JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 60, 227–235 (1977)

Thermal Instability in Compressible Fluids in the Presence of Rotation and Magnetic Field

R. C. SHARMA

Department of Mathematics, Himachal Pradesh University, Simla 171005, India

Submitted by C. L. Dolph

The thermal instability of compressible fluids pervaded by a uniform rotation and a uniform magnetic field, separately, is considered. For $(Cp/g)\beta < 1$, with Cp, g, and β denoting the specific heat at constant pressure, the acceleration due to gravity, and the uniform temperature gradient, respectively, the system is shown to be stable. The magnetic field as well as rotation introduces oscillatory modes in thermal instability of compressible fluids, which are completely missing for $(Cp/g)\beta > 1$ in the absence of rotation or magnetic field. The sufficient conditions which do not allow overstable modes are obtained.

1. INTRODUCTION

A detailed account of the theoretical and experimental study of the onset of Bénard convection in incompressible fluids has been given by Chandrasekhar [1]. The use of Boussinesq approximation has been made throughout, which states that the density may be treated as a constant in all the terms in the equations of motion except the term in the external force. The approximation is well justified in the case of incompressible fluids.

When the fluids are compressible, the equations governing the system become quite complicated. To simplify them, Boussinesq tried to justify the approximation for compressible fluids when the density variations arise principally from thermal effects. Spiegel and Veronis [3] have simplified the set of equations governing the flow of compressible fluids under the following assumptions.

(i) The depth of the fluid layer is much less than the scale height, as defined in [3], and (ii) the fluctuations in temperature, density, and pressure, introduced due to motion, do not exceed their total static variations. Under the above approximations, the flow equations are the same as those for incompressible fluids except that the static temperature gradient is replaced by its excess over the adiabatic.

Here we study the thermal instability of a layer of rotating compressible

ISSN 0022-247X

fluid and examine the role of rotation and compressibility in the thermal instability problem.

In Section 4, we consider the onset of Bénard convection in a layer of finitely conducting compressible fluid which is acted on by a uniform vertical magnetic field. The effects of compressibility and magnetic field on the instability are investigated. These aspects form the subject matter of the present paper.

2. The Physical Problem and Its Formulation

Consider an infinite horizontal layer of compressible, viscous, and heat-conducting fluid in which a uniform temperature gradient $\beta\,(= -dT/dz)$ is maintained. Let the origin be on the lower boundary $z = 0$ with the axis of z perpendicular to it along the vertical. The fluid is acted on by a uniform rotation $\mathbf{\Omega}(0, 0, \Omega)$ and a gravity force $\mathbf{g}(0, 0, -g)$.

The initial state is, therefore, a state in which the velocity, temperature, pressure, and density at any point in the fluid are given by

$$\mathbf{q} = 0, \qquad T = T(z), \qquad p = p(z), \qquad \rho = \rho(z), \tag{1}$$

respectively, where, following Spiegel and Veronis, we have

$$\begin{aligned} T(z) &= -\beta z + T_0\,, \\ p(z) &= p_{\mathrm{m}} - g\int_0^z (\rho_{\mathrm{m}} + \rho_0)\,dz, \\ \rho(z) &= \rho_{\mathrm{m}}[1 - \alpha_{\mathrm{m}}(T - T_{\mathrm{m}}) + K_{\mathrm{m}}(p - p_{\mathrm{m}})], \\ \alpha_{\mathrm{m}} &= -[(1/\rho)\,(\partial\rho/\partial T)]_{\mathrm{m}}\,, \\ K_{\mathrm{m}} &= [(1/\rho)\,(\partial\rho/\partial p)]_{\mathrm{m}}\,. \end{aligned} \tag{2}$$

Here we restrict our study to the infinitesimal perturbations so that the perturbations are smaller than their basic variations and the linearized stability theory is applicable. Also, the layer depth of the fluid is assumed to be small enough that it is much less than the scale height as defined by Spiegel and Veronis.

Spiegel and Veronis [3] expressed any space variable, say X, in the form

$$X = X_m + X_0(z) + X'(x, y, z, t),$$

where X_m stands for the constant space distribution of X, X_0 is the variation in X in the absence of motion, and $X'(x, y, z, t)$ stands for the fluctuations in X due to the motion of the fluid. p_m and ρ_m thus stand for the constant space distribution of p and ρ and T_0, ρ_0 stand for the temperature and density of the fluid at the lower boundary $z = 0$.

The linearized perturbation equations of momentum, continuity, and heat conduction, under the above-mentioned approximations, are

$$\partial \mathbf{q}/\partial t = -(1/\rho_m)\,\nabla\delta p - \mathbf{g}\alpha\theta + \nu\nabla^2\mathbf{q} + 2(\mathbf{q}\times\mathbf{\Omega}), \tag{3}$$

$$\nabla\cdot\mathbf{q} = 0, \tag{4}$$

$$(\partial\theta/\partial t) + w[-\beta + (g/C_p)] = \kappa\nabla^2\theta, \tag{5}$$

where

$$\alpha_m = 1/T_m = \alpha, \qquad \text{say,}$$

$$\nu = \mu/\rho_m\,,$$

$$\kappa = \kappa'/\rho_m C_p\,,$$

and g/C_p, μ, ν, κ', κ stand for the adiabatic gradient, the viscosity, the kinematic viscosity, the thermal conductivity, and the thermal diffusivity, respectively. Here $\mathbf{q}(u, v, w)$, δp, $\delta\rho$, and θ denote the perturbations in velocity, pressure p, density ρ, and temperature T, respectively. In writing Eq. (3), use has been made of the Boussinesq equation of state

$$\delta\rho = -\alpha\rho_m\theta, \tag{6}$$

where α is the coefficient of thermal expansion. Equations (3)–(5) give

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\nabla^2 w - g\alpha\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\theta + 2\Omega\frac{\partial\zeta}{\partial z} = 0, \tag{7}$$

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\zeta = 2\Omega\frac{\partial w}{\partial z}\,, \tag{8}$$

$$\left(\frac{\partial}{\partial t} - \kappa\nabla^2\right)\theta = -\left(\frac{g}{C_p} - \beta\right)w, \tag{9}$$

where $\zeta = (\partial v/\partial x) - (\partial u/\partial y)$, denotes the z component of vorticity.

3. Dispersion Relation and Discussion

Analyzing the disturbances into normal modes, we assume that the perturbation quantities are of the form

$$[w, \theta, \zeta] = [W(z), \Theta(z), Z(z)]\exp(ik_x x + ik_y y + nt), \tag{10}$$

where k_x, k_y are the wavenumbers in the x and y directions, respectively, $k = (k_x^2 + k_y^2)^{1/2}$ is the resultant wavenumber, and n is the frequency of oscillation, which is, in general, a complex constant.

Using expression (10), Eqs. (7)–(9) in nondimensional form become

$$(D^2 - a^2)(D^2 - a^2 - \sigma) W - \frac{g\alpha d^2}{\nu} a^2 \Theta - \frac{2\Omega d^3}{\nu} DZ = 0, \tag{11}$$

$$(D^2 - a^2 - \sigma) Z = -\left(\frac{2\Omega d}{\nu}\right) DW, \tag{12}$$

$$(D^2 - a^2 - p_1\sigma)\Theta = -\frac{d^2}{\kappa}\frac{g}{C_p}(H-1) W, \tag{13}$$

where we have written $a = kd$, $\sigma = nd^2/\nu$, $p_1 = \nu/\kappa$, $H = (C_p/g)\beta$, $x/d = x'$, $y/d = y'$, $z/d = z'$, and $D = d/dz'$.

Operating Eq. (11) by $(D^2 - a^2 - \sigma)(D^2 - a^2 - p_1\sigma)$ and using (12) and (13), thus eliminating Θ and Z, we obtain

$$\begin{aligned}(D^2 - a^2 - p_1\sigma)[(D^2 - a^2)(D^2 - a^2 - \sigma)^2 + T_A D^2] W \\ = -Ra^2\left(\frac{H-1}{H}\right)(D^2 - a^2 - \sigma) W,\end{aligned} \tag{14}$$

where $R = g\alpha\beta d^4/\nu\kappa$ stands for the Rayleigh number and $T_A = 4\Omega^2 d^4/\nu^2$ denotes the Taylor number. We consider the case in which both the boundaries are free and the medium adjoining the fluid is nonconducting. The case of two free boundaries is the most appropriate for stellar atmospheres [4].

The boundary conditions appropriate for the problem are

$$W = D^2 W = 0, \qquad \Theta = 0, \qquad DZ = 0 \qquad \text{at} \qquad z' = 0 \text{ and } 1. \tag{15}$$

Dropping the primes for convenience and using (15), we can show that all the even derivatives of W must vanish for $z = 0$ and 1 and hence the proper solution of Eq. (14) characterizing the lowest mode is

$$W = A \sin \pi z, \tag{16}$$

where A is a constant.

Substituting (16) in Eq. (14) and letting $a^2 = \pi^2 x$, $R_1 = R/\pi^4$, $T_1 = T_A/\pi^4$, we obtain the dispersion relation

$$R_1 = \frac{[1 + x + p_1(\sigma/\pi^2)]}{x[1 + x + (\sigma/\pi^2)]}\left(\frac{H}{H-1}\right)\left[(1+x)\left(1 + x + \frac{\sigma}{\pi^2}\right)^2 + T_1\right]. \tag{17}$$

For the stationary convection, $\sigma = 0$ and Eq. (17) reduces to

$$R_1 = (1/x)\,[H/(H - 0)]\,[(1 + x)^3 + T_1]. \tag{18}$$

Observing a perfect analogy between Eq. (18) and Chandrasekhar's Eq. (130) [1, Chap. III] for Bénard convection, we find that

$$R_c' = R_c \times [H/(H - 1)],$$

where R_c is the critical Rayleigh number in the absence of compressibility and R_c' stands for the critical Rayleigh number in the presence of compressibility.

Thus we obtain a stabilizing effect of compressibility, as its effect is to postpone the onset of thermal instability. The rotation also has a stabilizing effect.

THEOREM 1. *If $H < 1$, the system is stable.*

Proof. Multiplying Eq. (11) by W^*, the complex conjugate of W, integrating over the range of z, and making use of Eqs. (12) and (13), we obtain

$$I_1 + \sigma I_2 + d^2(I_5 + \sigma^* I_6) = \frac{C_p \alpha \kappa a^2}{\nu(H - 1)} (I_3 + p_1 \sigma^* I_4), \tag{19}$$

where

$$\begin{aligned}
I_1 &= \int_0^1 (|D^2W|^2 + 2a^2 |DW|^2 + a^4 |W|^2)\, dz,\\
I_2 &= \int_0^1 (|DW|^2 + a^2 |W|^2)\, dz,\\
I_3 &= \int_0^1 (|D\Theta|^2 + a^2 |\Theta|^2)\, dz, \qquad (20)\\
I_4 &= \int_0^1 |\Theta|^2\, dz,\\
I_5 &= \int_0^1 (|DZ|^2 + a^2 |Z|^2)\, dz,\\
I_6 &= \int_0^1 |Z|^2\, dz,
\end{aligned}$$

which are all positive definite. The real and imaginary parts of Eq. (19) give

$$\left[I_1 + d^2 I_5 + \frac{C_p \alpha \kappa a^2}{\nu(1 - H)} I_3\right] + \sigma_r \left[I_2 + d^2 I_6 + \frac{C_p \alpha \kappa a^2}{\nu(1 - H)} p_1 I_4\right] = 0, \tag{21}$$

and

$$\sigma_i \left[I_2 + \frac{C_p \alpha \kappa a^2}{\nu(H-1)} p_1 I_4 - d^2 I_6 \right] = 0. \tag{22}$$

It is clear from Eq. (21) that σ_r is negative if $H < 1$, meaning thereby that the system is stable.

THEOREM 2. *The modes may be oscillatory or nonoscillatory, in contrast to the case of no rotation where the modes are nonoscillatory, if $H > 1$.*

Proof. Equation (22) yields that $\sigma_i = 0$ or $\sigma_i \neq 0$, which means that the modes may be nonoscillatory or oscillatory. In the absence of rotation [2], Eq. (22) reduces to

$$\sigma_i \left[I_2 + \frac{C_p \alpha \kappa a^2}{\nu(H-1)} p_1 I_4 \right] = 0,$$

and the term in brackets is positive definite if $H > 1$. Thus $\sigma_i = 0$, which means that oscillatory modes are not allowed in the absence of rotation if $H > 1$. The presence of rotation therefore introduces oscillatory modes in thermal instability of compressible fluids.

We now find a sufficient condition which does not allow the overstable modes.

THEOREM 3. *$\nu > \kappa$ is a sufficient condition for the nonexistence of overstability.*

Proof. Assume that $i\sigma_1 = \sigma/\pi^2$, remembering that σ may be complex. Since for overstability, we wish to determine the critical Rayleigh number for the onset of instability via a state of pure oscillations, it will suffice to find conditions for which (17) will admit of solutions with σ_1 real.

Equation (17) becomes

$$R_1 = \frac{(1 + x + i p_1 \sigma_1)}{x(1 + x + i\sigma_1)} \left(\frac{H}{H-1} \right) [(1 + x)(1 + x + i\sigma_1)^2 + T_1]. \tag{23}$$

Equating the real and imaginary parts of Eq. (23), we obtain

$$R_1 x(1 + x) = \left(\frac{H}{H-1} \right) [(1 + x)^4 - \sigma_1^2 (1 + x)^2 (1 + 2p_1) + T_1(1 + x)], \tag{24}$$

and

$$R_1 x = \left(\frac{H}{H-1} \right) [(p_1 + 2)(1 + x)^3 - p_1(1 + x)\sigma_1^2 + p_1 T_1]. \tag{25}$$

Eliminating R_1 between Eqs. (24) and (25), we obtain

$$\sigma_1^2 = - \frac{(1 + p_1)(1 + x)^3 + T_1(p_1 - 1)}{(1 + p_1)(1 + x)^2}. \tag{26}$$

Equation (26) implies that σ_1^2 is negative if $p_1 > 1$. This is impossible since σ_1 is real.

Thus for $p_1 > 1$, overstability cannot occur and the principle of the exchange of stabilities is valid. $p_1 > 1$ or $\nu > \kappa$ is a sufficient condition for the nonexistence of overstability, the violation of which does not necessarily imply the occurrence of overstability.

4. Effect of Magnetic Field

Here we consider an infinite horizontal compressible, viscous, and finitely conducting fluid of depth d in which a uniform temperature gradient $\beta\,(= -dT/dz)$ is maintained. The uniform magnetic field $\mathbf{B}(0, 0, B)$ pervades the system. Then the nondimensional equations governing the motion are [1]

$$(D^2 - a^2)(D^2 - a^2 - \sigma)W + \left(\frac{\mu_e Bd}{4\pi\rho\nu}\right)(D^2 - a^2)DK = \left(\frac{g\alpha d^2}{\nu}\right) a^2\,\Theta, \quad (27)$$

$$(D^2 - a^2 - p_2\sigma)\,K = -(Bd/\eta)\,DW, \quad (28)$$

and the heat conduction equation, (13). Operating Eq. (27) by $(D^2 - a^2 - p_2\sigma)$ $(D^2 - a^2 - p_1\sigma)$ and using (28) and (13), thus eliminating Θ and K, we obtain

$$(D^2 - a^2 - p_1\sigma)\,[(D^2 - a^2)(D^2 - a^2 - \sigma)(D^2 - a^2 - p_2\sigma) - Q(D^2 - a^2)\,D^2]\,W$$
$$= -Ra^2[(H - 1)/H]\,(D^2 - a^2 - p_2\sigma)\,W, \quad (29)$$

where $p_2 = \nu/\eta$, $Q = \mu_e B^2 d^2/4\pi\rho\nu\eta$, the Chandrasekhar number, and we have written $b_z = K(z)\exp(ik_x x + ik_y y + nt)$.

Here again we consider the case of two free boundaries, but the medium adjoining the fluid is a perfect conductor, the boundary conditions appropriate for the problem are [1]

$$W = D^2W = 0, \qquad \Theta = 0, \qquad \text{and} \qquad DK = 0 \qquad \text{at} \qquad z = 0 \text{ and } 1. \quad (30)$$

The proper solution of (29) satisfying (30) is given by Eq. (16).

Substituting (16) in Eq. (29) and letting $a^2 = \pi^2 x$, $R_1 = R/\pi^4$, and $Q_1 = Q/\pi^2$, we obtain the dispersion relation

$$R_1 = \frac{(1 + x)\,[1 + x + p_1(\sigma/\pi^2)]}{x[1 + x + p_2(\sigma/\pi^2)]}\left(\frac{H}{H - 1}\right)$$
$$\times\left[\left(1 + x + \frac{\sigma}{\pi^2}\right)\left(1 + x + p_2\frac{\sigma}{\pi^2}\right) + Q_1\right]. \quad (31)$$

Putting $\sigma = 0$ in Eq. (31) and observing the perfect analogy between the resultant equation and Chandrasekhar's Eq. (165) [1, Chap. IV], we obtain the stabilizing effect of compressibility. The magnetic field has a stabilizing effect.

THEOREM 4. The system is stable for $H < 1$.

Proof. Multiplying Eq. (27) by W^*, the complex conjugate of W, and making use of Eqs. (13) and (28), we obtain

$$I_1 + \sigma I_2 + \frac{\mu_e \eta}{4\pi\rho\nu}(I_3 + p_2\sigma^* I_4) = \frac{a^2\kappa\alpha C_p}{\nu(H-1)}(I_5 + p_1\sigma^* I_6), \tag{32}$$

where

$$\begin{aligned} I_1 &= \int_0^1 (|D^2W|^2 + 2a^2|DW|^2 + a^4|W|^2)\,dz, \\ I_2 &= \int_0^1 (|DW|^2 + a^2|W|^2)\,dz, \\ I_3 &= \int_0^1 (|D^2K|^2 + 2a^2|DK|^2 + a^4|K|^2)\,dz, \\ I_4 &= \int_0^1 (|DK|^2 + a^2|K|^2)\,dz, \\ I_5 &= \int_0^1 (|D\Theta|^2 + a^2|\Theta|^2)\,dz, \\ I_6 &= \int_0^1 |\Theta|^2\,dz. \end{aligned} \tag{33}$$

The real and imaginary parts of Eq. (32) give

$$\left[I_1 + \frac{\mu_e\eta}{4\pi\rho\nu}I_3 + \frac{a^2\kappa\alpha C_p}{\nu(1-H)}I_5\right] + \sigma_r\left[I_2 + \frac{\mu_e\eta}{4\pi\rho\nu}p_2 I_4 + \frac{a^2\kappa\alpha C_p}{\nu(1-H)}p_1 I_6\right] = 0, \tag{34}$$

and

$$\sigma_i\left[I_2 + \frac{a^2\kappa\alpha C_p}{\nu(H-1)}p_1 I_6 - \frac{\mu_e\eta}{4\pi\rho\nu}p_2 I_4\right] = 0. \tag{35}$$

It is evident from Eq. (34) that, if $H < 1$, σ_r is negative, meaning thereby the stability of the system.

THEOREM 5. *The modes may be oscillatory or nonoscillatory, in contrast to the case of no magnetic field in which, for $H > 1$, the modes are nonoscillatory.*

Proof. It is evident from Eq. (35) that σ_r may be zero or nonzero. Thus the modes may be oscillatory or nonoscillatory. The oscillatory modes are introduced due to the presence of magnetic field. This is evident as in the absence of a magnetic field [2], the principle of the exchange of stabilities is valid for $H > 1$ and in the presence of a magnetic field, the overstable modes also come into play.

THEOREM 6. *$\kappa < \eta$ is a sufficient condition for the existence of the principle of the exchange of stabilities.*

Proof. For overstability, we put $\sigma/\pi^2 = i\sigma_1$, where σ_1 is real. Equation (31) becomes

$$R_1 = \frac{(1+x)(1+x+ip_1\sigma_1)}{x(1+x+ip_2\sigma_1)}\left(\frac{H}{H-1}\right)[(1+x+i\sigma_1)(1+x+ip_2\sigma_1)+Q_1]. \tag{36}$$

Equating the real and imaginary parts of Eq. (36), we have

$$R_1 x = \left(\frac{H}{H-1}\right)[(1+x)\{(1+x)^2+Q_1-p_2\sigma_1^2\}-p_1\sigma_1^2(1+x)(1+p_2)], \tag{37}$$

and

$$R_1 x p_2 = \left(\frac{H}{H-1}\right)(1+x)[(1+x)^2(1+p_2)+p_1\{(1+x)^2+Q_1-p_2\sigma_1^2\}]. \tag{38}$$

Eliminating R_1 between Eqs. (37) and (38), we obtain

$$\sigma_1^2 = \frac{(p_2-p_1)Q_1-(1+x)^2(1+p_1)}{p_2^2(1+p_1)}. \tag{39}$$

Equation (39) implies that σ_1^2 is negative if $p_2 < p_1$. This is impossible since σ_1^2 must be positive. Thus $p_2 < p_1$ or $\kappa < \eta$ is a sufficient condition for the nonexistence of overstability, and thus the principle of the exchange of stabilities is valid.

REFERENCES

1. S. CHANDRASEKHAR, "Hydrodynamic and Hydromagnetic Stability," Chaps. II–IV, Oxford Univ. Press, London, 1961.
2. J. SAINI AND S. C. AGRAWAL, *Math. Student* **XL**, No. 2 (1972), 158.
3. E. A. SPIEGEL AND G. VERONIS, *Astrophys. J.* **131** (1960), 442.
4. E. A. SPIEGEL, *Astrophys. J.* **141** (1965), 1068.