The geometric cone relations for simplicial and cubical complexes

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Abstract

This note gives several geometric cone relations for arbitrary polyhedron with a regular decomposition. When the decomposition is simplicial or cubical, we obtain the geometric analogs of Dehn–Sommerville and angle-sum relations for simplicial and cubical complexes.

1. Introduction

Let $P$ be a $d$-dimensional simplicial polytope of the Euclidean space $\mathbb{R}^d$. For each $i = -1, 0, ..., d - 1$, let $f_i(P)$ denote the number of $i$-faces of $P$. The empty set $\emptyset$ is considered as the unique face of $P$ of dimension $-1$. It is well known that the Dehn–Sommerville equations

$$f_i(P) = \sum_{j=i}^{d-1} (-1)^{d-1-j} \binom{j+1}{i+1} f_j(P), \quad i = -1, ..., d - 1 \quad (1)$$

are the only (independent) linear relations on the $f$-vectors $(f_0, f_1, ..., f_{d-1})$ for convex simplicial polytopes [5,8]. These relations are naturally generalized to cone relations in [4] for simplicial and cubical polytopes. The angle-sum relations of Perles and Shephard [5–7]

$$f_i(P) - x_i(P) = \sum_{j=i}^{d-1} (-1)^{d-1-j} \binom{j+1}{i+1} x_j(P), \quad i = 0, 1, ..., d - 1 \quad (2)$$

References:

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follow easily from the numerical form of these cone relations by taking certain in-
tegration and normalization [4], where \( x_i \) is the sum of angles extended over all
\( i \)-dimensional faces, see below.

The purpose of this note is to find the analogs of these relations for simplicial and
cubical complexes both in cone functions and angles.

Throughout we denote by \( V \) a finite-dimensional vector space over the field \( \mathbb{R} \) of
real numbers. The dual space of \( V \) is denoted by \( V^* \). The indicator function of
a subset \( E \) of \( V \) is the characteristic function \( 1_E \) on \( V \), that is, \( 1_E(x) = 1 \) for \( x \in E \) and
\( 1_E(x) = 0 \) otherwise. By a polyhedron we mean a subset of \( V \) which can be obtained by
taking unions, intersections, and complements finitely many times of open half-spaces
\( \{ x \in V \mid \varphi(x) < 0 \} \), where \( \varphi \) is an affine linear function on \( V \). The interior of a convex
polyhedron in the affine subspace that the polyhedron spans is called a relatively open
closed polyhedron. For a convex polyhedron \( Q \), the relative interior and closure of \( Q \)
are denoted by \( \operatorname{ri} Q \) and \( \operatorname{cl} Q \) (or \( Q^- \)), respectively. We use \( \mathcal{P} \) to denote the class of all
relatively open convex polyhedra.

Let \( C \) be a convex cone of \( V \) with the apex at the origin. Its dual cone \( C^* \) is the cone
of \( V^* \) defined by
\[
C^* := \{ y \in V^* \mid y(v) \leq 0, \forall v \in C \}.
\]
Let \( P \) be a relatively open convex polyhedron of \( V \). A relatively open convex poly-
hedron \( F \) is said to be a face of \( P \) if its closure \( \bar{F} \) is a face of \( P \) in the ordinary sense,
and this is denoted by \( F \preceq P \) or \( P^F \). We shall consider the interior cone \( C(F,P) \)
and exterior cone \( C^*(F,P) := C(F,P)^* \) of \( P \) near its face \( F \), and the cone \( C(\infty,P) \)
of \( P \) near \( \infty \), which are respectively defined by
\[
C(F,P) := \{ v \in V \mid \exists q \in F, p \in P, t > 0 \text{ s.t. } tv = p - q \},
\]
\[
C(\infty,P) := \{ -v \in V \mid \exists p \in P \text{ s.t. } p + tv \in P, \forall t > 0 \}.
\]
Note that when \( F = P \), \( \langle P \rangle := C(P,P) \) is a vector subspace of dimension \( \dim P \). The
intrinsic interior cone \( C^\prec(F,P) \) in [3] and the intrinsic exterior cone \( C^\succ(F,P) \) of \( P \)
near its face \( F \) are also considered and they are defined by
\[
C^\prec(F,P) := \langle F \rangle \cap C(F,P),
\]
\[
C^\succ(F,P) := C^\prec(F,P) \cap \langle P \rangle.
\]
We denote the indicator functions of the cones \( C(F,P) \), \( C^*(F,P) \), \( C^\prec(F,P) \), and
\( \operatorname{ri}[C^\succ(F,P)] \) by \( T(F,P) \), \( K(F,P) \), \( A(F,P) \), and \( B(F,P) \), respectively. For infinity \( \infty \),
we define
\[
T(\infty,P) := (-1)^{\dim P} 1_{\{ \emptyset \}} - (-1)^{\dim P} 1_{C(\infty,P)},
\]
\[
K(\infty,P) := (-1)^{\dim P} 1_{V^*} - (-1)^{\dim P - \dim C(\infty,P)} \cdot 1_{-\operatorname{ri}[C(\infty,P)]^*},
\]
where \( \dim_c C(\infty,P) \) denotes the largest dimension of the affine subspaces contained in
\( C(\infty,P) \). These functions on \( V \) are the elements of the Minkowski algebra \( S(V,\mathcal{P}) \).
which is the vector space generated by the indicator functions of member of \( \mathcal{R} \), and the multiplication is the convolution induced by the vector addition of \( V \), see [3]. There is a linear functional \( \chi \), called the Euler characteristic, on \( S(V, \mathcal{R}) \) such that \( \chi(1_P) = (-1)^{\dim P} \) for each relatively open convex polyhedron \( P \). There are three useful operators, reflection\(^-\), closure\(^-\), and dual\(*\) on \( S(V, \mathcal{R}) \), defined by

\[
\begin{align*}
  f^-(x) &:= f(-x), \quad \forall x \in V, \\
  \tilde{f}(x) &:= \lim_{r \to 0} \chi(f \cdot 1_{B(x,r)}), \quad \forall x \in V, \\
  f^*(x) &:= \chi(f \cdot 1_{\{x \in V \mid x(v) \leq 0\}}), \quad \forall x \in V^*,
\end{align*}
\]

where \( B(x,r) \) is the open ball of radius \( r \) centered at \( x \). For a relatively open convex polyhedron \( P \) it is clear that \( (1_P)^- = 1_{-P} \), where \( -P = \{-x \mid x \in P\} \), and \( 1_P = (-1)^{\dim P} \). While for a relatively open convex polyhedral cone \( C \), \( (1_C)^* = (-1)^{\dim C} 1_C^* \). The reflection and closure operators are involutions in the sense that they are idempotent, but the dual operator, when restricted from \( S(V, \mathcal{R}) \) to \( S(V^*, \mathcal{R}) \), is invertible and its inverse is given by the same definition, where \( \mathcal{C} \) denotes the class of relatively open convex cones with apex at the origin.

Let \( X \) be a polyhedron of \( V \). A regular decomposition of \( X \) is a collection \( \mathcal{D} \) of disjoint relatively open convex polyhedra \( P \) such that \( X = \bigcup_{P \in \mathcal{D}} \) and the collection \( \mathcal{F}(X, \mathcal{D}) = \{G \mid G \subseteq P \in \mathcal{D}\} \) is also disjoint. The set \( \mathcal{F} = \mathcal{F}(X, \mathcal{D}) \) is called the face system of \( X \) with respect to the decomposition \( \mathcal{D} \). Note that \( X \) is closed if and only if \( \mathcal{F}(X, \mathcal{D}) = \mathcal{D} \). In fact, \( \mathcal{F} \) is a regular decomposition of the closure \( \bar{X} \). For each face \( F \in \mathcal{F}(X, \mathcal{D}) \), the cone \( C(F, X) \) of \( X \) near the face \( F \) is the disjoint union of \( C(F, \mathcal{D}) = \{C(F, G) \mid F \subseteq G \in \mathcal{D}\} \). \( C(F, X) \) is a closed cone if and only if \( F \in \mathcal{D} \). The face system of \( C(F, \mathcal{D}) \) is isomorphic to the face poset \( \mathcal{F} = \{G \in \mathcal{F} \mid F \subseteq G\} \) of \( X \) near \( F \).

Let \( f \) be an incidence function with values in a commutative ring \( R \) with unit \( 1 \neq 0 \), that is, \( f \) is a map from \( \mathcal{F}(X, \mathcal{D}) \times \mathcal{F}(X, \mathcal{D}) \) to \( R \) such that \( f(F, P) = 0 \) if \( F \) is not a face of \( P \). We associate with \( f \) another incidence function \( f' \), defined by

\[
f'(F, P) := (-1)^{\dim P - \dim F} f(F, P), \quad \forall F \leq P.
\]

For each face \( F \in \mathcal{F}(X, \mathcal{D}) \) and \( i = 0, 1, \ldots, \dim X \), we define

\[
\begin{align*}
  f(F, X) &:= \sum_{F \in G \in \mathcal{D}} f(F, G), \\
  f_i(X) &:= \sum_{F \in \mathcal{F}, \dim F = i} f(F, X).
\end{align*}
\]

For example, if \( f \) is the function \( T \), then \( T(F, X) \) is the indicator function of the tangent cone of \( X \) near \( F \), and \( T_i(X) \) is the sum of the indicator functions of tangent cones of \( X \) near all its \( i \)-dimensional faces.
2. Cone relations

Let $X$ be a polyhedron with face system $\mathcal{F}(X, \mathcal{D})$. The generalized Gram–Sommerville theorem [1, Theorem 3.1] states that

$$\sum_{F \in \mathcal{F}} (-1)^{\dim F} T(F, X) + T(\infty, X) = \chi(X) 1_{\{o\}}$$  \hspace{1cm} (3)

and the generalized Gauss–Bonnet theorem [1, Theorem 4.2] states that

$$\sum_{F \in \mathcal{F}} K'(F, X) + K(\infty, X) = \chi(X) 1_{\{o\}}.$$  \hspace{1cm} (4)

These two theorems will be useful in deriving our cone relations in this section.

**Theorem 1.** Let $X$ be a polyhedron with a regular decomposition $\mathcal{D}$. Then for any fixed $F \in \mathcal{F}(X, \mathcal{D})$,

$$\sum_{G \in \mathcal{G}} (-1)^{\dim G} T(G, X) = \tilde{T}(-F, -X) = \tilde{T}(-F, -X).$$  \hspace{1cm} (5)

**Proof.** For each pair $(G, P)$ of relatively open faces such that $F \leq G \leq P \in \mathcal{D}$, it is clear that $C(G, P) = C(C(F, G), C(F, P))$. Then $C(G, X) = C(C(F, G), C(F, X))$. So the left side of (5) can be written as

$$\sum_{C(F, F) \in C(F, G) \in C(F, \mathcal{F})} (-1)^{\dim C(F, G)} T(C(F, G), C(F, X)).$$

With the generalized Gram–Sommerville formula (3) for the cone $C(F, X)$ and the definition of the cone near $\infty$, the left-hand side of (5) is then equal to

$$\chi(C(F, X)) 1_{\{o\}} - \sum_{C(F, F) \leq C(F, P) \leq C(F, \mathcal{F})} T(\infty, C(F, P)).$$

Note that $C(F, \mathcal{F})$ is in one-to-one correspondence to $\mathcal{F}$ and $\dim C(F, G) = \dim G$ for every $G \in \mathcal{F}$. Hence, the left-hand side can be written as

$$\chi(\hat{\mathcal{F}}) 1_{\{o\}} - \sum_{F \in \mathcal{F}(\mathcal{F})} [(-1)^{\dim P} - (-1)^{\dim P} 1_{C(\infty, C(F, P))}].$$

Since $C(F, P)$ are convex cones, then $C(\infty, C(F, P)) = -\text{cl}[C(F, P)]$. So the left side of (5) becomes

$$\sum_{F \in \mathcal{F}(\mathcal{F})} (-1)^{\dim P} 1_{\text{cl}[C(F, P)]} = \sum_{F \leq P \in \mathcal{F}} \tilde{T}(-F, -P) = \tilde{T}(-F, -\hat{\mathcal{F}}).$$

If the decomposition $\mathcal{D}$ is simplicial, that is, each cell of $\mathcal{D}$ is a relatively open simplex, we extend the sum of both sides of (5) over all $i$-dimensional faces of $X$. 

\[ \square \]
Note that each $j$-face has exact $\binom{j+1}{i+1}$ $i$-faces. Then

$$
\sum_{F \leq G \in \mathcal{P}, \dim F = i} (-1)^{\dim G} T(G, X) = \sum_{G \in \mathcal{P}, \dim G \geq i} (-1)^{\dim G} \sum_{F \leq G, \dim F = i} T(G, X)
$$

$$
= \sum_{j=i}^{\dim X} (-1)^j \binom{j+1}{i+1} T_j(X).
$$

Thus, we have obtained the geometric cone relations for an arbitrary polyhedron with a regular simplicial decomposition.

**Corollary 2.** Let $X$ be a polyhedron with a regular simplicial decomposition $\mathcal{D}$, then for each $i = 0, 1, \ldots, \dim X$,

$$
\tilde{T}_i^{-}(X) = \sum_{j=i}^{\dim X} (-1)^j \binom{j+1}{i+1} T_j(X). \quad (6)
$$

If the decomposition $\mathcal{D}$ is cubical, that is, each cell of $\mathcal{D}$ is a relatively open convex polytope isomorphic to a standard relatively open cube. Then each $j$-face has exact $\binom{j}{i} 2^{j-i}$ $i$-faces. Thus,

$$
\sum_{F \leq G \in \mathcal{P}, \dim F = i} (-1)^{\dim G} T(G, X) = \sum_{G \in \mathcal{P}, \dim G \geq i} (-1)^{\dim G} \sum_{F \leq G, \dim F = i} T(G, X)
$$

$$
= \sum_{j=i}^{\dim X} (-1)^j \binom{j}{i} 2^{j-i} T_j(X).
$$

**Corollary 3.** If $X$ is a polyhedron with a regular cubical decomposition $\mathcal{D}$, then for each $i = 0, 1, \ldots, \dim X$,

$$
\tilde{T}_i^{-}(X) = \sum_{j=i}^{\dim X} (-1)^j \binom{j}{i} 2^{j-i} T_j(X). \quad (7)
$$

Let the closure operator act on (5) and note that the closure operator is an involution. We then have

$$
\sum_{F \leq G \in \mathcal{P}} (-1)^{\dim G} \tilde{T}(G, X) = T(-F, -X) = T^{-}(F, X). \quad (8)
$$

With the generalized Gauss–Bonnet formula (4) there are dual versions of (5) and (8). These formulas can be obtained by applying the dual operator $*$ to both sides of (5). It is easy to see that $(1_C)^* = (-1)^{\dim C} 1_{C^*}$ and $(1_{C^*})^* = (-1)^{\dim C} 1_{r(C^*)}$ for any
relatively open convex polyhedral cone \( C \). Then

\[
[T(F, P)]^* = (-1)^{\dim C(F, P)} 1_{C^*(F, P)}
\]

\[
= (-1)^{\dim F} K'(F, P),
\]

\[
[T(F, P)]^* = [(1 - [C(F, P)])^*]
\]

\[
= (-1)^{\dim P + \dim, C(F, P) 1_{cl[C(F, P)]}}
\]

\[
= (-1)^{\dim V - \dim F} \tilde{K}'(F, P).
\]

Thus, we have the following identity for exterior cone functions:

\[
\sum_{F \leq G \in \mathcal{F}} K'(G, X) = (-1)^{\dim V - \dim F} \tilde{K}'(F, X).
\]

(9)

When we apply the closure operator on both sides of (9) we have

\[
\sum_{F \leq G \in \mathcal{F}} \tilde{K}'(G, X) = (-1)^{\dim V - \dim F} K'(F, X).
\]

(10)

If we take the sum again on both sides of (8)–(10) over all \( i \)-dimensional faces of \( X \), we obtain the geometric relations with the closed tangent cone, exterior cone, and closed exterior cone functions.

**Corollary 4.** Let \( X \) be a polyhedron with a regular decomposition \( \mathcal{D} \). For each \( i = 0, 1, \ldots, \dim X \), if \( \mathcal{D} \) is simplicial, then

\[
T_i^-(X) = \sum_{j=i}^{\dim X} (-1)^j \binom{j + 1}{i + 1} \tilde{T}_j(X),
\]

(11)

\[
\tilde{K}'_i(X) = (-1)^{\dim V - i} \sum_{j=i}^{\dim X} \binom{j + 1}{i + 1} K'_j(X),
\]

(12)

\[
K'_i(X) = (-1)^{\dim V - i} \sum_{j=i}^{\dim X} \binom{j + 1}{i + 1} \tilde{K}'_j(X),
\]

(13)

and if \( \mathcal{D} \) is cubical, then

\[
T_i^-(X) = \sum_{j=i}^{\dim X} (-1)^j \binom{j}{i} 2^{j-i} \tilde{T}_j(X),
\]

(14)

\[
\tilde{K}'_i(X) = (-1)^{\dim V - i} \sum_{j=i}^{\dim X} \binom{j}{i} 2^{j-i} K'_j(X),
\]

(15)

\[
K'_i(X) = (-1)^{\dim V - i} \sum_{j=i}^{\dim X} \binom{j}{i} 2^{j-i} \tilde{K}'_j(X),
\]

(16)
Remark 5. Whenever $X$ is a manifold without boundary and evaluate both sides of (11) and (14), we obtain the Dehn–Sommerville equations for simplicial and cubical complexes whose geometric realization is a manifold without boundary.

3. Angle-sum relations

We now consider numerical angles of these tangent cones of $X$. Let $C$ be a $k$-dimensional relatively open convex cone of $V$ with apex at the origin. The angle $\alpha(C)$ of $C$ is defined as the normalized $k$-dimensional Lebesgue measure of $C$ in the unit ball of $\langle C \rangle$, namely,

$$\alpha(C) := \frac{\text{vol}_k(C \cap B(o, 1))}{\text{vol}_k(B(o, 1))},$$

where $\text{vol}_k$ is the Lebesgue measure on the $k$-dimensional subspace $\langle C \rangle$ and $B(o, 1)$ is the unit ball centered at the origin $o$. For each relatively open convex polyhedron pair $F \leq P$, we denote the angles of the cones $C(F, P)$ and $C^*(F, P)$ by $\alpha(F, P)$ and $\beta(F, P)$, respectively. Note that the angle of $C^\| (F, P)$ is $\alpha(F, P)$ and the angle of $C^\wedge (F, P)$ is $\beta(F, P)$.

For a fixed relatively open convex polyhedron $P \in \mathcal{D}$ and a fixed face $F \leq P$, as a special case of a single $P$, Eq. (8) states that

$$\sum_{F \leq G \leq P} (-1)^{\dim G} \tilde{T}(G, P) = T^-(F, P).$$

Integrate both sides of the equation above, we then have

$$\sum_{F \leq G \leq P} \alpha'(G, P) = \alpha(F, P).$$

For each $F \in \mathcal{F}(X, \mathcal{D})$, if the tangent angle functions of $X$ near $F$ are defined as

$$\alpha(F, X) := \sum_{F \leq P \leq \mathcal{D}} \alpha(F, P),$$

$$\alpha'(F, X) := \sum_{F \leq P \leq \mathcal{D}} \alpha'(F, P),$$

we then have the following theorem.

**Theorem 6.** Let $X$ be a polyhedron with a regular decomposition $\mathcal{D}$. For each fixed face $F \in \mathcal{F}(X, \mathcal{D})$,

$$\sum_{F \leq G \leq X} \alpha'(G, X) = \alpha(F, X). \quad (17)$$

Let the face $F$ in (17) be extended over all $i$-dimensional faces of $X$. We then have the corollary.
Corollary 7. Let $X$ be a polyhedron with decomposition $\mathcal{D}$. For each $i=0,1,\ldots,$ $\dim X$, if $\mathcal{D}$ is simplicial, then

$$z_i(X) = \sum_{j=i}^{\dim X} \binom{j+1}{i+1} z'_j(X);$$

and if $\mathcal{D}$ is cubical, then

$$z_i(X) = \sum_{j=i}^{\dim X} \binom{j}{i} 2^{j-i} z'_j(X).$$

Remark 8. Formulas for $\beta$ similar to (17)–(19) are not found yet. Probably such formulas do not exist for $\beta$ on arbitrary polyhedra.

References