspace{1cm} (5.3)$$

$$I(\xi, s) = \Gamma(1-s) \sin \frac{\pi s}{2} + O \left( \frac{\xi^{2-\sigma}}{|t|(|t| - \xi)} \right) \quad \text{for} \ A_0|t| < \xi < |t|, \hspace{1cm} (5.4)$$

$$I(\xi, s) = O \left( \frac{\xi^{1-\sigma}}{|t|} \right) \quad \text{for} \ |t| < \xi < A_1|t|, \hspace{1cm} (5.5)$$

$$I(\xi, s) = O \left( \xi^{-\sigma} \right) \quad \text{for} \ |t| < A_1|t| < \xi, \hspace{1cm} (5.6)$$

and

$$I(\xi, s) = O \left( \xi^{-\sigma} |t|^{1/2} \right) \quad \text{in any case.} \hspace{1cm} (5.7)$$

Proof of Theorem 2. We may suppose that $t_2 > 0$. Let $\tau$ be a parameter which will be chosen later. We divide the sum (5.2) into five parts:

$$J = 2(2\pi)^{s_2-1} \left\{ \sum_{n = A_0 t_2}^{A_1 t_2} + \sum_{A_0 t_2 < n < \frac{t_2}{2\pi} - \tau} + \sum_{\frac{t_2}{2\pi} - \tau < n \leq \frac{t_2}{2\pi} + \tau} \right.$$

$$+ \sum_{\frac{t_2}{2\pi} + \tau < n \leq \frac{A_0 t_2}{2\pi}} + \sum_{\frac{A_1 t_2}{2\pi} < n} \right\} \frac{\sigma_{1-s_1-s_2}(n)}{n^{1-s_2}} I(2\pi n, s_2)$$

$$=: J_1 + J_2 + J_3 + J_4 + J_5,$$

say.
In \( J_1, \xi = 2\pi n < A_0 t_2 \). Hence by (5.3),

\[
J_1 = 2(2\pi)^{s_2-1} \sum_{n \leq \frac{A_0 t_2}{2\pi}} \frac{\sigma_{1-s_1-s_2}(n)}{n^{1-s_2}} \left\{ \sin \frac{\pi s_2}{2} \Gamma(1-s_2) + O \left( \frac{n^{1-\sigma_2}}{t_2} \right) \right\}
\]

\[
= \chi(s_2) \sum_{n \leq \frac{A_0 t_2}{2\pi}} \frac{\sigma_{1-s_1-s_2}(n)}{n^{1-s_2}} + O \left( t_2^{\delta+\varepsilon} \right).
\]

where \( \chi(s) = 2(2\pi)^{s-1} \sin(\frac{\pi}{2} s) \Gamma(1-s) \) and \( \delta = \max(0, 1 - \sigma_1 - \sigma_2) \) as before.

In \( J_2, A_0 t_2 < \xi = 2\pi n \leq t_2 - 2\pi \). Hence, by (5.4), we have

\[
J_2 = 2(2\pi)^{s_2-1} \sum_{n \geq \frac{A_0 t_2}{2\pi} < n \leq \frac{t_2}{2\pi} - \tau} \frac{\sigma_{1-s_1-s_2}(n)}{n^{1-s_2}} \left\{ \sin \frac{\pi s_2}{2} \Gamma(1-s_2) + O \left( \frac{n^{1-\sigma_2}}{t_2(t_2 - 2\pi n)} \right) \right\}
\]

\[
= \chi(s_2) \sum_{n \geq \frac{A_0 t_2}{2\pi} < n \leq \frac{t_2}{2\pi} - \tau} \frac{\sigma_{1-s_1-s_2}(n)}{n^{1-s_2}} + O \left( \frac{1}{t_2} \sum_{\frac{A_0 t_2}{2\pi} < n \leq \frac{t_2}{2\pi} - \tau} \frac{|\sigma_{1-s_1-s_2}(n)|n}{t_2 - 2\pi n} \right).
\]

The \( O \)-term above is estimated as

\[
\ll t_2^{\delta+\varepsilon} \sum_{0 \leq j < t_2} \frac{1}{\tau + j} \ll t_2^{\delta+\varepsilon}.
\]

Using (5.5) we have similarly that

\[
J_4 \ll t_2^{\delta+\varepsilon}.
\]

We consider \( J_3 \). By (5.7), we have

\[
J_3 \ll \sum_{|n-\frac{t_2}{2\pi}| \leq \tau} \frac{|\sigma_{1-s_1-s_2}(n)|n^{-\sigma_2}t_2^{1/2}}{n^{1-\sigma_2}} \ll t_2^{\delta-1/2+\varepsilon} \tau.
\]

Finally for \( J_5 \), we use

\[
I(2\pi n, s_2) = s_2 \int_{2\pi n}^{\infty} u^{-s_2-1} \sin u \, du \ll t_2 n^{-\sigma_2-1},
\]

to obtain

\[
J_5 \ll t_2 \sum_{n \geq \frac{A_1 t_2}{2\pi}} \frac{|\sigma_{1-s_1-s_2}(n)|n^{-1-\sigma_2}}{n^{1-\sigma_2}} \ll t_2^{\delta+\varepsilon}.
\]

Choosing \( 1 \ll \tau \ll t_2^{1/2} \) and noting that

\[
|\chi(s_2)| \ll t_2^{\delta+\varepsilon},
\]

we get

\[
J = \chi(s_2) \sum_{n \leq \frac{t_2}{2\pi}} \frac{\sigma_{1-s_1-s_2}(n)}{n^{1-s_2}} + O(t_2^{\delta+\varepsilon}).
\]
\(s_1\) only occurs in the \(J_i\ (i = 1, \ldots, 5)\) terms through \(\sigma_1\) in \(\delta\) and through \(\sigma_1 - s_1 - s_2\) in \(n\), and the latter is estimated trivially, so the factors \(d^{n} t_1\) are estimated by 1. Hence, we can see that the \(O\)-constant is independent of \(t_1\).

\[\square\]

6. Applications

We shall give proofs of Corollaries 1 and 2.

**Proof of Corollary 1.** We recall that

\[
\zeta_2(s_1, s_2) = \frac{\zeta(s_1 + s_2 - 1)}{s_2 - 1} - \frac{1}{2} \zeta(s_1 + s_2) + O(|t_2|^{1/2 + \delta + \varepsilon}).
\]

Using the functional equation \(\zeta(s) = \chi(s)\zeta(1 - s), \chi(s) \asymp |t|^{1/2 - \sigma}\) and Theorem 9.2 of Ivić [9], we get

\[
\zeta(s) \asymp |t|^{1/2 - \sigma} \quad \text{if} \quad \sigma < 0,
\]

and

\[
\zeta(s) = \Omega(|t|^{1/2} \log \log |t|) \quad \text{if} \quad \sigma = 0.
\]

The assumption in the corollary implies that

\[
\left| \frac{\zeta(s_1 + s_2 - 1)}{s_2 - 1} \right| \gg |t_2|^{1/2 + \delta + \varepsilon}.
\]

It is also seen easily that under this assumption the order of \(|\zeta(s_1 + s_2)|\) is smaller than that of \(|\frac{\zeta(s_1 + s_2 - 1)}{s_2 - 1}|\), and hence we get the assertion.

\[\square\]

**Remark 2.** In [10], we conjectured that under the assumptions \(|t_1| \asymp |t_2|\) and \(|t_1 + t_2| \gg 1\),

\[
\zeta_2(s_1, s_2) \ll |t_1|^{\mu(\sigma_1) + \mu(\sigma_2)} \log^A |t_1| \tag{6.1}
\]

with some positive constant \(A\), where \(\mu(\sigma)\) means the infimum of a number \(c\) such that \(\zeta(\sigma + it) \ll |t|^c\). Let us consider the case \(\sigma_1 = \sigma_2 = 1/2\). In Corollary 1.2 in [10], we showed that

\[
\zeta_2\left(\frac{1}{2} + it_1, \frac{1}{2} + it_2\right) \ll |t_1|^{1/2} \log^2 |t_1|
\]

under the condition stated above. Now we take \(|t_2| \ll |t_1|^{1/6 - \varepsilon'}\) in our Corollary 1 which clearly satisfies the assumption. Then it gives us

\[
\zeta_2\left(\frac{1}{2} + it_1, \frac{1}{2} + it_2\right) = \Omega(|t_1|^{1/2 + \varepsilon'}).
\]

This means that the estimate (6.1) does not hold in general. The assumption that \(|t_1| \asymp |t_2|\) is indispensable for the estimate (6.1).

**Remark 3.** The bound (1.6) is also obtained by applying the so-called “second-derivative test” (Ivić [9, Lemma 2.2]) to the expression (1.4). We note that if we apply the result of Matsumoto and Tanigawa [13, Lemma 2] to (4.4), which is the Mellin–Barnes integral representation of \(J(s_1, s_2)\), we can get only

\[
J(s_1, s_2) \ll |t_2|^{1+\eta+\varepsilon} (|t_2| + |t_1 + t_2|)^{\mu(\sigma_1 + \sigma_2 + \eta)} \tag{6.2}
\]
for $0 < \eta < 1$. The estimate (6.2) is not good enough, because it contains $t_1$. We should note that our estimate (1.6) is uniform on $t_1$. Furthermore if we take $\sigma_1 + \sigma_2 = 1/2$ and $\eta$ very small and $t_1$ constant, the estimate (6.2) gives us $J(s_1, s_2) \ll |t_2|^{1+\varepsilon}$ even if we assume that $\zeta(1/2 + it) \ll |t|^\varepsilon$.

**Proof of Corollary 2.** From Theorem 2, we get

$$J(s_1, s_2) = \chi(s_2) \sum_{n \leq |t_2| \pi} \frac{\sigma_{1-s_1-1}^s(n)}{n^{s_1}} + O(|t_2|^\delta + \varepsilon),$$

and

$$J(1 - s_2, 1 - s_1) = \chi(1 - s_1) \sum_{n \leq |t_1| / 2\pi} \frac{\sigma_{s_1+s_2-1}(n)}{n^{s_1}} + O(|t_1|^{\delta' + \varepsilon}),$$

where $\delta = \max(0, 1 - \sigma_1 - \sigma_2)$ and $\delta' = \max(0, \sigma_1 + \sigma_2 - 1)$. Since $\sigma_{\alpha}(n) = n^\alpha \sigma_{-\alpha}(n)$, the sum in $J(1 - s_2, 1 - s_1)$ has the same form as that in $J(s_1, s_2)$, but the range of $n$ is changed to $1 \leq n \leq |t_1| / 2\pi$. The difference of the ranges in these two sums is less than $|t_1|^{1/2}$ by the assumption (1.6); hence we have for $|t_1| < |t_2|$, \[
\chi(s_2) \sum_{\frac{|t_1|}{2\pi} < n < \frac{|t_2|}{2\pi}} \frac{\sigma_{1-s_1-s_2}^s(n)}{n^{1-s_2}} \ll |t_2|^{1/2-\sigma_2} |t_2|^{\sigma_2-1+\delta + \varepsilon} \sqrt{|t_2|} \ll |t_2|^{\delta + \varepsilon}.
\]

The case $|t_1| > |t_2|$ gives the same bound. This means that

$$J(s_1, s_2) = \chi(s_2) \sum_{n \leq |t_2| \pi} \frac{\sigma_{1-s_1-s_2}^s(n)}{n^{1-s_2}} + O(|t_2|^\delta + \varepsilon),$$

and so eliminating the summation part we have

$$\frac{J(s_1, s_2)}{\chi(s_2)} - \frac{J(1 - s_2, 1 - s_1)}{\chi(1 - s_1)} \ll |t_2|^{\sigma_2-1/2+\delta + \varepsilon} + |t_1|^{1/2-\sigma_1+\delta' + \varepsilon} \ll \begin{cases} |t_1|^{\sigma_2-1/2+\varepsilon} & \text{if } \sigma_1 + \sigma_2 \geq 1 \\ |t_1|^{1/2-\sigma_1+\varepsilon} & \text{if } \sigma_1 + \sigma_2 < 1. \end{cases}$$

This completes the proof of Corollary 2. □

The bound (6.3) is expected to hold true under the assumption $|t_1| \asymp |t_2|$. 

**Remark 4.** Matsumoto proved a “functional equation” of $\zeta_2(s_1, s_2)$ in terms of the function $g$, which has the following form in the present case [12, Theorem 1]:

$$\frac{g(s_1, s_2)}{(2\pi)^{s_1+s_2-1} \Gamma(1-s_1)} = \frac{g(1-s_2, 1-s_1)}{i^{s_1+s_2-1} \Gamma(s_2)} + 2i \sin \left(\frac{\pi}{2} (s_1 + s_2 - 1)\right) F_+(s_1, s_2),$$

where

$$F_+(s_1, s_2) = \sum_{k=1}^{\infty} \sigma_{s_1+s_2-1}(k) \psi(s_2, s_1 + s_2; 2\pi ik).$$
The series $F_+ (s_1, s_2)$ is convergent only in the region $\text{Re} \ s_1 < 0$, $\text{Re} \ s_2 > 1$ but it can be continued analytically to the whole $\mathbb{C}^2$ space. His method is essentially a combination of (2.5) and the transformation formula (2.9).

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