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New expansions of numerical eigenvalues by Wilson's element

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ABSTRACT

The paper explores new expansions of eigenvalues for $-\Delta u = \lambda \rho u$ in S with Dirichlet boundary conditions by Wilson's element. The expansions indicate that Wilson's element provides lower bounds of the eigenvalues. By the extrapolation or the splitting extrapolation, the $O(h^4)$ convergence rate can be obtained, where h is the maximal boundary length of uniform rectangles. Numerical experiments are carried to verify the theoretical analysis made. It is worth pointing out that these results are new, compared with the recent book, Lin and Lin [Q. Lin, J. Lin, Finite Element Methods; Accuracy and Improvement, Science Press, Beijing, 2006].

1. Introduction

In this paper, we consider the eigenvalue problem

$$-\Delta u = \lambda \rho u \quad \text{in } S, \tag{1.1}$$

$$u = 0 \quad \text{in } \partial S, \tag{1.2}$$

where $S = [0, 1]^2$, and the function $\rho = \rho(x, y) > 0$ and $\rho \in C^2(S)$. Then Eqs. (1.1) and (1.2) can be written in a weak form: To seek $(\lambda, u) \in R \times H_0^1(S)$ with $u \neq 0$ such that

$$a(u, v) = \lambda(u, v), \quad \forall v \in H_0^1(S), \tag{1.3}$$

where $H_0^1(S) = \{v | v \in H^1(S), v|_{\partial S} = 0\}$, and

$$a(u, v) = \iint_S \nabla u \nabla v, \tag{1.4}$$

$$(u, v) = \iint_S \rho u v. \tag{1.5}$$

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For Wilson's element [24,8], the piecewise interpolation functions $u_I \in W = P_2 = \text{span}\{1, x, y, xy, x^2, y^2\}$ are formulated by

$$u(Z_\ell) = u_I(Z_\ell), \quad \ell = 1, 2, 3, 4, \tag{1.6}$$

$$u_{xx}(O) = (u_I)_{xx}(O), \qquad u_{yy}(O) = (u_I)_{yy}(O), \tag{1.7}$$

where Z_i are the four corners of $\square_{ij}$, O is the center of $\square_{ij}$, and $\square_{ij} = \{(x, y)|x_i - h_i \le x \le x_i + h_i, y_j - k_j \le y \le y_j + k_j\}$. Choose the affine transformation:

$$\xi = \frac{x - x_i}{h_i}, \qquad \eta = \frac{y - y_j}{k_j}.$$

The admissible functions on $\square_{ij}$ can be expressed as

$$v(x, y) = \sum_{t=1}^{4} v_t \phi_t(\xi, \eta) + h_i^2 u_{xx}(O)\phi_5(\xi, \eta) + k_j^2 u_{yy}(O)\phi_6(\xi, \eta),$$

where the nodal points 1, 2, 3, 4 denote (i, j), $(i + 1, j)$, $(i, j + 1)$, $(i + 1, j + 1)$, respectively, and the six basis functions on $[-1, 1]^2$ are given explicitly by

$$\phi_1(\xi, \eta) = \frac{1}{4}(1 - \xi)(1 - \eta), \qquad \phi_2(\xi, \eta) = \frac{1}{4}(1 + \xi)(1 - \eta),$$

$$\phi_3(\xi, \eta) = \frac{1}{4}(1 - \xi)(1 + \eta), \qquad \phi_4(\xi, \eta) = \frac{1}{4}(1 + \xi)(1 + \eta),$$

$$\phi_5(\xi, \eta) = \frac{1}{8}(-1 + \xi)(1 + \xi), \qquad \phi_6(\xi, \eta) = \frac{1}{8}(-1 + \eta)(1 + \eta).$$

Let $S = \cup_{ij} \square_{ij}$, where $\square_{ij}$ are quasi-uniform. Denote by $V_h^0 \subset L^2(S)$ the finite-dimensional collection of the admissible functions defined in Wilson's element. The nonconforming elements, such as Wilson's element, are used to seek $(\lambda_h, u_h) \in R \times V_h^0$[1] such that

$$a_h(u_h, v) = \lambda_h(u_h, v), \quad \forall v \in V_h^0, \tag{1.8}$$

where

$$a_h(u, v) = \sum_{ij} \iint_{\square_{ij}} \nabla u \, \nabla \, v. \tag{1.9}$$

In this paper, we explore the expansions of the eigenvalues λ_h by Wilson's element. When $\square_{ij}$ are uniform rectangles with the boundary lengths h and k, we obtain the following formula,

$$\lambda_h - \lambda = -\frac{h^2 + k^2}{3} \iint_S u_{xx} u_{yy} - \frac{h^2}{3} \iint_S u_{xx}(u_h)_{yy} - \frac{k^2}{3} \iint_S u_{yy}(u_h)_{xx} + O(h^4), \tag{1.10}$$

where $k = O(h)$. The detailed proof for the expansions in (1.10) by Wilson's element is deferred to Section 3. For uniform $\square_{ij}$, Wilson's element provides a lower estimation on eigenvalues, whose proof is also deferred to Section 4. By the extrapolation or the splitting extrapolation with $h \ne k$, we may reach the $O(h^4)$ convergence rate, which is validated by our numerical experiments in Section 5.

Let us mention the references related to this paper. Numerical eigenvalues are discussed in [1–3,6,11,19–22,25–27]. The Wilson's element is studied in [7,18,15], and the extrapolation for eigenvalues are explored in [4,13,16,17]. Here let us mention other works for the expansions of numerical eigenvalues for $-\Delta u = \lambda \rho u$ in S with Dirichlet boundary conditions. We report the new expansions for the conforming bilinear elements Q_1 in [10], and for the nonconforming elements, such as the rotated bilinear elements Q_1^{rot} and the extended rotated bilinear elements EQ_1^{rot}, in Lin, Huang and Li [14]. More results of this subject are given in [16].

Asymptotic lower bounds for eigenvalues have been obtained by the finite difference method (FDM) in [9,23]. In [9], for a convex S, the numerical eigenvalues by the standard five-node finite difference equations have lower bounds, and upper and lower bounds of numerical eigenvalues by FDM are also discussed in [23]. Since the FDM can be regarded as a special kind of FEM involving different integration rules in [12], the variational crimes, the terminology used in [22] for FEM with nonconforming elements and numerical integration, may produce the lower bounds of approximate eigenvalues. Based on the error expansions of numerical eigenvalues, the Wilson's elements yield the lower bounds, the same conclusion made for $\rho = 1$ in the recent paper of Zhang, Yang and Zhen [28].

[1] Here V_h^0 is not a subset of $H_0^1(S)$.

2. Basic theorems

We rewrite (1.3) as:

$$a(u, v) = (f, v), \quad \forall v \in H_0^1(S), \tag{2.1}$$

where $f = \lambda u$. Define the finite element projection R_h by

$$a_h(R_h u, v) = (f, v), \quad \forall v \in V_h^0. \tag{2.2}$$

For simplicity, we assume the simple eigenvalues, and consider only a few leading eigenvalues

$$\lambda_1 < \lambda_2 \le \cdots \le \lambda_k, \tag{2.3}$$

where k is a small integer. Note that the minimal eigenvalue $\lambda_1 = \lambda_{\min}$ is of great interest in practical application.

For the above elements, we cite the known results in [26,27] as a lemma.

Lemma 2.1. *For the quasi-uniform $\square_{ij}$ with the maximal boundary length h, there exist the following bound for leading eigenvalues λ and their corresponding eigenfunctions u,*

$$|\lambda - \lambda_h| + \|u - u_h\|_{0,S} + \|u - R_h u\|_{0,S} \le Ch^2, \tag{2.4}$$

where C is a constant independent of h, and (λ_h, u_h) are the FEM solutions by Wilson's element.

Below we give a new theorem, whose proof is given in [14,15].

Theorem 2.1 (*Nonconforming*). *Let $\square_{ij}$ be quasi-uniform with the maximal boundary length h. For the nonconforming elements, there exists the error formula,*

$$\lambda_h - \lambda = \lambda(u - u_I, u_h) - a_h(u - u_I, u_h) + a_h(u - R_h u, u_h) + O(h^4), \tag{2.5}$$

where u and u_I are the true solution (i.e., eigenfunction) and the FEM interpolation of u, respectively, and u_h and $R_h u$ are the FEM solution of (1.8) and the FEM projection in (2.2), respectively.

In Theorem 2.1, in order to derive the errors $\lambda_h - \lambda$, we need to evaluate the following interpolation errors:

$$(u - u_I, v), a_h(u - u_I, v), \quad \forall v \in V_h^0, \tag{2.6}$$

and the projection error

$$a_h(u - R_h u, v), \quad \forall v \in V_h^0. \tag{2.7}$$

Note that the projection error (2.7) is null for the conforming elements and that the estimation of (2.6) is similar to that for Poisson's equation. Hence the key analysis of the nonconforming elements is to derive the expansions of (2.7). In error estimates, we often use the Bramble–Hilbert lemma [5]: Denote $B(u)$ a bounded linear function from $H^k(S)$ to R. If for all polynomials P_k of degree k, $B(P_k) = 0$, then there exists a constant C independent of u such that

$$|B(u)| \le C|u|_{k+1,S}. \tag{2.8}$$

In this paper, we need more expansions of higher terms of degree $k+1$. We solicit the generalized Bramble–Hilbert Lemma. Denote

$$B(u) = \sum_{|\alpha|=k+1} \frac{B(x^\alpha)}{\alpha!|S|} \iint_S D^\alpha u + H(u), \tag{2.9}$$

where $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2}$, $\alpha_1 + \alpha_2 = \alpha$, and $\alpha! = \alpha_1!\alpha_2!$. In (2.9) $H(u)$ is also a bounded linear function from $H^{k+1}(S)$ to R. We write the following lemma, whose proof is given in [15].

Lemma 2.2 (*Generalized Bramble–Hilbert Lemma*). *Let $u \in H^{k+2}(S)$ and $B(P_k) = 0$. Suppose that $H(P_{k+1}) = 0$ in (2.9). There exists a bound,*

$$|H(u)| \le C|u|_{k+2,S}, \tag{2.10}$$

where C is a constant independent of u.

By using the techniques in this section, the expansions of numerical eigenvalues can be derived for the bilinear element (denoted by Q_1) in [10,15], the rotated Q_1 element (denoted by Q_1^{rot}) and the extension of rotated Q_1 element (denoted by EQ^{rote}) in [14,15], and Wilson's element in this paper. The Q_1 is conforming, but the Q^{rot}, the EQ^{rot} and Wilson's element are nonconforming. However, since the Wilson element is a benchmark of nonconforming elements, its analysis can be found in many references, such as [8,7,15,17,18,22,24,28]. It is worthy to derive the new expansions of its numerical solutions. More importantly, Theorems 3.1 and 4.1 given later are the new developments of [15], and the proof of the lower bound (4.2) in Theorem 4.1 is more intriguing than that in [10,14], since it is completed via the errors of eigenvalues in Section 4.1.

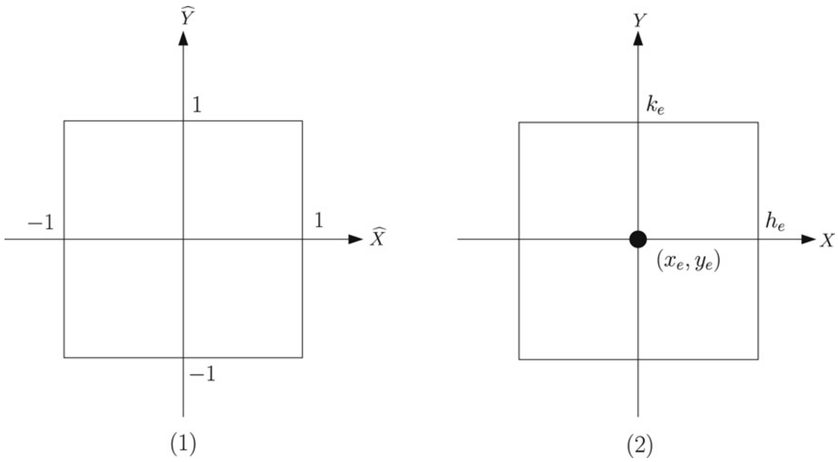

Fig. 1. (1) $\widehat{e} = [-1, 1] \times [-1, 1]$. (2) $e = \square_{ij} = [x_i - h_i, x_i + h_i] \times [y_j - k_j, y_j + k_j]$.

3. Wilson's element

Denote $\widehat{e} = [-1, 1] \times [-1, 1]$ and $e = \square_{ij} = [x_e - h_e, x_e + h_e] \times [y_e - k_e, y_e + k_e]$ (see Fig. 1). Next, we give four lemmas whose proof is deferred to Sections 3.1–3.3.

Lemma 3.1. *For $v \in W(e)$*

$$\begin{aligned}\iint_e (u - u_I)_x v_x &= -\frac{k_e^2}{3} \iint_e u_{xyy} v_x + \frac{h_e^4}{30} \iint_e u_{xxxx} v_{xx} \\ &\quad + \frac{2h_e^2 k_e^2}{9} \iint_e u_{xxyy} v_{xx} + \frac{14k_e^4}{45} \iint_e u_{xyyy} v_{xy} + O(h^5)|u|_{5,e}|v|_{2,e},\end{aligned} \tag{3.1}$$

where $|v|_{m,h} = \sqrt{\sum_e |v|_{m,e}^2}$ $(m = 1, 2)$, and $e = \square_{ij} = [x_e - h_e, x_e + h_e] \times [y_e - k_e, y_e + k_e]$.

Lemma 3.2. *For $v \in W(e)$*

$$\iint_S (u - u_I) v = -\frac{1}{45} \sum_e \iint_e (h_e^4 u_{xxx} v_x + k_e^4 u_{yyy} v_y) - \frac{1}{9} \sum_e \iint_e h_e^2 k_e^2 (u_{xyy} v_x + u_{xxy} v_y) + O(h^5)\|u\|_4 \|v\|_1. \tag{3.2}$$

Lemma 3.3. *For $v = u_h \in W(e)$*

$$a_h(u - R_h u, u_h) = -\frac{1}{3} \sum_e \iint_e \left(k_e^2 u_{xx} (u_h)_{xx} + h_e^2 u_{yy} (u_h)_{xx}\right) + O(h^4)|u|_4 |u_h|_{2,h}. \tag{3.3}$$

Lemma 3.4. *For $v = u_h \in W(e)$*

$$\sum_e \left(\int_{\ell_1} - \int_{\ell_3}\right) u_x u_h \, \mathrm{d}y = -\frac{k_e^2}{3} \iint_e u_{xx} (u_h)_{yy} + O(h^4)|u|_4 |u_h|_{2,h}, \tag{3.4}$$

where ℓ_i are the edges of ∂e shown in Fig. 2.

Theorem 3.1. *For Wilson's element, there exists the eigenvalue error*

$$\lambda_h - \lambda = -\frac{1}{3} \sum_e (h_e^2 + k_e^2) \iint_e u_{xx} u_{yy} - \frac{1}{3} \sum_e \iint_e \left(k_e^2 u_{xx} (u_h)_{yy} + h_e^2 u_{yy} (u_h)_{xx}\right) + O(h^4). \tag{3.5}$$

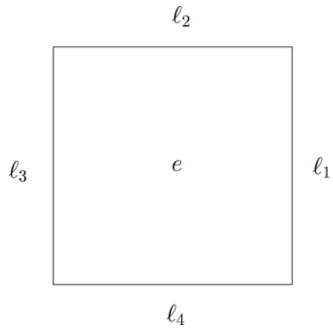

Fig. 2. The rectangle.

Proof. There exists the bound

$$|u_h|_{2,h} \le |u_h - u_I|_{2,h} + |u_I - u|_{2,h} + |u|_2 \le Ch^{-1}|u_h - u_I|_{1,h} + C|u|_2 \le C|u|_2. \tag{3.6}$$

From Lemma 3.1 and (3.6),

$$a_h(u - u_I, u_h) = \frac{1}{3}\sum_e \iint_e \left(k_e^2 u_{xyy}(u_h)_x + h_e^2 u_{xxy}(u_h)_y\right) + O(h^4). \tag{3.7}$$

By using integration by parts we have

$$\begin{aligned} a_h(u - u_I, u_h) &= \frac{1}{3}\sum_e \iint_e \left(k_e^2 u_{xxyy} u_h + h_e^2 u_{xxyy} u_h\right) \\ &\quad - \frac{1}{3}\sum_e \left(\int_{\ell_1} - \int_{\ell_3}\right) k_e^2 u_{xyy} u_h - \frac{1}{3}\sum_e \left(\int_{\ell_2} - \int_{\ell_4}\right) h_e^2 u_{xxy} u_h + O(h^4). \end{aligned} \tag{3.8}$$

Based on Lemma 3.4, there exist the bounds,

$$\sum_e \left(\int_{\ell_1} - \int_{\ell_3}\right) k_e^2 u_{xyy} u_h = -\frac{k_e^4}{3}\iint_e u_{xxyy}(u_h)_{yy} + O(h^6) = O(h^4), \tag{3.9}$$

$$\sum_e \left(\int_{\ell_2} - \int_{\ell_4}\right) h_e^2 u_{xxy} u_h = O(h^4). \tag{3.10}$$

Hence, we have from (3.8) and Lemma 2.1

$$\begin{aligned} a_h(u - u_I, u_h) &= \frac{1}{3}\sum_e \iint_e \left(k_e^2 u_{xxyy} u_h + h_e^2 u_{xxyy} u_h\right) + O(h^4) \\ &= \frac{1}{3}\sum_e \iint_e \left(k_e^2 u_{xxyy} u + h_e^2 u_{xxyy} u\right) + O(h^4). \end{aligned} \tag{3.11}$$

Since $u = 0$ on ∂S, from integration by parts again, Eq. (3.11) leads to

$$a_h(u - u_I, u_h) = \frac{1}{3}\sum_e (k_e^2 + h_e^2) \iint_e u_{xx} u_{yy} + O(h^4). \tag{3.12}$$

Moreover, based on Theorem 2.1, we have from Lemmas 3.2 and 3.3 and (3.12)

$$\begin{aligned} \lambda_h - \lambda &= a_h(u - R_h u, u_h) + \lambda(u - u_I, u_h) - a_h(u - u_I, u_h) + O(h^4) \\ &= -\frac{1}{3}\sum_e (h_e^2 + k_e^2) \iint_e u_{xx} u_{yy} - \frac{1}{3}\sum_e \iint_e \left(k_e^2 u_{xx}(u_h)_{yy} + h_e^2 u_{yy}(u_h)_{xx}\right) + O(h^4). \end{aligned} \tag{3.13}$$

This is the desired equation (3.5), and completes the proof of Theorem 3.1. ∎

Table 1
The integration $\iint_{\widehat{e}}(u-u_I)_x v_x$ for $u \in P_3 \setminus P_2$, $v \in \text{span}\{1, x, y, xy, x^2, y^2\}$ and $v_x \in \text{span}\{0, 1, 0, y, 2x, 0\}$, where $\widehat{e} = [-1, 1]^2$ and the sign 0^+ denotes that the computed integrals are zero

u	x^3	x^2y	xy^2	y^3	Note
u_I	x	y	x	y	/
$(u-u_I)_x$	$3x^2-1$	$2xy$	y^2-1	0	/
$\iint_{\widehat{e}}(u-u_I)_x$	0^+	0	$-\frac{8}{3}$	0	$v=x$
$\iint_{\widehat{e}}(u-u_I)_x y$	0	0	0	0	$v=xy$
$\iint_{\widehat{e}}(u-u_I)_x(2x)$	0	0	0	0	$v=x^2$

u	x^4	x^3y	x^2y^2	xy^3	y^4	Note
u_I	1	xy	1	xy	1	/
$(u-u_I)_x$	$4x^3$	$3x^2y-y$	$2xy^2$	y^3-y	0	/
$\iint_{\widehat{e}}(u-u_I)_x$	0	0	0	0	0	$v=x$
$\iint_{\widehat{e}}(u-u_I)_x y$	0	0^+	0	$-\frac{8}{15}$	0	$v=xy$
$\iint_{\widehat{e}}(u-u_I)_x(2x)$	$\frac{32}{5}$	0	$\frac{16}{9}$	0	0	$v=x^2$

3.1. *Proof of* Lemma 3.1

When $u \in P_2$, $\iint_e (u-u_I)_x v_x = 0$. For $u \in P_3 \setminus P_2$, we list in Table 1 the integration terms, where $v \in \text{span}\{1, x, y, xy, x^2, y^2\}$ and $v_x \in \text{span}\{0, 1, 0, y, 2x, 0\}$. In Table 1, the zero values can be easily seen by checking odd polynomials with respect to x or y, and the zero values with "+" in the tables are confirmed by real integral evaluation.

First, when $u = x^3$ and $v = x$, we have

$$\iint_{\widehat{e}}(u-u_I)_x v_x = \iint_{\widehat{e}}(3x^2-1) = 0.$$

Next, we examine the non-trivial term in Table 1. When $u = xy^2$ and $v = x$, we have

$$\iint_{\widehat{e}}(u-u_I)_x v_x = \iint_{\widehat{e}}(y^2-1) = -\frac{8}{3} = -\frac{1}{3}\iint_{\widehat{e}} u_{xyy} v_x.$$

Denote a functional

$$H(u, v) = B(u, v) + \frac{1}{3}\iint_{\widehat{e}} u_{xyy} v_x, \tag{3.14}$$

where $B(u, v) = \iint_{\widehat{e}}(u-u_I)_x v_x$. Hence for $u \in P_3$, $H(u, v) = 0$ and $v \in W(\widehat{e})$, we obtain from Lemma 2.2

$$|H(u, v)| \le C|u|_{4,\widehat{e}}|v|_{1,\widehat{e}}.$$

Then we have

$$\iint_{\widehat{e}}(u-u_I)_x v_x = -\frac{1}{3}\iint_{\widehat{e}} u_{xyy} v_x + O(1)|u|_{4,\widehat{e}}|v|_{1,\widehat{e}}. \tag{3.15}$$

In what follows, we consider the more terms in $P_4 \setminus P_3$, whose results are also listed in Table 1. First, let us check the zero term with 0^+. When $u = x^3y$ and $v = xy$, we have

$$\iint_{\widehat{e}}(u-u_I)_x v_x = \iint_{\widehat{e}}(3x^2-1)y^2 = 0. \tag{3.16}$$

Next, we examine the non-trivial terms for $P_4 \setminus P_3$. For $u = x^4$ and $v = x^2$, we have

$$\iint_{\widehat{e}}(u-u_I)_x v_x = \iint_{\widehat{e}} 4x^3(2x) = \frac{32}{5}. \tag{3.17}$$

For $u = x^2y^2$ and $v = x^2$,

$$\iint_{\widehat{e}}(u-u_I)_x v_x = 4\iint_{\widehat{e}} x^2y^2 = \frac{16}{9}, \tag{3.18}$$

and for $u = xy^3$ and $v = xy$,

$$\iint_{\widehat{e}}(u-u_I)_x v_x = \iint_{\widehat{e}}(y^3-y)y = -\frac{8}{15}. \tag{3.19}$$

Now we have to re-count $H(u, v)$ for those more non-trivial terms of $P_3 \setminus P_2$, and obtain from (3.14):

(1) When $u = x^4$ and $v = x^2$

$$H(u, v) = B(x^4, x^2) + \frac{1}{3}\iint_{\widehat{e}} u_{xyy}v_x = \frac{32}{5} + 0 = \frac{32}{5} = \frac{1}{30}\iint_{\widehat{e}} u_{xxxx}v_{xx}.$$

(2) When $u = x^2y^2$ and $v = x^2$,

$$H(u, v) = B(x^2y^2, x^2) + \frac{1}{3}\iint_{\widehat{e}} u_{xyy}v_x = \frac{16}{9} + \frac{32}{9} = \frac{16}{3} = \frac{1}{6}\iint_{\widehat{e}} u_{xxyy}v_{xx}.$$

(3) When $u = xy^3$ and $v = xy$

$$H(u, v) = B(xy^3, xy) + \frac{1}{3}\iint_{\widehat{e}} u_{xyy}v_x = -\frac{8}{15} + \frac{8}{3} = \frac{32}{15} = \frac{4}{45}\iint_{\widehat{e}} u_{xyyy}v_{xy}.$$

Hence we define a new functional

$$X(u, v) = H(u, v) - \frac{1}{30}\iint_{\widehat{e}} u_{xxxx}v_{xx} - \frac{1}{6}\iint_{\widehat{e}} u_{xxyy}v_{xx} - \frac{4}{45}\iint_{\widehat{e}} u_{xyyy}v_{xy}. \tag{3.20}$$

Obviously, for $u \in P_4$, $H(u, v) = 0$, $v \in P_2$, and then from Lemma 2.2,

$$X(u, v) \le C|u|_{5,\widehat{e}}|v|_{2,\widehat{e}}.$$

Then, we conclude that

$$B(u, v) = -\frac{1}{3}\iint_{\widehat{e}} u_{xyy}v_x + \frac{1}{30}\iint_{\widehat{e}} u_{xxxx}v_{xx} + \frac{1}{6}\iint_{\widehat{e}} u_{xxyy}v_{xx} + \frac{4}{45}\iint_{\widehat{e}} u_{xyyy}v_{xy} + O(1)|u|_{5,\widehat{e}}|v|_{2,\widehat{e}}.$$

Define an affine transformation $T : (x, y) \to (\widehat{x}, \widehat{y})$ with

$$\widehat{x} = \frac{x - x_e}{h_e}, \qquad d\widehat{y} = \frac{y - y_e}{k_e}. \tag{3.21}$$

Then under T, we have that $e \to \widehat{e} = [-1, 1]^2$ and the following equations,

$$\widehat{u}(\widehat{x}, \widehat{y}) = u(x, y), \qquad \widehat{u}_I(\widehat{x}, \widehat{y}) = u_I(x, y),$$
$$d\widehat{x} = \frac{dx}{h_e}, \qquad d\widehat{y} = \frac{dy}{k_e},$$
$$\widehat{u}_{\widehat{x}} = h_e u_x, \qquad \widehat{u}_{\widehat{y}} = k_e u_y.$$

By the affine transformation T in (3.21), we have[2]

$$\begin{aligned}\iint_e (u - u_I)_x v_x &= \frac{k_e}{h_e}\left[\iint_{\widehat{e}} (u - u_I)_x v_x\right]\\ &= \frac{k_e}{h_e}\left[-\frac{1}{3}\iint_{\widehat{e}} u_{xyy}v_x + \frac{1}{30}\iint_{\widehat{e}} u_{xxxx}v_{xx} + \frac{1}{6}\iint_{\widehat{e}} u_{xxyy}v_{xx} + \frac{4}{45}\iint_{\widehat{e}} u_{xyyy}v_{xy} + O(1)|u|_{5,\widehat{e}}|v|_{2,\widehat{e}}\right]\\ &= -\frac{k_e^2}{3}\iint_e u_{xyy}v_x + \frac{h_e^4}{30}\iint_e u_{xxxx}v_{xx} + \frac{h_e^2k_e^2}{6}\iint_e u_{xxyy}v_{xx} + \frac{4k_e^4}{45}\iint_e u_{xyyy}v_{xy} + O(h^5)|u|_{5,e}|v|_{2,e}.\end{aligned} \tag{3.22}$$

This yields the desired result (3.1), and completes the proof of Lemma 3.1. ∎

3.2. *Proof of* Lemma 3.2

When $u \in P_2$, $\iint_e (u - u_I)v = 0$. We list in Table 2 the integration $\iint_{\widehat{e}} (u - u_I)v$ for $u \in P_3 \setminus P_2$ and $v \in P_2$. We examine the non-trivial values in Table 2. First, for $u = x^3$ and $v = x$,

$$\iint_{\widehat{e}} (u - u_I)v = \iint_{\widehat{e}} (x^3 - x)x = -\frac{8}{15} = -\frac{1}{45}\iint_{\widehat{e}} u_{xxx}v_x,$$

and for $u = y^3$ and $v = y$, similarly

$$\iint_{\widehat{e}} (u - u_I)v = -\frac{1}{45}\iint_{\widehat{e}} u_{yyy}v_y.$$

[2] For simplicity, we omit the hat notation on the top for the integration on $\widehat{e}$. For instant, the integration $\iint_{\widehat{e}} \widehat{u}_{\widehat{x}\widehat{y}\widehat{y}}\widehat{v}_{\widehat{x}} d\widehat{x}d\widehat{y}$ is simplified as $\iint_{\widehat{e}} u_{xyy}v_x$ in (3.22).

Table 2
The integration $\iint_{\widehat{e}}(u-u_I)v$ for $u \in P_3 \setminus P_2$, $v \in \text{span}\{1, x, y, xy, x^2, y^2\}$, where $\widehat{e} = [-1, 1]^2$ and the sign 0^+ denotes that the computed integrals are zero.

u u_I $u-u_I$	x^3 x x^3-x	x^2y y x^2y-y	xy^2 x xy^2-x	y^3 y y^3-y	Note / /
$\iint_{\widehat{e}}(u-u_I)$	0	0	0	0	$v=1$
$\iint_{\widehat{e}}(u-u_I)x$	$-\frac{8}{15}$	0	$-\frac{8}{9}$	0	$v=x$
$\iint_{\widehat{e}}(u-u_I)y$	0	$-\frac{8}{9}$	0	$-\frac{8}{15}$	$v=y$
$\iint_{\widehat{e}}(u-u_I)xy$	0	0	0	0	$v=xy$
$\iint_{\widehat{e}}(u-u_I)x^2$	0	0	0	0	$v=x^2$
$\iint_{\widehat{e}}(u-u_I)y^2$	0	0	0	0	$v=y^2$

Next for $u = x^2y$ and $v = y$,

$$\iint_{\widehat{e}}(u-u_I)v = \iint_{\widehat{e}}(x^2y-y)y = -\frac{8}{9} = -\frac{1}{9}\iint_{\widehat{e}} u_{xxy}v_y,$$

and for $u = y^2x$ and $v = x$, similarly

$$\iint_{\widehat{e}}(u-u_I)v = -\frac{1}{9}\iint_{\widehat{e}} u_{xyy}v_x.$$

Denote

$$H(u,v) = B(u,v) + \frac{1}{45}\iint_{\widehat{e}}(u_{xxx}v_x + u_{yyy}v_y) + \frac{1}{9}\iint_{\widehat{e}}(u_{xxy}v_y + u_{xyy}v_x), \tag{3.23}$$

where $B(u,v) = \iint_{\widehat{e}}(u-u_I)v$. We can see that for $u \in P_3$, $H(u,v) = 0$, $\forall v \in P_2$. Based on Lemma 2.2,

$$H(u,v) \le C|u|_{4,\widehat{e}}|v|_{1,\widehat{e}}.$$

This gives

$$B(u,v) = -\frac{1}{45}\iint_{\widehat{e}}(u_{xxx}v_x + u_{yyy}v_y) - \frac{1}{9}\iint_{\widehat{e}}(u_{xxy}v_y + u_{xyy}v_x) + O(1)|u|_{4,\widehat{e}}|v|_{1,\widehat{e}}. \tag{3.24}$$

By using the affine transformation T in (3.21), and by following the proof in Section 3.1, the desired result (3.2) is obtained. This completes the proof of Lemma 3.2. ■

3.3. *Proof of* Lemmas 3.3 *and* 3.4

We have from (2.1) and (2.2) and the Green formula,

$$a_h(u - R_hu, u_h) = \sum_e \int_{\partial e} \frac{\partial u}{\partial n} v\mathrm{d}s = \sum_e \left(\int_{\ell_1} - \int_{\ell_3}\right) u_x u_h\,\mathrm{d}y + \sum_e \left(\int_{\ell_2} - \int_{\ell_4}\right) u_y u_h\,\mathrm{d}x, \tag{3.25}$$

where ℓ_i are the edges of e in Fig. 2. The desired result (3.3) follows from Lemma 3.4, and completes the proof of Lemma 3.3. ■

Below we prove Lemma 3.4. Denote by $I_h(v)$ the bilinear interpolation of $v \in W(e)$. Hence $I_h(v)$ is continuous in S and $I_h(v)|_{\partial S} = 0$. Then we obtain

$$\sum_e \left(\int_{\ell_1} - \int_{\ell_3}\right) u_x I_h(v)\,\mathrm{d}y = 0, \tag{3.26}$$

and

$$\sum_e \left(\int_{\ell_1} - \int_{\ell_3}\right) u_x v\,\mathrm{d}y = \sum_e \left(\int_{\ell_1} - \int_{\ell_3}\right) u_x (v - I_h(v))\,\mathrm{d}y. \tag{3.27}$$

Moreover, there exist the equations

$$(v - I_h(v))|_{\ell_i} = F(y)v_{yy}|_{\ell_i}, \quad i = 1, 3, \tag{3.28}$$

where the function

$$F(y) = \frac{1}{2}\left((y-y_e)^2 - k_e^2\right), \qquad F(y)|_{\ell_i} = 0, \quad i = 2, 4, \tag{3.29}$$

$$F(y) = -\frac{k_e^2}{3} + \frac{1}{6}(F^2(y))''.$$

Hence we obtain repeatedly from the integration by parts

$$\begin{aligned}
\left(\int_{\ell_1} - \int_{\ell_3}\right) u_x(v - I_h(v))\,\mathrm{d}y &= \left(\int_{\ell_1} - \int_{\ell_3}\right) u_x(F(y)v_{yy})\,\mathrm{d}y \\
&= \iint_e (u_x F(y) v_{yy})_x = \iint_e u_{xx}\left[-\frac{k_e^2}{3} + \frac{1}{6}(F^2(y))''\right] v_{yy} \\
&= -\frac{k_e^2}{3}\iint_e u_{xx}v_{yy} - \frac{1}{6}\iint_e u_{xxy}(F^2(y))' v_{yy} + \frac{1}{6}\left(\int_{\ell_2} - \int_{\ell_4}\right) u_{xx}(F^2(y))' v_{yy}\,\mathrm{d}x \\
&= -\frac{k_e^2}{3}\iint_e u_{xx}v_{yy} - \frac{1}{6}\iint_e u_{xxy}(F^2(y))' v_{yy} \\
&= -\frac{k_e^2}{3}\iint_e u_{xx}v_{yy} + \frac{1}{6}\iint_e u_{xxyy}(F^2(y)) v_{yy} - \frac{1}{6}\left(\int_{\ell_2} - \int_{\ell_4}\right) u_{xxy}(F^2(y)) v_{yy}\,\mathrm{d}x \\
&= -\frac{k_e^2}{3}\iint_e u_{xx}v_{yy} + \frac{1}{6}\iint_e u_{xxyy}(F^2(y)) v_{yy},
\end{aligned} \tag{3.30}$$

where we have used $(F^2(y))'|_{\ell_2\cup\ell_4} = (F^2(y))|_{\ell_2\cup\ell_4} = 0$. When $v = u_h \in W(e)$, combining (3.27) and (3.30) gives

$$\begin{aligned}
\sum_e \left(\int_{\ell_1} - \int_{\ell_3}\right) u_x u_h\,\mathrm{d}y &= -\frac{k_e^2}{3}\iint_e u_{xx}(u_h)_{yy} + \frac{1}{6}\iint_e u_{xxyy}(F^2(y))(u_h)_{yy} \\
&= -\frac{k_e^2}{3}\iint_e u_{xx}(u_h)_{yy} + O(h^4)|u|_4|u_h|_{2,h}.
\end{aligned} \tag{3.31}$$

This is the desired result (3.4), and completes the proof of Lemma 3.4. ■

4. Lower bounds for eigenvalues

Theorem 4.1. *Let the rectangles $\square_{ij}$ be uniform. For the eigenvalue by Wilson's element, there exists the equality*

$$\lambda_h - \lambda = -\frac{(h^2+k^2)}{3}\iint_S u_{xx}u_{yy} - \frac{h^2}{3}\iint_S u_{yy}(u_h)_{xx} - \frac{k^2}{3}\iint_S u_{xx}(u_h)_{yy} + O(h^4). \tag{4.1}$$

Moreover, there exists the following bound,

$$\lambda_h - \lambda \le -\frac{2(h^2+k^2)}{3}\iint_S u_{xy}^2 + O(h^4). \tag{4.2}$$

For the extension of rotated Q_1 element (denoted by EQ_1^{rot}), there exists the formula in [14,15].

$$\lambda_h - \lambda = -\frac{h^2+k^2}{3}\iint_S u_{xy}^2 + O(h^4). \tag{4.3}$$

From Theorem 4.1, and by comparing it with (1.10) for (4.3) of EQ_1^{rot}, we have the following corollary.

Corollary 4.1. *Let the rectangles $\square_{ij}$ be uniform. When h is small, Wilson's element provides the lower bound for λ. Also there exist the following bounds,*

$$\lambda_h|_{\mathrm{Wilson}} < \lambda_h|_{EQ_1^{\mathrm{rot}}} < \lambda, \tag{4.4}$$

$$|\lambda_h|_{\mathrm{Wilson}} - \lambda| \ge 2\left|\lambda_h|_{EQ_1^{\mathrm{rot}}} - \lambda\right|. \tag{4.5}$$

Evidently, for seeking eigenvalues, the popular nonconforming Wilson's element is less efficient than EQ_1^{rot}.

4.1. Error expansions for eigenfunctions

In the above discussions, we seek only the expansions of eigenvalues by the FEMs. In this section, let us consider the errors of eigenfunctions.

Theorem 4.2. *Let $\Box_{ij}$ be quasi-uniform, there exists the equality,*

$$\|u-u_h\|_{1,h}^2=\lambda_h-\lambda-2a_h(u-R_hu,u_h)+O(h^4). \tag{4.6}$$

Proof. Since R_hu is the FEM projection in (2.2) to satisfy

$$a_h(R_hu,v)=\lambda(u,v),\forall v\in V_h^0, \tag{4.7}$$

we have

$$a_h(u,u_h)=a_h(u-R_hu,u_h)+a_h(R_hu,u_h)=a_h(u-R_hu,u_h)+\lambda(u,u_h). \tag{4.8}$$

Then we obtain from $(u,u)=(u_h,u_h)=1$, (1.8), (4.7) and (4.8),

$$\begin{aligned}|u-u_h|_{1,h}^2&=a_h(u-u_h,u-u_h)=a_h(u,u)+a_h(u_h,u_h)-2a_h(u,u_h)\\&=\lambda+\lambda_h-2a_h(u-R_hu,u_h)-2a_h(R_hu,u_h)\\&=\lambda+\lambda_h-2a_h(u-R_hu,u_h)-2\lambda(u,u_h)\\&=\lambda_h-\lambda+\lambda\|u-u_h\|_0^2-2a_h(u-R_hu,u_h).\end{aligned} \tag{4.9}$$

Due to Lemma 2.1 and

$$\|u-u_h\|_{1,h}^2=|u-u_h|_{1,h}^2+\|u-u_h\|_0^2=|u-u_h|_{1,h}^2+O(h^4),$$

we obtain the desired result (4.6). This completes the proof of Theorem 4.2. ■

Based on Theorem 4.2, the error $\|u-u_h\|_{1,h}^2$ can be obtained from $\lambda_h-\lambda$ in (1.10) and $a_h(u-R_hu,u_h)$ whose expansions are given in Section 3.3. Here we provide the error expansions of eigenfunctions by Wilson's element,

$$\|u-u_h\|_{1,h}^2=-\frac{h^2+k^2}{3}\iint_S u_{xx}u_{yy}+\frac{h^2}{3}\iint_S u_{yy}(u_h)_{xx}+\frac{k^2}{3}\iint_S u_{xx}(u_h)_{yy}+O(h^4). \tag{4.10}$$

4.2. *Proof of* Theorem 4.1

When $\Box_{ij}$ are uniform, for Wilson's element Eq. (4.1) is obtained from (3.13). Hence we have from (4.1) and (4.10)

$$\lambda_h-\lambda=-A-B+O(h^4), \tag{4.11}$$

$$\|u-u_h\|_{1,h}^2=-A+B+O(h^4)\ge 0, \tag{4.12}$$

where

$$A=\frac{h^2+k^2}{3}\iint_S u_{xx}u_{yy}, \tag{4.13}$$

$$B=\frac{h^2}{3}\iint_S u_{yy}(u_h)_{xx}+\frac{k^2}{3}\iint_S u_{xx}(u_h)_{yy}. \tag{4.14}$$

By summing (4.11) and (4.12), we obtain

$$\lambda_h-\lambda+\|u-u_h\|_{1,h}^2=-2A+O(h^4), \tag{4.15}$$

and therefore

$$\lambda_h-\lambda=-\|u-u_h\|_{1,h}^2-2\frac{h^2+k^2}{3}\iint_S u_{xx}u_{yy}+O(h^4). \tag{4.16}$$

Hence by the integration by parts, we have

$$\begin{aligned}\lambda_h-\lambda&=-\|u-u_h\|_{1,h}^2-2\frac{h^2+k^2}{3}\iint_S u_{xy}^2+O(h^4)\\&\le-\frac{2(h^2+k^2)}{3}\iint_S u_{xy}^2+O(h^4).\end{aligned}$$

This is the desired result (4.2), and completes the proof of Theorem 4.1. ■

Table 3

The first eigenvalue solutions $\lambda_{1,h}$ by extrapolation from the Wilson's solutions, where the true $\lambda_1 = 2\pi^2 \doteq 19.73920880217872$, $\lambda_{1,h}^{(k)} = \frac{2^{2k}\lambda_{1,h}^{(k-1)} - \lambda_{1,2h}^{(k-1)}}{2^{2k}-1}$, $\varepsilon_h^{(k)} = \frac{\lambda_{1,h}^{(k)} - \lambda_1}{\lambda_1}$, Ratio$(k) = |\varepsilon_{2h}^{(k)}/\varepsilon_h^{(k)}|$, $\lambda_{1,h}^{(0)} = \lambda_{1,h}$ and $h = \frac{1}{2N}$

N	2	4	8	16	32
$\lambda_{1,h}$	13.3210127367	17.2960110470	19.0232226313	19.5519189423	19.6918333914
$\lambda_{1,h}^{(1)}$	/	18.6210104838	19.5989598260	19.7281510459	19.7384715410
$\lambda_{1,h}^{(2)}$	/	/	19.6641564488	19.7367637939	19.7391595741
$\lambda_{1,h}^{(3)}$	/	/	/	19.7379162915	19.7391976023
$\lambda_{1,h}^{(4)}$	/	/	/	/	19.7392026271
$\varepsilon_h^{(0)}$	−0.325	−0.124	−0.363 (−1)	−0.949 (−2)	−0.240 (−2)
$\varepsilon_h^{(1)}$	/	−0.566 (−1)	−0.711 (−2)	−0.560 (−3)	−0.374 (−4)
$\varepsilon_h^{(2)}$	/	/	−0.380 (−2)	−0.124 (−3)	−0.249 (−5)
$\varepsilon_h^{(3)}$	/	/	/	−0.655 (−4)	−0.567 (−6)
$\varepsilon_h^{(4)}$	/	/	/	/	−0.313 (−6)
Ratio (0)	/	2.63	3.41	3.82	3.95
Ratio (1)	/	/	7.97	12.7	15.0
Ratio (2)	/	/	/	30.7	49.7
Ratio (3)	/	/	/	/	115

Table 4

The second eigenvalue solutions $\lambda_{h,2}$ by the Wilson element and the true second eigenvalue $\lambda_2 = 5\pi^2 \doteq 49.34802200544679$, where $\lambda_{2,h}^{(k)} = \frac{2^{2k}\lambda_{2,h}^{(k-1)} - \lambda_{2,2h}^{(k-1)}}{2^{2k}-1}$, $\varepsilon_h^{(k)} = \frac{\lambda_{2,h}^{(k)} - \lambda_2}{\lambda_2}$, Ratio$(k) = |\varepsilon_{2h}^{(k)}/\varepsilon_h^{(k)}|$, $\lambda_{2,h}^{(0)} = \lambda_{2,h}$ and $h = \frac{1}{2N}$

N	3	6	12	24
$\lambda_{2,h}$	33.8900820581	43.4379825024	47.6033857887	48.8905854494
$\lambda_{2,h}^{(1)}$	/	46.6206159838	48.9918535508	49.3196520030
$\lambda_{2,h}^{(2)}$	/	/	49.1499360552	49.3415052331
$\lambda_{2,h}^{(3)}$	/	/	/	49.3445460137
$\varepsilon_h^{(0)}$	−0.313	−0.120	−0.354 (−1)	−0.927 (−2)
$\varepsilon_h^{(1)}$	/	−0.553 (−1)	−0.722 (−2)	−0.575 (−3)
$\varepsilon_h^{(2)}$	/	/	−0.401 (−2)	−0.132 (−3)
$\varepsilon_h^{(3)}$	/	/	/	−0.704 (−4)
Ratio (0)	/	2.62	3.39	3.81
Ratio (1)	/	/	7.66	12.6
Ratio (2)	/	/	/	30.4

5. Numerical experiments

In this section, for solving (1.1) and (1.2) we provide numerical experiments for Wilson's element.

5.1. Function $\rho = 1$

We consider the eigenvalue problem of Laplace's operator with $\rho = 1$,

$$-\Delta u = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \lambda u \quad \text{in } S,$$
$$u = 0 \quad \text{on } \Gamma = \partial S,$$

where $S = \{(x, y) | 0 \le x, y \le 1\}$. Then we have the exact eigenvalues and eigenfunctions,[3]

$$u_{k,\ell} = 2\sin(k\pi x)\sin(\ell\pi y), \qquad \lambda_{k,\ell} = (k^2 + \ell^2)\pi^2, \quad 1 \le k, \ell \le N-1. \tag{5.1}$$

The minimal and the next minimal eigenvalues, denoted by λ_1 and λ_2, are most interesting. We list in Tables 3 and 4 their numerical eigenvalues by Wilson's element. In the tables, their errors and the ratios $= |\frac{\varepsilon_{2h}}{\varepsilon_h}|$ are given, where $\varepsilon_h = \lambda_h - \lambda$, and λ_h and λ are the approximate and the true eigenvalues, respectively. For simplicity, we only choose the uniform squares

[3] The constant 2 of eigenfunctions in (5.1) is used for $(u, u) = 1$.

with $h = k$. Denote $h = 1/(2N)$ from Fig. 1, and $N = 2^m$, $m = 1, 2, \ldots$. When $|\frac{\varepsilon_{2h}}{\varepsilon_h}| \approx 2^p$, we may conclude the empirical convergence rates $O(h^p)$.

We can see from Tables 3 and 4,

$$\lambda_{\ell,h} - \lambda = O(h^2), \tag{5.2}$$

where $\lambda_{\ell,h}$ denote the computed λ_ℓ ($\ell = 1, 2$) at the mesh size h. Eq. (5.2) agrees with (1.10). From Tables 3 and 4, we can find the following relative errors of λ_1 and λ_2,

$$\frac{\lambda_{1,h} - \lambda_1}{\lambda_1} = -0.240(-2), \quad \text{at } N = 32, \tag{5.3}$$

$$\frac{\lambda_{2,h} - \lambda_2}{\lambda_2} = -0.927(-2), \quad \text{at } N = 24, \tag{5.4}$$

to indicate the lower bounds due to negative relative errors. More importantly, the expansions of eigenvalues can be applied to raise the accuracy by the extrapolation or the splitting extrapolation techniques. For simplicity, we only report the numerical results for the extrapolation techniques. Based on the computed eigenvalues in Tables 3 and 4, we use the following extrapolation formulas for $\lambda_{1,h}$,

$$\lambda_h^{(k)} = \frac{2^{2k}\lambda_h^{(k-1)} - \lambda_{2h}^{(k-1)}}{2^{2k} - 1}, \quad k = 1, 2, 3, \tag{5.5}$$

where $\lambda_h^0 = \lambda_h$. Note that in (5.5), $\lambda_h^{(1)}$ denotes the first level of extrapolation. In computation, we have computed from the first to the fourth levels of extrapolation. Such a procedure is like that in the Romberg integration. The extrapolation results are also listed in Tables 3 and 4 for $\lambda_{1,h}$ and $\lambda_{2,h}$ by Wilson's element. We can see that all approximate eigenvalues are less than the true eigenvalues, to verify Theorem 4.1 and Corollary 4.1. Moreover from Table 3 there exist the numerical errors,

$$\lambda_{1,h}^{(1)} - \lambda = O(h^4), \tag{5.6}$$

where $\lambda_{1,h}^{(1)}$ is the better approximation of $\lambda_{1,h}$ at the first level of extrapolation. Below, from Table 3 we list the following eigenvalues at the first and fourth levels of extrapolation,

$$\frac{\lambda_{1,h}^{(1)} - \lambda_1}{\lambda_1} = -0.374(-4), \qquad \frac{\lambda_{1,h}^{(4)} - \lambda_1}{\lambda_1} = -0.313(-6), \tag{5.7}$$

for Wilson's elements at $N = 32$. Evidently, the errors in (5.7) are much smaller than $-0.240(-2)$ in (5.3).

5.2. Function $\rho \neq 1$

The error analysis is valid for the function $\rho = \rho(x, y) \geq \rho_0 > 0$. To verify the analysis made, we also carry out the numerical experiments for $\rho \neq 1$. Choose

$$\rho = \rho(x, y) = 1 + \left(x - \frac{1}{2}\right)\left(y - \frac{1}{2}\right), \tag{5.8}$$

which is symmetric with respect to x and y. Consider

$$-\Delta u = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \lambda \rho u \quad \text{in } S,$$
$$u = 0 \quad \text{on } \Gamma = \partial S,$$

where S is also the unit square. For the ρ in (5.8), we may evaluate $\iint_S \rho u v$ in (1.5) exactly. The FEM as (1.3) can be easily performed. We provide the numerical results for λ_1, and list them in Table 5. Since for ρ in (5.8), the true solution of λ_1 is unknown, we may compute the ratios of sequential errors, to display the empirical convergence rates. The numerical solutions, the sequential errors and their ratios are listed in Table 5 for Wilson's element. Since only the sign of $\varepsilon^{(0)}$ is significant, this is also listed in Table 5.

From Table 5, we can see the sequential errors

$$\frac{\lambda_{1,2h} - \lambda_{1,4h}}{\lambda_{1,h} - \lambda_{1,2h}} = O(h^2), \tag{5.9}$$

for Wilson's element. The empirical convergence rate of λ_1 is exactly the same as that in Section 5.1 for $\rho = 1$. All those theoretical results have been verified by the numerical results.

Table 5

The first eigenvalue solutions $\lambda_{1,h}$ for $-\Delta u = \lambda\rho u$ by Wilson's element, where $\lambda_{1,h}^{(k)} = \frac{2^{2k}\lambda_{1,h}^{(k-1)} - \lambda_{1,2h}^{(k-1)}}{2^{2k}-1}$, $\varepsilon_h^{(k)} = \lambda_{1,h}^{(k)} - \lambda_{1,2h}^{(k)}$, Ratio$(k) = |\varepsilon_{2h}^{(k)}/\varepsilon_h^{(k)}|$, $\lambda_{1,h}^{(0)} = \lambda_{1,h}$ and $h = \frac{1}{2N}$

N	2	4	8	16	32
$\lambda_{1,h}$	12.8079589980	17.2866245950	19.0155981774	19.5447728619	19.6848297584
$\lambda_{1,h}^{(1)}$	/	18.7795131273	19.5919227049	19.7211644234	19.7315153906
$\lambda_{1,h}^{(2)}$	/	/	19.6460833434	19.7297805380	19.7322054551
$\lambda_{1,h}^{(3)}$	/	/	/	19.7311090649	19.7322439458
$\lambda_{1,h}^{(4)}$	/	/	/	/	19.7322483963
$\varepsilon_h^{(0)}$	/	−4.479	−1.729	−0.529	−0.140
$\lvert\varepsilon_h^{(1)}\rvert$	/	/	0.812	0.129	0.104 (−1)
$\lvert\varepsilon_h^{(2)}\rvert$	/	/	/	0.837 (−1)	0.242 (−2)
$\lvert\varepsilon_h^{(3)}\rvert$	/	/	/	/	0.113 (−2)
Ratio (0)	/	/	2.59	3.27	3.78
Ratio (1)	/	/	/	6.29	12.49
Ratio (2)	/	/	/	/	34.52

Remark 5.1. Strictly speaking, the derivatives $(u_h)_{xx}$ and $(u_h)_{yy}$ in (1.10) may depend on h. However, based on the numerical results in (5.6) and (5.9), the principal integrals $\iint_S u_{xx}(u_h)_{yy}$ and $\iint_S u_{yy}(u_h)_{xx}$ seem to be independent of h. Hence from Theorem 4.1 we may assume

$$\lambda_h - \lambda = -\frac{2(h^2+k^2)}{3}\alpha \iint_S u_{xy}^2 + O(h^4),$$

where $\alpha(>0)$ is a constant independent of h, but may depend on λ. From the numerical data in Tables 3 and 4, we can see that $\alpha = 1.5$ and $\alpha = 2$ for λ_1 and λ_2, respectively.

6. Concluding remarks

The new expansions of numerical eigenvalues by Wilson's element is summarized in (1.10) for uniform $\square_{ij}$; the strict proof of Wilson's element is provided in this paper. Not only can Wilson's element display a lower bound of the FEM solution of λ_i, but also can lead to higher superconvergence rates by the extrapolation or the splitting extrapolation techniques. All the theoretical analyses have been verified by the numerical experiments in Section 5. Moreover, the higher accurate solutions can also be obtained numerically by multiple levels of extrapolation for Wilson's element. It is worth pointing out that not only these results are new to [15], but also the proof for the lower bound is more intriguing than that in [14,15,28].

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