Note

Diameter series of lattice covering simplices

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Abstract

We develop an interesting relationship between finite sets in a lattice and the minimal density of simplex coverings of n-space. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The lower bound problem for diameters of finite, directed Abelian Cayley graphs has provoked renewed interest in the corresponding minimum density problem for lattice-simplex coverings of n-dimensional space (see [1,2]). In particular, the minimum density \( \nu_L \) of a covering \( L + S = \mathbb{R}^n \), where \( L \) is a lattice and \( S \) is a simplex, is unknown for \( n \geq 2 \).

By a simplex in \( \mathbb{R}^n \), we mean the convex hull of \( N + 1 \) points in general position. A lattice will mean a discrete, rank-n subgroup of \( \mathbb{R}^n \). A fundamental cell of a lattice \( L \) is the closed parallelepiped defined by any basis for the lattice. Let \( S \) be a simplex, \( L \) a lattice in \( \mathbb{R}^n \) and \( x \) an element of \( \mathbb{R}^n \). If \( x \notin L + S \) then \( S - x \) (the set \( \{ s - x | s \in S \} \)) is disjoint from \( L \). In other words, \( L + S \) fails to cover \( \mathbb{R}^n \) iff some translate of \( S \) is disjoint from \( L \). Therefore, the smallest dilation \( \lambda S \) which covers with \( L \) (i.e. for which \( L + \lambda S = \mathbb{R}^n \)) is also the largest dilation which can be translated so that \( x + \lambda S \) contains no \( L \)-points in its topological interior, for some \( x \in \mathbb{R}^n \). Clearly such a translate must have \( L \)-points in the \((n - 1)\)-dimensional interior of each facet else the dilation could be made larger. Thus, our interest is attracted to those configurations of \( L \)-points which can lie on the surface of a translated dilation \( x + \lambda S \), such that the simplex \( x + \lambda S \) contains no interior \( L \)-points. We demonstrate an interesting, possibly useful relationship among such configurations.

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2. Diameter series

If $\Lambda$ is a non-singular linear transformation and $L + S = \mathbb{R}^n$ is a lattice-simplex covering, then $\Lambda(L) + \Lambda(S) = \mathbb{R}^n$ is also a lattice-simplex covering with the same density. Thus, we may, in our search for small covering densities, fix any simplex we like and vary the lattice. Accordingly, for ease of computation in the following, we shall use the base simplex $D$ — the convex hull of the standard unit vectors and the origin, in $\mathbb{R}^n$ — with $n$-volume $\text{vol}(D) = 1/n!$.

By a configuration $C(A)$ in a lattice $L$, we mean the translation-equivalence-class of any finite subset $A \subset L$. The diameter $\rho(C)$ of a configuration $C = C(A)$ will be the smallest real $\rho$ such that some translate $x + \rho D$ contains $A$. This is consistent with the Manhattan diameter used in [1–3]. If the translate $x + \rho D$ contains no $L$-points in its interior, we will call $C$ a proper configuration.

Let $\text{vol}(W)$ denote the $n$-dimensional volume of a subset $W \subset \mathbb{R}^n$. If $L$ is a lattice, let $\text{vol}(L)$ denote the $n$-volume of a fundamental cell of $L$. Let $\#(A)$ denote the cardinality of a finite set $A$, and let $\#(C)$ denote the cardinality of $A$ when $C = C(A)$.

If a dilation $\lambda D$ is used to cover (partly or wholly) $\mathbb{R}^n$ with the lattice $L$, we may define the fraction of coverage $f(\lambda) = \text{vol}((L + \lambda D) \cap U)/\text{vol}(U)$ where $U$ is any fundamental cell of the lattice. Thus $0 \leq f(\lambda) \leq 1$.

**Theorem 1.** Let $L$ be a lattice in $\mathbb{R}^n$. For $\lambda > 0$ let $f(\lambda)$ be the fraction of $\mathbb{R}^n$ which is covered by $\lambda D + L$. Then

$$f(\lambda) = \frac{\text{vol}(D)}{\text{vol}(L)} \sum_{\rho(C) < \lambda} (-1)^{\#(C) - 1}(\lambda - \rho(C))^n,$$

the sum being taken over all configurations with diameter less than $\lambda$.

**Proof.** Let $U$ be a fundamental cell of $L$ and let $r(\lambda) = f(\lambda)\text{vol}(L)$ be the volume of that part of $U$ which is covered by $L + \lambda D$. Let $U_k$ be the part of $U$ which is covered with multiplicity $k$ by $L + \lambda D$. Then

$$r(\lambda) = \sum_{k \geq 1} \text{vol}(U_k) = \sum_{k \geq 1} \frac{\text{vol}(G_k)}{k},$$

where $G_k$ denotes the portion of $G = \lambda D$ which is covered with multiplicity $k$ by $L + \lambda D$.

Each $G_k$ is partitioned into atoms which are intersections of $G_k$ with $k$ distinct translates of $\lambda D$ (including $\lambda D$ itself). If $A$ is a finite subset of $L$, then $\bigcap_{x \in A}(x + \lambda D)$ is a translation of $\alpha D$, where $\alpha = \max\{0, \lambda - \rho(C(-A))\}$. Thus,

$$\text{vol}\left(\bigcap_{x \in A}(x + \lambda D)\right) = (\lambda - \rho(C(-A)))^n/n!,$$

when $\rho(C(-A)) < \lambda$ (and zero otherwise). Let $v_\lambda(A)$ denote $\text{vol}(\bigcap_{x \in A}(x + \lambda D))$. 


Consider the sum
\[ S = \sum_{(0) \in A} \frac{v_{\lambda}(A)(-1)^{\#(A)-1}}{\#(A)} \]
taken over all finite subsets \( A \subseteq L \) which contain the origin. (Note that only finitely many such \( A \) will have \( v_{\lambda}(A) > 0 \).) Now each atom with multiplicity \( k \) in \( G_k \) will be included in this sum once for each set \( A \) such that \( A \) contains the origin and \( A \) is included in the translates \( (k \text{ of them}) \) of \( \lambda D \) which determine the atom. In other words, the atom’s volume will be counted with total multiplicity
\[ \sum_{i=1}^{k} \left( \frac{k-1}{i-1} \right) (-1)^{i-1} = \frac{1}{k}, \]
which means
\[ S = \sum_{k \geq 1} \frac{\text{vol}(G_k)}{k} = r(\lambda). \]

Finally, note that by multiplying each summand in \( S \) by \( 1/\#(A) \), we are effectively summing over configurations rather than subsets, since each configuration \( C \) with \( \#(C) = i \) will be represented by \( i \) different subsets \( A \) with a point at the origin. Thus (letting \( C = C(-A) \)),
\[ S = \sum_{\rho(C) < \lambda} (-1)^{\#(C)-1}(\lambda - \rho(C))^n \text{vol}(D). \]
Substituting \( r(\lambda) \) for \( S \), and dividing by \( \text{vol}(L) \) gives the desired equation for \( f(\lambda) \).

Note that if \( C(A) \) is not a proper configuration, then there must be an \( L \)-point \( y \) in the interior of the smallest \( x + \lambda D \) which covers \( A \). By either adding \( y \) to \( A \) (or removing it if it is already in \( A \)), we may construct a set \( A' \) so that \( \rho(C(A')) = \rho(C(A)) \) but \( \#(A) \) and \( \#(A') \) differ by one. Thus, the two terms corresponding to \( C(A) \) and \( C(A') \) will cancel each other out in the summation for \( f(\lambda) \). By well-ordering \( L \) and by always using the smallest (by that ordering) interior \( L \)-point, we may systematically pair up all non-proper configurations and cancel them out of the summation. Thus we have the following.

**Corollary 1.** Let \( L \) be a lattice in \( \mathbb{R}^n \). For \( \lambda > 0 \) let \( f(\lambda) \) be the fraction of \( \mathbb{R}^n \) which is covered by \( \lambda D + L \). Then
\[ f(\lambda) = \frac{\text{vol}(D)}{\text{vol}(L)} \sum_{C \in \mathcal{P}(\lambda)} (-1)^{\#(C)-1}(\lambda - \rho(C))^n, \]
where \( \mathcal{P}(\lambda) \) denotes the set of all proper configurations with diameter less than \( \lambda \).

Let \( \lambda_0 \) be the minimum covering diameter with \( D \) (the smallest \( \lambda \) such that \( L + \lambda D \supseteq \mathbb{R}^n \)). Since the above polynomial in \( \lambda \) becomes constant (equal to 1) when \( \lambda \geq \lambda_0 \), and
since no proper configuration can have \( \rho(C) > \lambda_0 \), we obtain the following by isolating the coefficients of the polynomial. Let \( \mathcal{P} \) denote the set of all proper configurations.

**Corollary 2.** Let \( k \) be a non-negative integer \( \leq n \), and \( L \) a lattice. Then

\[
\sum_{C \in \mathcal{P}} (-1)^{\theta(C)}(\rho(C))^k = \begin{cases} 0 & \text{if } k = 0, 1, \ldots, n-1, \\ (-1)^{n-1} \frac{\text{vol}(L)}{\text{vol}(D)} & \text{if } k = n, \end{cases}
\]

(where we apply the combinatorial convention that \( 0^0 = 1 \)).

Let \( \theta(C) \) denote the density of \( L + \rho(C)D \) (whether it is a covering of \( \mathbb{R}^n \) or not). In other words, \( \theta(C) = \rho(C)^n/n! \text{vol}(L) \). Note that the largest value of \( \theta(C) \) for \( C \in \mathcal{P} \) is precisely the smallest density achievable by a lattice-simplex covering of the form \( L + \lambda D \). Then one may re-write the above equations (multiplying both sides by \( \frac{\text{vol}(D)}{\text{vol}(L)} \)) as follows:

**Corollary 3.** Let \( k \) be a non-negative integer \( \leq n \), and \( L \) a lattice. Then

\[
\sum_{C \in \mathcal{P}} (-1)^{\theta(C)}(\theta(C))^k = \begin{cases} 0 & \text{if } k = 0, 1, \ldots, n-1, \\ (-1)^{n-1} & \text{if } k = n. \end{cases}
\]

Note that the largest summands on the left have absolute value \( \theta_D(L) \), the smallest covering density achievable by using \( L \) plus a dilation of \( D \). Minimizing \( \theta_D(L) \) over all lattices \( L \) in \( \mathbb{R}^n \) would provide the minimum lattice-simplex covering density in \( n \) dimensions.

### 3. Example

In \( \mathbb{R}^2 \), let \( L \) be the lattice generated by \((2,2)\) and \((-1,4)\). Thus \( \text{vol}(L) = 10 \) (and \( \text{vol}(D) = \frac{1}{2} \)). There are six proper configurations (obtained quite crudely by hand-checking all small configurations): \( \{(0,0)\} \) with one point, \( \{(0,0),(2,2)\} \), \( \{(0,0),(3,-2)\} \) and \( \{(0,0),(-1,4)\} \) with two points, \( \{(0,0),(2,2),(-1,4)\} \) and \( \{(0,0),(2,2),(3,-2)\} \) with three points.

The diameters of these configurations are \( 0,4,3,4,5,6 \), respectively. Thus,

\[
f(\lambda) = \begin{cases} \lambda^2 & \text{for } \lambda < 3, \\ 0.05(\lambda^2 - (\lambda - 3)^2) = 0.05(6\lambda - 9) & \text{for } 3 \leq \lambda < 4, \\ 0.05(\lambda^2 - (\lambda - 3)^2 - 2(\lambda - 4)^2) = 0.05(-2\lambda^2 + 22\lambda - 41) & \text{for } 4 \leq \lambda < 5, \\ 0.05(\lambda^2 - (\lambda - 3)^2 - 2(\lambda - 4)^2 + (\lambda - 5)^2) = 0.05(-\lambda^2 + 12\lambda - 16) & \text{for } 5 \leq \lambda < 6, \\ 0.05(\lambda^2 - (\lambda - 3)^2 - 2(\lambda - 4)^2 + (\lambda - 5)^2 + (\lambda - 6)^2) = 0.05(20) = 1 & \text{for } 6 \leq \lambda. \end{cases}
\]
Note also that
\[-0^0 + 4^0 + 3^0 + 4^0 - 5^0 - 6^0 = 0\]
(again using the convention \(0^0 = 1\)),
\[-0^1 + 4^1 + 3^1 + 4^1 - 5^1 - 6^1 = 0\]
and
\[-0^2 + 4^2 + 3^2 + 4^2 - 5^2 - 6^2 = -20 = -\frac{\text{vol}(L)}{\text{vol}(D)}\].

References