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Singular sets for curvature equation of order k

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Abstract

We consider the removability of singular sets for the curvature equations of the form $H_k[u] = \psi$, which is determined by the kth elementary symmetric function, in an n-dimensional domain Ω. We prove that, for $1 \leqslant k \leqslant n-1$ and a compact set K whose $(n-k)$-dimensional Hausdorff measure is zero, any generalized solution to the curvature equation on $\Omega \setminus K$ is always extendable to a generalized solution on the whole domain Ω.

Keywords: Curvature equations; Fully nonlinear equations; Singular set; Generalized solutions

1. Introduction

This paper is a sequel to [27]. We study the removability of singular sets of solutions to the curvature equations of the form

$$H_k[u] = S_k(\kappa_1, \ldots, \kappa_n) = \psi \tag{1.1}$$

in $\Omega \setminus K$, where Ω is a bounded domain in $\mathbb{R}^n$ and K is a compact set contained in Ω. For a function $u \in C^2(\Omega)$, $\kappa = (\kappa_1, \ldots, \kappa_n)$ denotes the principal curvatures of the graph of the function u, namely, the eigenvalues of the matrix

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$$\mathcal{C} = D\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \frac{1}{\sqrt{1+|Du|^2}}\left(I - \frac{Du \otimes Du}{1+|Du|^2}\right)D^2u, \tag{1.2}$$

and S_k, $k = 1, \dots, n$, denotes the kth elementary symmetric function, that is,

$$S_k(\kappa) = \sum \kappa_{i_1} \dots \kappa_{i_k}, \tag{1.3}$$

where the sum is taken over increasing k-tuples, $i_1, \dots, i_k \subset \{1, \dots, n\}$. The mean, scalar and Gauss curvatures correspond respectively to the special cases $k = 1, 2, n$ in (1.3). We call Eq. (1.1) "k-curvature equation."

Here we consider *generalized solutions* to k-curvature equation, which are solutions in a certain weak sense. In the previous paper [26] the author introduced the notion of generalized solutions to

$$H_k[u] = \nu, \tag{1.4}$$

where ν is a nonnegative Borel measure. Generalized solutions form a wider class than classical solutions or viscosity solutions under the convexity assumptions.

In [27], we considered the removability of isolated singularities for solutions to homogeneous k-curvature equation (i.e. (1.1) with $\psi \equiv 0$), both in the viscosity sense and in the generalized sense. In this paper we establish results concerning the removability of a singular set of a generalized solution to (1.4). We state our main theorem.

Theorem 1.1. *Let Ω be a convex domain in $\mathbb{R}^n$ and $K \Subset \Omega$ be a compact set whose $(n-k)$-dimensional Hausdorff measure is zero. Let $1 \leqslant k \leqslant n-1$, $\psi \in L^1(\Omega)$ be a nonnegative function, and u be a continuous function in $\Omega \setminus K$. We assume that for any convex subdomain $\Omega' \subset \Omega \setminus K$, u is a convex function in Ω' and a generalized solution to $H_k[u] = \psi\, dx$ in Ω'. Then u can be defined in the whole domain Ω as a generalized solution to $H_k[u] = \psi\, dx$ in Ω.*

For the case of $k = 1$, which corresponds to the mean curvature equation in (1.1), such removability problems were extensively studied. Bers [2], Nitsche [22], and De Giorgi and Stampacchia [14] proved the removability of isolated singularities for solutions to the equation of minimal surface ($\psi \equiv 0$) or constant mean curvature (ψ is a constant function). Serrin [24,25] studied the same problem for a more general class of quasilinear equations of mean curvature type. He proved that any weak solution u to the mean curvature type equation in $\Omega \setminus K$ can be extended to a weak solution in Ω if the singular set K is a compact set of vanishing $(n-1)$-dimensional Hausdorff measure. For various semilinear and quasilinear equations, there are a number of papers concerning removability results. See [4,5,32] and references therein.

Here we remark that (1.1) is a quasilinear equation for $k = 1$ while it is a *fully nonlinear* equation for $k \geqslant 2$. It is much harder to study the fully nonlinear equations' case. For Monge–Ampère equations' case, there are some results on the removability of isolated singularities (see, for example, [3,16,23]). However, until recently, there were no results for other types of fully nonlinear equations. For solutions to uniformly elliptic equations and Hessian equations, such removability problems were studied by Labutin [18–20]. In this paper we obtained Serrin type removability result for generalized solutions to k-curvature equations.

For the case $k = n$ which corresponds to the Gauss curvature case, one has a solution to (1.1) with nonremovable singularities at a single point. For example,

$$u(x) = \alpha|x|, \quad x \in \Omega = B_1(0) = \{|x| < 1\}, \tag{1.5}$$

where $\alpha > 0$, satisfies Eq. (1.1) with $k = n$, $\psi \equiv 0$ and $K = \{0\}$, in the classical sense as well as in the generalized sense. However, u does not satisfy $H_n[u] = 0$ in $\Omega = B_1(0)$ in the generalized sense (see Example 2.1). Accordingly the case $k = n$ is excluded from Theorem 1.1.

This paper is divided as follows. In the next section, we give a definition of generalized solutions to k-curvature equation with some examples. Then we prove that the notion of generalized solutions is weaker than that of viscosity solutions under the convexity assumptions. Section 3 is devoted to the proof of Theorem 1.1. Finally, in Section 4, we state some remarks and open problems on k-curvature equations.

2. The notion of generalized solutions

In this section we give the definition of generalized solutions to (1.1) which was introduced in [26].

For a large class of elliptic PDEs, it is well known that one can consider a function which is not necessarily differentiable in a usual (classical) sense as a solution to the equation. Many mathematicians have investigated solutions in a generalized sense, such as *weak solutions* for quasilinear equations of divergence type and *distributional solutions* for semilinear equations. For fully nonlinear equations, the theory of *viscosity solutions* provides existence and uniqueness theorem under mild hypotheses (we refer to [10–12, 21]). Weak solutions and distributional solutions have an integral nature, while viscosity solutions do not have. It is difficult to define solutions with an integral nature for fully nonlinear PDEs. However, for some special types of fully nonlinear PDEs, one can introduce an appropriate notion of solutions that have such property, such as *generalized solutions* for Monge–Ampère type equations (see [1,7]) and for Hessian equations (see [9,29–31]). Recently, the author [26] introduced the notion of *generalized solutions* for k-curvature equations which form a wider class than viscosity solutions under the convexity assumptions (we prove this in Proposition 2.3).

Let Ω be an open, convex and bounded subset of $\mathbb{R}^n$ and we look for solutions in the class of convex and (uniformly) Lipschitz functions defined in Ω. For a point $x \in \Omega$, let $\mathrm{Nor}(u; x)$ be the set of downward normal unit vectors to u at $(x, u(x))$. For a nonnegative number ρ and a Borel subset η of Ω, we set

$$Q_\rho(u; \eta) = \{z \in \mathbb{R}^n \mid z = x + \rho v,\ x \in \eta,\ v \in \gamma_u(x)\}, \tag{2.1}$$

where $\gamma_u(x)$ is a subset of $\mathbb{R}^n$ defined by

$$\gamma_u(x) = \{(a_1, \ldots, a_n) \mid (a_1, \ldots, a_n, a_{n+1}) \in \mathrm{Nor}(u; x)\}. \tag{2.2}$$

The following theorem, which is an analogue of the so-called Steiner type formula, plays an important part in the definition of generalized solutions.

Theorem 2.1 [26, Theorem 1.1]. *Let Ω be an open convex bounded set in $\mathbb{R}^n$, and let u be a convex and Lipschitz function defined in Ω. Then the following hold.*

(i) *For every Borel subset η of Ω and for every $\rho \geqslant 0$, the set $Q_\rho(u;\eta)$ is Lebesgue measurable.*
(ii) *There exist $n+1$ nonnegative, finite Borel measures $\sigma_0(u;\cdot),\dots,\sigma_n(u;\cdot)$ such that*

$$\mathcal{L}^n\big(Q_\rho(u;\eta)\big) = \sum_{m=0}^{n} \binom{n}{m} \sigma_m(u;\eta)\rho^m \tag{2.3}$$

for every $\rho \geqslant 0$ and for every Borel subset η of Ω, where $\mathcal{L}^n$ denotes the n-dimensional Lebesgue measure.

Remark 2.1. The measures $\sigma_k(u;\cdot)$ determined by u are characterized by the following two properties.

(i) If $u \in C^2(\Omega)$, then for every Borel subset η of Ω,

$$\binom{n}{k}\sigma_k(u;\eta) = \int_\eta H_k[u](x)\,dx. \tag{2.4}$$

(The proof is given in [26, Proposition 2.1].)
(ii) If u_i converges uniformly to u on every compact subset of Ω, then

$$\sigma_k(u_i;\cdot) \rightharpoonup \sigma_k(u;\cdot) \quad \text{(weakly)}. \tag{2.5}$$

Therefore we can say that for $k = 1,\dots,n$, the measure $\binom{n}{k}\sigma_k(u;\cdot)$ generalizes the integral of the function $H_k[u]$.

Now we state the definition of a generalized solution to k-curvature equation.

Definition 2.2. Let Ω be an open convex bounded set in $\mathbb{R}^n$ and let ν be a nonnegative, finite Borel measure in Ω. A convex and Lipschitz function $u \in C^{0,1}(\Omega)$ is said to be a *generalized solution* to

$$H_k[u] = \nu \quad \text{in } \Omega, \tag{2.6}$$

if it holds that

$$\binom{n}{k}\sigma_k(u;\eta) = \nu(\eta) \tag{2.7}$$

for every Borel subset η of Ω.

It is easy to see that $C^2(\Omega)$ generalized solution is also a classical solution. Here we note that one can also define the notion of generalized solutions stated above in the case where Ω is not necessarily convex. Indeed, we shall say that u is a generalized solution to (2.6) if for any point $x \in \Omega$ and for any ball $B = B_R(x) \subset \Omega$, (2.7) holds for every Borel subset η of $B_R(x)$.

Here are some examples.

Example 2.1. Let α be a positive constant.

(1) $u_1(x) = \alpha|x|$, which is a function we have already seen in (1.5), is a generalized solution to

$$H_n[u_1] = \left(\frac{\alpha}{\sqrt{1+\alpha^2}}\right)^n \omega_n \delta_0 \quad \text{in } \mathbb{R}^n, \tag{2.8}$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$, and δ_0 is the Dirac measure at 0.

(2) $u_2(x) = \alpha\sqrt{x_1^2 + \cdots + x_k^2}$, where $x = (x_1, \ldots, x_n)$, is a generalized solution to

$$H_k[u_2] = \left(\frac{\alpha}{\sqrt{1+\alpha^2}}\right)^k \omega_k \mathcal{L}^{n-k} \lfloor T \quad \text{in } \mathbb{R}^n, \tag{2.9}$$

where ω_k denotes the k-dimensional measure of the unit ball in $\mathbb{R}^k$ and $T = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 = \cdots = x_k = 0\}$. We note that Hausdorff dimension of T is $n-k$. Hence, as far as k-curvature equation is concerned, we cannot expect that the removability theorem holds for the set with nonzero $(n-k)$-dimensional Hausdorff measure.

There is a notion of generalized solutions to the Gauss curvature equation which corresponds to the case of $k = n$ in (2.6), since they are in a class of Monge–Ampère type. As far as the Gauss curvature equation, namely,

$$\frac{\det(D^2 u)}{(1+|Du|^2)^{(n+2)/2}} = \nu \tag{2.10}$$

is concerned, the definition of generalized solutions for Monge–Ampère type equations coincides with the one introduced in Definition 2.2. The proof is given in [26, Theorem 3.3].

In the last part of this section, we prove that the notion of generalized solutions is weaker than that of viscosity solutions in some sense.

Proposition 2.3. *Let $1 \leqslant k \leqslant n$ and Ω be a domain in $\mathbb{R}^n$. Let ψ be a positive function with $\psi^{1/k} \in C^{0,1}(\bar{\Omega})$, and u be a convex function in $\bar{\Omega}$. If u is a viscosity solution to $H_k[u] = \psi$ in Ω, then u is a generalized solution to $H_k[u] = \nu$ in Ω, where $\nu = \psi(x)\,dx$.*

Proof. Let x_0 be any point in Ω. We wish to show that u is a generalized solution to $H_k[u] = \nu\,dx$ in some ball centered at x_0. We fix a sufficiently small constant $r > 0$ such that

$$\|\psi\|_{L^{n/k}(B_r(x_0))} < \frac{1}{2}\binom{n}{k}\omega_n^{k/n}, \tag{2.11}$$

which assures C^0-a priori bound for a solution to $H_k[u] = \psi$ (see [28]). We may assume that $\Omega = B_r(x_0)$.

First we extend the function u to a convex function defined in $\mathbb{R}^n$, which is proved in [8]. Let φ be a nonnegative function in $C_0^\infty(\mathbb{R}^n)$ vanishing outside $B_1(0)$ and satisfying $\int_{B_1(0)} \varphi\,dx = 1$. We define

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right), \tag{2.12}$$

and set $u_i = \varphi_{1/i} * u$, the regularization of u. It turns out that u_i converges uniformly to u in Ω as $i \to \infty$.

Next, let $\{\Omega_i\}_{i=1}^{\infty}$ be a sequence of convex domains such that $\Omega_1 \Subset \Omega_2 \Subset \cdots$ and that $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$. In the case of $1 \leqslant k \leqslant n-1$, we take $\{\psi_i\}_{i=1}^{\infty} \subset C^{\infty}(\bar{\Omega})$ which satisfies that

$$\psi_i \to \psi \text{ in } L^1(\Omega) \text{ and uniformly in } C^0(\bar{\Omega}_j) \quad \text{for every } j \in \mathbb{N}, \tag{2.13}$$

$$\text{for every } j \in \mathbb{N}, \quad \sup_{i=1,2,\dots} |D\psi_i| \text{ is bounded in } \Omega_j, \tag{2.14}$$

$$S_k(\kappa_1', \dots, \kappa_{n-1}', 0) \geqslant \psi_i \quad \text{on } \partial\Omega, \tag{2.15}$$

where $\kappa' = (\kappa_1', \dots, \kappa_{n-1}')$ denotes the principal curvatures of the boundary $\partial\Omega$ and that

$$\psi_i > 0 \quad \text{in } \bar{\Omega}. \tag{2.16}$$

For $k = n$, the condition (2.16) is replaced by

$$\psi_i > 0 \quad \text{in } \Omega \quad \text{and} \quad \psi_i = 0 \quad \text{on } \partial\Omega. \tag{2.17}$$

One can get $\{\psi_i\}_{i=1}^{\infty}$ by using the regularizations of ψ.

Now we consider the following Dirichlet problem:

$$\begin{cases} H_k[v_i] = \psi_i & \text{in } \Omega, \\ v_i = u_i & \text{on } \partial\Omega. \end{cases} \tag{2.18}$$

By virtue of the results in [15,28], there exists a unique classical solution $v_i \in C^{\infty}(\bar{\Omega})$ to (2.18), for sufficiently large i. From the maximum principle [28], the sequence $\{v_i\}$ is uniformly bounded. We also see that for any open set $\Omega' \Subset \Omega$, the interior gradient bound by Korevaar [17] implies that $\{v_i\}$ is equicontinuous in Ω'. Therefore, using the diagonal argument, we deduce from Ascoli–Arzelà's theorem that there exists a subsequence of $\{v_i\}$ (we relabel it as $\{v_i\}$ again) converging uniformly to some function $v \in C^0(\Omega)$ on every compact subset of Ω. By the stability property of viscosity solutions, it follows that v is a viscosity solution to

$$\begin{cases} H_k[v] = \psi & \text{in } \Omega, \\ v = u & \text{on } \partial\Omega. \end{cases} \tag{2.19}$$

The uniqueness of solutions to the Dirichlet problem (2.19) implies that $u \equiv v$ in Ω.

We set

$$\mu_i(\eta) = \int_{\eta} \psi_i(x)\, dx \tag{2.20}$$

for Borel subset η of Ω. From (2.13), we obtain

$$\mu_i \to \nu \quad \text{(strongly)}. \tag{2.21}$$

On the other hand, from the uniform convergence of $\{v_i\}$ on every compact subset of Ω and Remark 2.1(ii) (see also [26, Proposition 3.2]), we see that

$$\mu_i \rightharpoonup \binom{n}{k} \sigma_k(u; \cdot) \quad \text{(weakly)}. \tag{2.22}$$

Then, the uniqueness of the weak limit yields

$$\binom{n}{k}\sigma_k(u;\eta) = \int\limits_{\eta} \psi(x)\,dx \tag{2.23}$$

for every Borel subset η of Ω. Hence the proposition is proved. □

Remark 2.2. It is not known whether a viscosity solution to the Dirichlet problem

$$\begin{cases} H_k[u] = \psi & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{2.24}$$

where $\varphi \in C^0(\partial\Omega)$, is unique or not for general nonnegative ψ. We note here also that Cranny [13] proved the uniqueness of a viscosity solution to (2.24) for "highly degenerate" case, that is, $\psi \equiv 0$.

3. Proof of Theorem 1.1

Before giving a proof of Theorem 1.1, we introduce some notations. We write $x = (x_1, \dots, x_{n-1}, x_n) = (x', x_n)$. $B_r^{n-1}(x') \subset \mathbb{R}^{n-1}$ denotes the $(n-1)$-dimensional open ball of radius r centered at x'.

Proof. The proof is split into two steps.

Step 1 (Extension of u to a convex function in Ω). Here we prove that u can be extended to a convex function in the whole domain Ω. The idea of the proof is adapted from that of Yan [33].

Let y, z be any two distinct points in $\Omega \setminus K$. Without loss of generality we may assume that y is the origin and $z = (0, \dots, 0, 1)$. First we prove the following lemma.

Lemma 3.1. *There exist sequences* $\{y_j\}_{j=1}^{\infty}, \{z_j\}_{j=1}^{\infty} \subset \Omega \setminus K$ *such that* $y_j \to y, z_j \to z$ *as* $j \to \infty$ *and*

$$[y_j, z_j] = \bigl\{ty_j + (1-t)z_j \mid 0 \leqslant t \leqslant 1\bigr\} \subset \Omega \setminus K. \tag{3.1}$$

Proof. To the contrary, we suppose that there exists $\delta > 0$ such that for every $\tilde{y} \in B_\delta(y)$ and for every $\tilde{z} \in B_\delta(z)$, there exists $\tilde{t} \in (0,1)$ such that $\tilde{t}\tilde{y} + (1-\tilde{t})\tilde{z} \in K$. Here we note that $\tilde{t}\tilde{y} + (1-\tilde{t})\tilde{z}$ must be in Ω since Ω is assumed to be convex. In particular, if we set $\tilde{y} = (a_1, \dots, a_{n-1}, 0), \tilde{z} = (a_1, \dots, a_{n-1}, 1)$ with $a' = (a_1, \dots, a_{n-1}) \in B_\delta^{n-1}(0)$, one sees that there exists $t_{a'} \in (0,1)$ such that $(a', t_{a'}) \in K$. We define the set V by

$$V = \bigl\{(a', t_{a'}) \mid a' \in B_\delta^{n-1}(0)\bigr\}. \tag{3.2}$$

Clearly $V \subset K$.

The assumption on K implies that the $(n-1)$-dimensional Hausdorff measure of K is zero. Hence there exist countable balls $\{B_{r_i}(x_i)\}_{i=1}^{\infty}$ such that

$$K \subset \bigcup_{i=1}^{\infty} B_{r_i}(x_i) \quad \text{and} \quad \sum_{i=1}^{\infty} r_i^{n-1} < \delta^{n-1}. \tag{3.3}$$

It follows that V is also covered by $\{B_{r_i}(x_i)\}_{i=1}^{\infty}$. By projecting both V and $\{B_{r_i}(x_i)\}_{i=1}^{\infty}$ onto $\mathbb{R}^{n-1}\times\{0\}$, we have that

$$B_{\delta}^{n-1}(0)\subset\bigcup_{i=1}^{\infty}B_{r_i}^{n-1}(x_i'). \tag{3.4}$$

Taking $(n-1)$-dimensional measure of each side of (3.4), we obtain that

$$\omega_{n-1}\delta^{n-1}\leqslant\sum_{i=1}^{\infty}\omega_{n-1}r_i^{n-1}<\omega_{n-1}\delta^{n-1}, \tag{3.5}$$

which is a contradiction. Lemma 3.1 is thus proved. □

Let $\lambda\in[0,1]$ and set $x=\lambda y+(1-\lambda)z\in\Omega\setminus K$. From the above lemma and the local convexity of u, it follows that

$$u(x)\leqslant\lambda u(y_j)+(1-\lambda)u(z_j) \tag{3.6}$$

for all $j\in\mathbb{N}$, where $\{y_j\}_{j=1}^{\infty}$ and $\{z_j\}_{j=1}^{\infty}$ are sequences which we obtained in Lemma 3.1. Since u is locally convex in $\Omega\setminus K$, u is continuous in $\Omega\setminus K$. Taking $j\to\infty$,

$$u(x)\leqslant\lambda u(y)+(1-\lambda)u(z). \tag{3.7}$$

Next let U be the supergraph of u, that is,

$$U=\big\{(x,w)\mid x\in\Omega\setminus K,\ w\geqslant u(x)\big\}\subset\mathbb{R}^{n+1}, \tag{3.8}$$

and for every set $X\subset\mathbb{R}^{n+1}$, $\operatorname{co}X$ denotes the convex hull of X. Now we define the function $\tilde{u}$ by

$$\tilde{u}(x)=\inf\big\{w\in\mathbb{R}\mid(x,w)\in\operatorname{co}U\big\}. \tag{3.9}$$

One can easily show that the convex hull of $\Omega\setminus K$ (in $\mathbb{R}^n$) is Ω, so that $\tilde{u}$ is defined in the whole Ω. Moreover, $\tilde{u}$ is a convex function due to the convexity of $\operatorname{co}U$. Finally, we show that $\tilde{u}$ is an extension of u defined in $\Omega\setminus K$. To see this, fix a point $x\in\Omega\setminus K$. The definition of $\tilde{u}$ follows that $\tilde{u}(x)\leqslant u(x)$. Taking the infimum of the right-hand side of (3.7) over all $y,z\in\Omega\setminus K$, we have that $u(x)\leqslant\tilde{u}(x)$. Consequently, it holds that $u\equiv\tilde{u}$ in $\Omega\setminus K$. $\tilde{u}$ is the desired function.

Step 2 (Removability of the singular set K). We denote the extended function constructed in Step 1 by the same symbol u. Theorem 2.1 implies that there exists a nonnegative Borel measure ν whose support is contained in K such that

$$H_k[u]=\psi\,dx+\nu\quad\text{in }\Omega \tag{3.10}$$

in the generalized sense. We fix arbitrary $\varepsilon>0$. By the assumption we can cover K by countable open balls $\{B_{r_i}(x_i)\}_{i=1}^{\infty}$ such that

$$\sum_{i=1}^{\infty}r_i^{n-k}<\varepsilon. \tag{3.11}$$

For any $\rho\geqslant 0$,

$$\begin{aligned}\omega_n(r_i+\rho)^n &\geqslant \mathcal{L}^n\big(Q_\rho\big(u; B_{r_i}(x_i)\big)\big) = \sum_{m=0}^{n}\binom{n}{m}\sigma_m\big(u; B_{r_i}(x_i)\big)\rho^m \\ &\geqslant \binom{n}{k}\sigma_k\big(u; B_{r_i}(x_i)\big)\rho^k = \Bigg(\int\limits_{B_{r_i}(x_i)}\psi\,dx + \nu\big(B_{r_i}(x_i)\big)\Bigg)\rho^k \\ &\geqslant \nu\big(B_{r_i}(x_i)\big)\rho^k. \end{aligned} \tag{3.12}$$

The first inequality in (3.12) is due to the fact that $Q_\rho(u; B_{r_i}(x_i)) \subset B_{r_i+\rho}(x_i)$, since taking any $z \in Q_\rho(u; B_{r_i}(x_i))$ we obtain

$$|z - x_i| = |y + \rho v - x_i| \leqslant |y - x_i| + \rho|v| < r_i + \rho, \tag{3.13}$$

for some $y \in B_{r_i}(x_i)$, $v \in \gamma_u(y)$. Inserting $\rho = r_i$ in (3.12), we obtain that

$$\omega_n 2^n r_i^n \geqslant \nu\big(B_{r_i}(x_i)\big) r_i^k. \tag{3.14}$$

Consequently, it holds that

$$\nu\big(B_{r_i}(x_i)\big) \leqslant \omega_n 2^n r_i^{n-k}. \tag{3.15}$$

Now taking the summation for $i \geqslant 1$, we have that

$$\nu(K) \leqslant \nu\Bigg(\bigcup_{i=1}^{\infty} B_{r_i}(x_i)\Bigg) \leqslant \sum_{i=1}^{\infty}\nu\big(B_{r_i}(x_i)\big) \leqslant \sum_{i=1}^{\infty}\omega_n 2^n r_i^{n-k} < \omega_n 2^n \varepsilon. \tag{3.16}$$

Since we can take $\varepsilon > 0$ arbitrarily, we see that $\nu(K) = 0$. Therefore, $\nu \equiv 0$. We conclude that K is a removable set. □

4. Final remarks

There are a number of results concerning the Dirichlet problem for k-curvature equation (1.1) in the literature, for general $k = 1, 2, \ldots, n$. Such problems were investigated by Caffarelli et al. [6] and Ivochkina [15] in the classical sense. Trudinger [28] established the existence and uniqueness of Lipschitz solutions to the Dirichlet problem in the viscosity sense, under natural geometric restrictions and under relatively weak regularity hypotheses on ψ, for instance, $\psi^{1/k} \in C^{0,1}(\bar{\Omega})$.

Therefore, it seems an interesting problem to study the solvability of the Dirichlet problem

$$\begin{cases} H_k[u] = \nu & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

in the generalized sense, where ν is a nonnegative Borel measure. For $k = n$ (the Gauss curvature case) which is an equation of Monge–Ampère type, the existence and uniqueness of generalized solutions to the Dirichlet problem (4.1) in a bounded convex domain have been studied. We refer the reader to [1], for example. We would like to seek appropriate conditions on ν which guarantee the solvability of generalized solutions to (4.1) for the case of $1 \leqslant k \leqslant n-1$. However, we obtain few results about that so far. Theorem 1.1

in this paper implies that, for example, there exist no generalized solutions to (4.1) when $1 \leqslant k \leqslant n-1$ and $\nu = C\delta_{x_0}$, where C is a positive constant and δ_{x_0} is a Dirac delta measure at $x_0 \in \Omega$. In fact, if we write $\nu = \psi\, dx + \mu$, where ψ is a nonnegative $L^1(\Omega)$ function and μ is the singular part of ν with respect to the Lebesgue measure, then either of the two alternatives must hold:

(i) the $(n-k)$-dimensional Hausdorff measure of the support of μ is nonzero; or
(ii) $\mu = 0$.

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