Zero-divisor graphs of non-commutative rings

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Abstract

In a manner analogous to the commutative case, the zero-divisor graph of a non-commutative ring 
\( R \) can be defined as the directed graph \( \Gamma(R) \) that its vertices are all non-zero zero-divisors of \( R \) in 
which for any two distinct vertices \( x \) and \( y \), \( x \rightarrow y \) is an edge if and only if \( xy = 0 \). We investigate 
the interplay between the ring-theoretic properties of \( R \) and the graph-theoretic properties of \( \Gamma(R) \).

In this paper it is shown that, with finitely many exceptions, if \( R \) is a ring and \( S \) is a finite semisimple 
ring which is not a field and \( \Gamma(R) \cong \Gamma(S) \), then \( R \cong S \). For any finite field \( F \) and each integer \( n \geq 2 \), 
we prove that if \( R \) is a ring and \( \Gamma(R) \cong \Gamma(M_n(F)) \), then \( R \cong M_n(F) \). Redmond defined the simple 
undirected graph \( \bar{\Gamma}(R) \) obtaining by deleting all directions on the edges in \( \Gamma(R) \). We classify all 
ring \( R \) whose \( \bar{\Gamma}(R) \) is a complete graph, a bipartite graph or a tree.

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1. Introduction

The concept of zero-divisor graph of a commutative ring with identity was introduced 
by Beck in [6] and has been studied in [1–5]. Recently Redmond in [9] has extended this 
concept to any arbitrary ring. Throughout the paper, \( R \) denotes a ring (not necessarily with
and \( \psi \) between the vertex set of \( Y \) and \( X \), the form \( \phi \mapsto \psi \) is an edge in \( \Gamma(R) \). For an undirected graph \( G \), if there is an edge between every pair of the vertices, \( G \) is said to be \( \text{bipartite} \) if its vertex set can be partitioned into two sets \( X \) and \( Y \) such that for any two vertices \( x \in X \) and \( y \in Y \), there is an edge between \( x \) and \( y \). A graph \( G \) is called \( \text{complete} \) if there is an edge between every pair of the vertices. A subset \( X \) of the vertices of a graph \( G \) is called \( \text{independent} \) if there is no edge with two endpoints in \( X \). A graph \( G \) is called \( \text{bipartite} \) if its vertex set can be partitioned into \( X \) and \( Y \) such that every edge of \( G \) has one endpoint in \( X \) and other endpoint in \( Y \). A graph \( G \) is said to be \( \text{star} \) if \( G \) contains one vertex in which all other vertices are joined to this vertex and no other edges. Two graphs \( G_1 \) and \( G_2 \) are said to be \( \text{isomorphic} \) if there is a bijective map \( \varphi \) between the vertex set of \( G_1 \) and the vertex set of \( G_2 \) such that for any two vertices \( x \in G_1 \) and \( y \in G_2 \), \( x \) and \( y \) are adjacent in \( G_1 \) if and only if \( \varphi(x) \) and \( \varphi(y) \) are adjacent in \( G_2 \). For an undirected graph \( G \), the degree of a vertex \( v \) in \( G \) is the number of edges incident to \( v \). A graph \( G \) is \( \text{complete} \) if there is an edge between every pair of the vertices. A subset \( X \) of the vertices of a graph \( G \) is called \( \text{independent} \) if there is no edge with two endpoints in \( X \). A graph \( G \) is called \( \text{bipartite} \) if its vertex set can be partitioned into \( X \) and \( Y \) such that every edge of \( G \) has one endpoint in \( X \) and other endpoint in \( Y \). A graph \( G \) is said to be \( \text{star} \) if \( G \) contains one vertex in which all other vertices are joined to this vertex and no other edges. Two graphs \( G_1 \) and \( G_2 \) are said to be \( \text{isomorphic} \) if there is a bijective map \( \varphi \) between the vertex set of \( G_1 \) and the vertex set of \( G_2 \) such that the adjacency relation is preserved.

For any field \( F \), we denote \( M_n(F) \) the ring of all \( n \times n \) matrices over \( F \). Also for any \( i \) and \( j \), \( 1 \leq i, j \leq n \), we denote by \( E_{ij} \) the element of \( M_n(F) \) whose \((i, j)\)-entry is 1 and other entries are 0. Moreover, a ring \( R \) is called \( \text{semisimple} \) if its Jacobson radical is zero. By the Wedderburn–Artin Theorem, we know that every semisimple Artinian ring is isomorphic to the direct product of finitely many full matrix rings over division rings. If \( e \) is an idempotent element in \( R (e^2 = e) \), by Peirce decomposition [8, p. 308], we can write \( R = eRe \oplus eR(1 - e) \oplus (1 - e)Re \oplus (1 - e)(1 - e)R(1 - e) \) (if \( R \) has no identity, then \( eR(1 - e) \) denotes the subring \( \{ex - ex | x \in R\} \), and similar definitions are used for the sets \( (1 - e)Re \) and \( (1 - e)R(1 - e) \)). An idempotent element \( e \) is said to be \( \text{non-trivial} \) if
e \neq 0,1. A ring \( R \) is called reduced if it has no non-zero nilpotent elements. Also a ring \( R \) is called a null ring if \( R^2 = \{0\} \). We denote the finite field of order \( q \) by \( \mathbb{F}_q \).

In this article we prove that for any finite field \( F \) and each integer \( n \geq 2 \), if \( R \) is a ring and \( \Gamma(R) \cong \Gamma(M_n(F)) \), then \( R \cong M_n(F) \). This shows that the zero-divisor graph of \( M_n(F) \) determines the ring up to isomorphism. Using later results, we obtain that if \( S \) is a finite semisimple ring which is not a field and \( \Gamma(R) \cong \Gamma(S) \), then apart from finitely many exceptions, we have \( R \cong S \). In relation to the graph \( \Gamma(R) \), we prove that this graph is complete if and only if either \( R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \) or \( \mathbb{D}(R)^2 = \{0\} \). Also we establish that there are just two finite non-commutative rings \( R \) for which \( \Gamma(R) \) is bipartite. We show that if \( \Gamma(R) \) has no cycle, then either \( \Gamma(R) \) is a star graph or \( R \) is isomorphic to one of the rings \( \mathbb{Z}_2 \times \mathbb{Z}_4 \) or \( \mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2) \). Furthermore we prove that for any left Artinian ring \( R \), if \( \Gamma(R) \) has at least four vertices, then \( \Gamma(R) \) is a star graph if and only if \( R \) is isomorphic to the direct product of a division ring and a ring of order 2.

2. Some properties of the zero-divisor graphs of non-commutative rings

In this section we characterize rings with respect to their zero-divisor graphs. We start with some properties of the zero-divisors of a ring. The following remark is useful in our proofs.

**Remark 1.** For a ring \( R \), we note that if \( \mathbb{D}(R) \) is finite, then \( R \) is either a finite ring or a domain. Moreover, if \( R \) is not a domain, then \( |R| \leq |\mathbb{D}(R)|^2 \). To see this, suppose that \( \mathbb{D}(R) \) is finite and non-zero. Let \( x \in \mathbb{D}(R)^* \). Since \( \text{Ann}_\ell(x) \) and \( Rx \) are contained in \( \mathbb{D}(R) \), these sets are finite. Since \( [R : \text{Ann}_\ell(x)] = |Rx| \), we conclude that \( |R| = |\text{Ann}_\ell(x)||Rx| \leq |\mathbb{D}(R)|^2 \).

**Lemma 2.** Let \( R \) be a left Artinian ring and \( R \neq \mathbb{D}(R) \). Then \( R \) has identity and any element of \( R \setminus \mathbb{D}(R) \) is unit.

**Proof.** Let \( a \in R \setminus \mathbb{D}(R) \). So the map \( \phi : R \to R \), defined by \( \phi(x) = xa \), is a left \( R \)-module monomorphism. Since \( R \) is a left Artinian ring and \( \phi \) is injective, it is well known that \( \phi \) is surjective. Thus there is an element \( e \in R \) such that \( ea = a \). Now since \( a \notin \mathbb{D}(R) \) and \( (xe - x)a = 0 \), we conclude that \( xe = x \), for each \( x \in R \). Also we have \( a(ex - x) = 0 \). This yields that \( ex = x \), for any \( x \in R \). Hence \( 1 = e \) is the identity of \( R \). Moreover, since \( \phi \) is surjective, there is an element \( a' \in R \) such that \( a'a = 1 \). Again we have \( a \notin \mathbb{D}(R) \) and \( (aa' - 1)a = 0 \), so \( aa' = 1 \). Thus \( a \) is a unit, as desired. \( \square \)

**Theorem 3.** Let \( R \) be a ring such that \( |\mathbb{D}(R)| = p \) is a prime number. If \( R \) is not a reduced ring, then either \( R \) is isomorphic to \( \mathbb{Z}_{p^2} \), \( \mathbb{Z}_p[x]/(x^2) \) or the null ring of order \( p \).

**Proof.** Since \( \mathbb{D}(R) \) is finite, by Remark 1, \( R \) is also finite. If \( R \) has no identity, then by Lemma 2 we have \( R = \mathbb{D}(R) \). Thus the additive group of \( R \) is cyclic. Clearly, this implies that \( R \) is the null ring of order \( p \). So assume that \( R \) has identity. Let \( J(R) \) be the Jacobson radical of \( R \). If \( J(R) = \{0\} \), then by the Wedderburn–Artin Theorem, we may
write \( R = M_{n_1}(F_{q_1}) \times \cdots \times M_{n_r}(F_{q_r}) \), where \( q_i \)'s are prime powers and \( n_1, \ldots, n_r, r \) are natural numbers. Since \( R \) is not reduced, at least one of \( n_i \)'s, say \( n_1 \), is greater than 1. By Lemma 2, all non-invertible elements of \( R \) are zero-divisors, so we find that

\[
|\mathcal{D}(R)| = \prod_{i=1}^{r} q_i^{n_i} - \prod_{i=1}^{r} (q_i^{n_i} - 1)(q_i^{n_i} - q_i) \cdots (q_i^{n_i} - q_i^{n_i-1}).
\]

This implies that

\[
q_1^{n_1(n_1-1)/2} \cdots q_r^{n_r(n_r-1)/2} \left( \prod_{i=1}^{r} q_i^{n_i(n_i+1)/2} - \prod_{i=1}^{r} (q_i^{n_i} - 1)(q_i^{n_i-1} - 1) \cdots (q_i - 1) \right) = p.
\]

Now since \( p \) is a prime and \( n_1 > 1 \), we conclude that \( q_1 = p \). But the number given in the above parenthesis is greater than 1, a contradiction. Hence \( J(R) \neq \{0\} \). It is well known that any unit of the ring \( R/J(R) \) is of the form \( u + J(R) \), where \( u \) is a unit of \( R \). Since \( R \) is finite, this implies that any zero-divisor of \( R/J(R) \) is of the form \( z + J(R) \), where \( z \) is a zero-divisor of \( R \). On the other hand, for any \( z \in \mathcal{D}(R) \) and \( j \in J(R) \), we have \((z + j) + J(R) = z + J(R) \) and hence \( z + J(R) \subseteq \mathcal{D}(R) \). So \( z + J(R) \subseteq \mathcal{D}(R) \). Thus we obtain that \( |\mathcal{D}(R)| = |\mathcal{D}(R/J(R))| \). It follows from \( |\mathcal{D}(R)| = p \) and \( J(R) \neq \{0\} \) that \( |\mathcal{D}(R/J(R))| = 1 \) and \( |J(R)| = p \). So \( R/J(R) \) is a domain and hence its characteristic is a prime number. Since \( J(R) \) is a nilpotent ideal of order \( p \), we conclude that \( J(R)^2 = \{0\} \). Therefore \( |R| \) is a power of \( p \). Since \( |R| \) is divisible by \( p \), we find that \( |R| = p^2 \). Furthermore, \( |R/J(R)| \geq p \) and so \( |R| = p^2 \). If char \( R = p^2 \), then \( R \simeq \mathbb{Z}_p^2 \).

Suppose that char \( R = p \) and \( a \) is an element of \( J(R)^* \). Then \( R = \{\lambda + \mu a \mid \lambda, \mu \in \mathbb{Z}_p^* \} \). Now since \( a^2 = 0 \), it is easily checked that \( R \simeq \mathbb{Z}_p[x]/(x^2) \) and the proof is complete. \( \square \)

**Corollary 4.** Let \( R \) be a ring. If \( \Gamma(R) \) has one vertex, then either \( R \) is isomorphic to one of the rings \( \mathbb{Z}_4, \mathbb{Z}_2[x]/(x^2) \) or the null ring of order 2.

**Proof.** By Remark 1, \( R \) is finite. Since every finite reduced ring is isomorphic to the direct product of finitely many fields, any finite reduced ring is either a field or has at least three zero-divisors. Hence \( R \) is not reduced and by Theorem 3 the assertion is proved. \( \square \)

**Theorem 5.** Let \( R \) be a ring. Then \( \Gamma(R) \) is a complete graph if and only if either \( R \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \) or \( \mathcal{D}(R)^2 = \{0\} \). Moreover in the latter case \( \mathcal{D}(R) \) is an ideal of \( R \).

**Proof.** For one direction, the proof is straightforward. For the other direction assume that \( \Gamma(R) \) is a complete graph. First suppose that there exists a zero-divisor \( e \) with \( e^2 \neq 0 \). We show that \( e^2 = e \). If not, then since the vertices \( e \) and \( e^2 \) are adjacent, we have \( e^3 = ee^2 = 0 \). Hence \( e^2(e - e^2) = 0 \) and so \( e - e^2 \) is a non-zero zero-divisor different from \( e \). Now since \( \Gamma(R) \) is a complete graph, we conclude that \( e^2 = e(e - e^2) = 0 \), a contradiction. So \( e \) is an idempotent element and hence we can write \( R = eRe \oplus eR(1 - e) \oplus (1 - e)R \oplus (1 - e)(1 - e) \). We show that \( eR(1 - e) = \{0\} \). Let \( x \) be an element of \( eR(1 - e) \). Since \( x(e + x) = 0, e + x \) is a vertex of \( \Gamma(R) \). But \( e \) and \( e + x \) are not adjacent, which is
a contradiction. Similarly \((1 - e)Re = \{0\}\). Since none of the vertices of \(eRe^*\) is adjacent to \(e\), we have \(eRe = \{0, e\}\). Now we show that \((1 - e)R(1 - e)\) is a domain. Suppose \(yy' = 0\), for some \(y, y' \in (1 - e)R(1 - e)^*\). Since \((e + y)y' = 0\), \(e + y\) is a vertex of \(\overline{Γ}(R)\). But \(e\) and \(e + y\) are not adjacent, a contradiction. So \((1 - e)R(1 - e)^*\) is an independent set. This yields that \((1 - e)R(1 - e) \simeq \mathbb{Z}_2\). Hence in this case \(R \simeq \mathbb{Z}_2 \times \mathbb{Z}_2\).

Next suppose that \(z^2 = 0\), for any \(z \in \mathcal{D}(R)\). Clearly this implies that \(\mathcal{D}(R)\) is an ideal of \(R\). Assume that there exist two zero-divisors \(a\) and \(b\) such that \(ab \neq 0\). Since \(\overline{Γ}(R)\) is a complete graph, \(ba = 0\). Hence \(b \in \text{Ann}_e(a + b) \cap \text{Ann}_r(a + b)\). Now since \(\mathcal{D}(R) = \text{Ann}_e(a + b) \cup \text{Ann}_r(a + b)\) is an additive group, \(\mathcal{D}(R) = \text{Ann}_e(a + b)\). Thus \(a(a + b) = 0\) and so \(ab = 0\), a contradiction. Therefore \(\mathcal{D}(R)^2 = \{0\}\) and the proof is complete. \(\square\)

The following theorem has been established in [5] for every commutative ring with identity. Now using Theorem 5 we generalize it for any arbitrary ring.

**Theorem 6.** Let \(R\) be a ring which is not isomorphic to \(\mathbb{Z}_2 \times \mathbb{Z}_2\). If one of the graphs \(\overline{Γ}(R)\), \(\overline{Γ}(R[x])\), and \(\overline{Γ}(R[[x]])\) is a complete graph, then the other graphs are also complete graphs.

**Proof.** Since \(\overline{Γ}(R)\) is an induced subgraph of \(\overline{Γ}(R[x])\) and \(\overline{Γ}(R[x])\) is an induced subgraph of \(\overline{Γ}(R[[x]])\), it is enough to prove that if \(\overline{Γ}(R)\) is a complete graph, then \(\overline{Γ}(R[[x]])\) is also a complete graph. Suppose that \(\overline{Γ}(R)\) is complete. Since \(R \not\simeq \mathbb{Z}_2 \times \mathbb{Z}_2\), Theorem 5 implies that \(\mathcal{D}(R)\) is an ideal of \(R\) with \(\mathcal{D}(R)^2 = \{0\}\). To complete the proof, one needs only verify that \(\mathcal{D}(R[[x]]) \subseteq \mathcal{D}(R)[[x]]\). Assume that \(f(x) = \sum_{n=0}^{∞} f_n x^n\) is an element of \(\mathcal{D}(R[[x]])\setminus \mathcal{D}(R)[[x]]\). Then there exists a non-zero element \(g(x) = \sum_{n=0}^{∞} g_n x^n\) such that \(f(x)g(x) = 0\). Since \(\overline{Γ}(R)\) is a domain, then \(\overline{Γ}(R[[x]]) \simeq \overline{Γ}(R)[[x]]\) is also a domain. Since \(f(x)g(x) = 0\) and \(f(x) \notin \mathcal{D}(R)[[x]]\), we conclude that \(g(x) \in \mathcal{D}(R)[[x]]\). Assume that \(r\) and \(s\) are the smallest indices such that \(f_r \notin \mathcal{D}(R)\) and \(g_s \neq 0\). Since \(f(x)g(x) = 0\) and \(g(x) \in \mathcal{D}(R)[[x]]\) and \(\mathcal{D}(R)^2 = \{0\}\), we have \(f_l g(x) = 0\), for each \(i < r\). Thus \((f(x) - \sum_{n=0}^{r-1} f_n x^n)g(x) = 0\) and therefore \(f_r g_s = 0\). It follows from \(g_s \neq 0\) that \(f_r \in \mathcal{D}(R)\), a contradiction. \(\square\)

Note that in [5] it has been shown that if \(\mathcal{R} = \mathbb{Z}_2 \times \mathbb{Z}_2\), then \(\overline{Γ}(\mathcal{R}[x])\) is not complete, while \(\overline{Γ}(\mathcal{R})\) is a complete graph with two vertices. It has been also proved that between each pair of the vertices of \(\overline{Γ}(\mathcal{R}[[x]])\) there is a path of the length at most 2.

**Example 7.** Let \(\mathcal{R} = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{Z}_2 \right\}\). Then \(Γ(\mathcal{R})\) is a directed graph with three vertices and the edges \(E_{12} \rightarrow E_{11}\) and \(E_{12} \rightarrow E_{11} + E_{12}\). Note that \(Γ(\mathcal{R})\) is the smallest zero-divisor graph associated to a non-commutative ring.

**Theorem 8.** Let \(R\) be a ring such that \(Γ(R)\) has no multiple edges. If \(Γ(R)\) has more than one vertex, then \(R \simeq \mathcal{R}\) or \(R \simeq \mathcal{R}^{op}\).

**Proof.** If \(R\) is a reduced ring, then for any \(x, y \in R\), \(xy = 0\) if and only if \(yx = 0\). Thus in this case if there is an edge between two vertices \(x\) and \(y\), then there is a multiple
edge between $x$ and $y$. So $R$ is not reduced. Suppose that $a$ is a non-zero nilpotent element of $R$ and $n$ is the smallest natural number such that $a^n = 0$. If $n \geq 3$, then there is a multiple edge between $a$ and $a^{n-1}$, a contradiction. Hence $a^2 = 0$. Without loss of generality, suppose that $a \to e$ is an edge in $\Gamma(R)$ (if there is no such edge, then we consider $R^{\text{op}}$ instead of $R$). Since $ea \neq 0$ and $a(ea) = (ea)a = 0$, we have $ea = a$. Thus $(e - e^2)a = a(e - e^2) = 0$. Now since $ea \neq ae$, $e^2 = e$. So $e$ is a non-trivial idempotent and hence we can write $R = eRe \oplus eR(1-e) \oplus (1-e)Re \oplus (1-e)R(1-e)$. Since $(1-e)R(1-e) \subseteq \text{Ann}_e(e) \cap \text{Ann}_r(e)$, we conclude that $(1-e)R(1-e) = \{0\}$. Assume that $x \in eR(1-e)$ and $y \in (1-e)Re$ are non-zero elements. We have $yx \in (1-e)R(1-e)$, so $yx = 0$ and therefore $xy \neq 0$. Since $x(xy) = (xy)x = 0$ and $y(xy) = (xy)y = 0$ and $x \neq y$, we get a contradiction. Now since $e \neq 1$, exactly one of the sets $eR(1-e)$ and $(1-e)Re$ is non-zero. Suppose $eR(1-e) \neq \{0\}$ and $(1-e)Re = \{0\}$. Since $eR(1-e)$ is a null ring, we obtain that $|eR(1-e)| = 2$. Let $eR(1-e) = \{0, b\}$. Since $b(eRe) = \{0\}$ and $\Gamma(R)$ has no multiple edges, $eRe \cap \text{Ann}_e(b) = \{0\}$. Using $R = eRe \oplus eR(1-e)$, we conclude that $\text{Ann}_e(b) = eR(1-e)$. Also since $eb = b$ and $Rb \subseteq \text{Ann}_e(b)$, we obtain $Rb = \{0, b\}$. Now the equation $[R : \text{Ann}_e(b)] = |Rb|$ implies that $|R| = 4$. Thus $R = \{0, e, b, e + b\}$. We have $e^2 = e$, $b^2 = 0$, $eb = b$, and $be = 0$. Now it is easy to see that $R \cong \mathbb{F}_4$. Similarly, if $eR(1-e) = \{0\}$ and $(1-e)Re \neq \{0\}$, then we find that $R \cong \mathbb{F}_4^{\text{op}}$ and the proof is complete.

In the next theorem, we determine all left Artinian rings $R$ for which $\overline{\Gamma}(R)$ is a star graph with at least four vertices.

**Theorem 9.** Let $R$ be a left Artinian ring and $\overline{\Gamma}(R)$ has at least four vertices. Then $\overline{\Gamma}(R)$ is a star graph if and only if $R$ is isomorphic to the direct product of a division ring and a ring of order 2.

**Proof.** For one direction, the proof is clear. For the other direction let $a$ be that vertex of $\overline{\Gamma}(R)$ which is adjacent to all other vertices. First suppose that $a^2 \neq 0$. Since $a^2$ is also adjacent to all other vertices of $\overline{\Gamma}(R)$, $a^2 = a$ and therefore we can write $R = aRa \oplus aR(1-a) \oplus (1-a)Ra \oplus (1-a)R(1-a)$. If $(1-a)R(1-a) = \{0\}$, then for any elements $x$ and $y$ of the set $aR(1-a)^* \cup (1-a)Ra^*$ we have either $xy = 0$ or $yx = 0$. Since $a \neq 1$, this set cannot be empty and hence it has just one element. Without loss of generality, assume that $aR(1-a) = \{0, b\}$ and $(1-a)Ra = \{0\}$. We have $b(a-b) = 0$. So $a - b$ is a vertex of $\overline{\Gamma}(R)$ which is not adjacent to $a$, a contradiction. Thus $(1-a)R(1-a) \neq \{0\}$. Since all vertices of the set $(1-a)R(1-a)^*$ are adjacent to all vertices of $aRa^* \cup aR(1-a)^* \cup (1-a)Ra^*$ and noting that $\overline{\Gamma}(R)$ is a star graph, we conclude that $aRa = \{0, a\}$ and $aR(1-a) = (1-a)Ra = \{0\}$. Therefore $aRa$ and $(1-a)R(1-a)$ are two ideals of $R$. Furthermore, because of $\overline{\Gamma}(R)$ is a star graph, $(1-a)R(1-a)^*$ is an independent set. Suppose that $(1-a)R(1-a)$ is not a domain. Since $\overline{\Gamma}((1-a)R(1-a))$ is a connected graph [9, Theorem 3.2] with no edge, the ring $(1-a)R(1-a)$ has exactly one non-zero zero-divisor, say $c$. Clearly $c^2 = 0$ and so $(a + c) - c$ is an edge of $\overline{\Gamma}(R)$, a contradiction. Hence $(1-a)R(1-a)$ is a domain. Now since $R = \{0, a\} \oplus (1-a)R(1-a)$, the ring $(1-a)R(1-a)$ is a left Artinian domain and so it is a division ring. Moreover, $a^2 = a$ and therefore in this case $R$ is isomorphic to the direct product of a division ring and $\mathbb{Z}_2$. 


Next suppose that $a^2 = 0$. Assume that $R$ has a non-trivial idempotent, say $e$. We have $R = eRe \oplus eR(1-e) \oplus (1-e)R \oplus (1-e)R(1-e)$. Since all vertices of the set $eR(1-e)^* \cup (1-e)R(1-e)^*$ are adjacent to $e$, this set is a subset of $\{a\}$. Moreover $e \neq 1$, so $eR(1-e) \oplus (1-e)R \oplus (1-e)R(1-e) = \{0, a\}$ and thus we have

$$R = eRe \oplus \{0, a\}.$$ (*)&

Also, since $\overline{T}(R)$ is a star graph, $eRe^*$ is an independent set. If we repeat the above proof as we did for $(1-a)R(1-a)$ before, we find that $eRe$ must be a domain. Now if $eR(1-e) = \{0, a\}$, then $Ra \subseteq \text{Ann}_R(ea, a)$. This implies that $Ra = \{0, a\}$. Moreover in this case $R = eRe \oplus eR(1-e)$. Thus $aR = \{0\}$. This shows that $\{0, a\}$ is an ideal of $R$. So by (*), we have $R_{[a]} \simeq eRe$ and thus $eRe$ is a left Artinian ring and so it is a division ring. Assume that $za = 0$, for some $z \in eRe$. If $z' = z^{-1}$ in the division ring $eRe$, then $a = ea = (z')a = 0$, a contradiction. By (*), we conclude that $\text{Ann}_R(a) = \{0, a\}$. Since $[R : \text{Ann}_R(a)] = |Ra| = 2$, we have $|R| = 4$. This is a contradiction, since $\overline{T}(R)$ has at least four vertices. By a similar argument, $(1-e)R(1-e) = \{0\}$ and therefore $(1-e)R(1-e) = \{0, a\}$. Since $\{0, a\}$ is an ideal of $R$, by (*), $eRe$ is a division ring and so in this case $R$ is isomorphic to the direct product of a division ring and the null ring of order 2.

Now suppose that $R$ has no non-trivial idempotent. Let $x \in \mathcal{D}(R) \setminus \{0, a\}$. Since $R$ is a left Artinian ring with no non-trivial idempotent, by [8, Corollary 19.19], $R$ is a left Artinian local ring. So there exists the smallest natural number $n$ such that $x^n = 0$. If $n \geq 4$, then since $x = x^{n-1}$ and $x^2 - (x^{n-2} - a)$ are two edges in $\overline{T}(R)$, we easily get a contradiction. If $x^2 = 0$, then $x$ and $a - x$ are adjacent in $\overline{T}(R)$, so we again obtain a contradiction. Thus $n = 3$ and $x^3 = 0$. Hence $x^2 = a$, for any $x \in \mathcal{D}(R) \setminus \{0, a\}$. Therefore $x(xy) = ay = y^3 = 0$, for all zero-divisors $x$ and $y$. So we conclude that $xy = a$, for any $x, y \in \mathcal{D}(R) \setminus \{0, a\}$. Now let $b, c, d$ be distinct vertices of degree 1 in $\overline{T}(R)$. We have $bc = bd = cb = cd = a$ and so $b(c - d) = c(b - d) = 0$. Since $b^2 \neq 0$ and $c^2 \neq 0$, we conclude that $c - d = b - d = a$, a contradiction. This completes the proof. □

Let $\mathcal{R}$ be a ring of order 2, $p$ be a prime, and $n$ be a natural number. Then $\overline{T}(\mathcal{R} \times \mathbb{F}_p^n)$ is a star graph with $p^n$ vertices, if $\mathcal{R}$ has identity; and it is a star graph with $2p^n - 1$ vertices, if $\mathcal{R}$ has no identity. So by Corollary 4 and Theorem 9 we get the following.

**Corollary 10.** A finite star graph can be realized as $\overline{T}(R)$ if and only if it has $p^n$ or $2p^n - 1$ vertices, for some prime $p$ and integer $n \geq 0$.

**Example 11.** Theorem 9 is not valid for an arbitrary ring. To see this let $\mathbb{Z}[x, y]$ be the polynomial ring with non-commuting variables and $\mathcal{I} = (x^2, 2x, xy, yx - x)$. Assume that $\mathcal{R} = \mathbb{Z}[x, y]/\mathcal{I}$. Then it is easy to see that $\overline{T}(\mathcal{R})$ is a star graph and $x + \mathcal{I}$ is that vertex which is adjacent to all other vertices of the graph. We note that $\mathcal{R}$ has no non-trivial idempotent and so $\mathcal{R}$ is not isomorphic to the ring given in Theorem 9.

**Theorem 12.** Let $R$ be a left Artinian ring which is not a division ring and $\overline{T}(R)$ has no cycle of length 3. If $\overline{T}(R)$ is not a star graph, then $R$ is isomorphic to either the direct product of two division rings or the direct product of a division ring and one of the rings $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$. 


Proof. If $R$ is a local ring, then $\mathcal{D}(R)$ is equal to the Jacobson radical of $R$, and hence it is a nilpotent ideal. This implies that there is a vertex adjacent to all other vertices of $\overline{\Gamma}(R)$. Hence either $\overline{\Gamma}(R)$ has a cycle of length 3 or $\overline{\Gamma}(R)$ is a star graph. Thus assume that $R$ is not a local ring. By [8, Corollary 19.19], $R$ has a non-trivial idempotent, say $e$. We have

$$R = eR e \oplus eR (1-e) \oplus (1-e) Re \oplus (1-e) R(1-e).$$

(*)

If $(1-e) R (1-e) = \{0\}$, then $((1-e) Re)(eR(1-e)) = \{0\}$ and so every three vertices of $\{e\} \cup eR(1-e)^* \cup (1-e) Re^*$ are mutually adjacent. This implies that the set $eR(1-e)^* \cup (1-e) Re^*$ has at most one vertex. Since $e \neq 1$, by (*) we conclude that this set is not empty. So without loss of generality, let $eR(1-e) = \{0, x\}$ and $(1-e) Re = \{0\}$. Since $R = eR e \oplus eR (1-e)$, we have $xR = \{0\}$. This yields that $x$ is adjacent to all other vertices of $\overline{\Gamma}(R)$. Hence again either $\overline{\Gamma}(R)$ has a cycle of length 3 or $\overline{\Gamma}(R)$ is a star graph. Thus we have $(1-e) R (1-e) \neq \{0\}$. Since the vertices of $\{e\}$, $(1-e) R (1-e)^*$ and $eR(1-e)^* \cup (1-e) Re^*$ are mutually adjacent, we have $eR(1-e) = (1-e) Re = \{0\}$. Therefore by (*), we can write $\overline{\Gamma} \simeq \overline{\Gamma}_1 \times \overline{\Gamma}_2$, where $\overline{\Gamma}_1 = eRe$ and $\overline{\Gamma}_2 = (1-e) R(1-e)$. If $|\mathcal{D}(R_1)| \geq 3$, then by [9, Theorem 3.2], $\overline{\Gamma}(R_1)$ is a connected graph with at least two vertices. So there are two distinct elements $u, v \in \mathcal{D}(R_1)^*$ such that $uv = 0$. Now if $w$ is an arbitrary element of $R_2$, then the vertices $(u, 0)$, $(v, 0)$ and $(0, w)$ of $\overline{\Gamma}(R)$ are mutually adjacent, a contradiction. This implies that $|\mathcal{D}(R_1)| \leq 2$ and $|\mathcal{D}(R_2)| \leq 2$. If there are two elements $a \in \mathcal{D}(R_1)^*$ and $b \in \mathcal{D}(R_2)^*$, then $a^2 = b^2 = 0$. So the vertices $(a, 0)$, $(0, b)$ and $(a, b)$ are mutually adjacent, a contradiction. Thus at least one of the rings $R_1$ and $R_2$ is domain. If both $R_1$ and $R_2$ are domains, then $R$ is the direct product of two division rings. Otherwise we may assume that $|\mathcal{D}(R_1)| = 1$ and $|\mathcal{D}(R_2)| = 2$. Now $R_1$ is a division ring and by Corollary 4 and Theorem 9, $R_2$ is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_2[x]/(x^2)$, as desired. □

Theorem 13. Let $R$ be a ring which is not a domain and $\overline{\Gamma}(R)$ has no cycle. If $\overline{\Gamma}(R)$ is not a star graph, then either $R \simeq \mathbb{Z}_2 \times \mathbb{Z}_4$ or $R \simeq \mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$.

Proof. Suppose that $\overline{\Gamma}(R)$ is not a star graph. So there exists a path $a = x = y = b$ in $\overline{\Gamma}(R)$ of length 3. Since two vertices $a$ and $y$ are not adjacent, we have $x^2 \notin \{a, y\}$. If $x^2 \notin \{0, x\}$, then $a = x = y = x^2 = a$ is a cycle in $\overline{\Gamma}(R)$, a contradiction. Hence $x^2 \in \{0, x\}$ and similarly $y^2 \in \{0, y\}$. If $x^2 = y^2 = 0$, then $x = y = (x + y) = x$ is a cycle in $\overline{\Gamma}(R)$, a contradiction. Thus either $x^2 = x$ or $y^2 = y$. Hence $R$ has a non-trivial idempotent. Now using the second paragraph of the proof of Theorem 12, $R$ is isomorphic to either the direct product of two domains or the direct product of a domain and one of the rings $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$. Since $\overline{\Gamma}(R)$ has no cycle and is not a star graph, the assertion clearly holds. □

3. The zero-divisor graphs of matrix rings

We begin this section with the following lemma.

Lemma 14. Let $F$ be a finite field and $n \geq 2$. Suppose $a \in M_n(F)$ is a non-zero matrix and rank $a = k < n$. Then the in-degree and the out-degree of $a$ in $\Gamma(M_n(F))$ are $|F|^{n(n-k)} - \varepsilon$, $|F|^{n(n-k)} - \varepsilon$. 

and the degree of a in $\overline{Γ}(M_n(F))$ is equal to $2|F|^{n(n-k)} - |F|^{(n-k)^2} - \varepsilon$, where $\varepsilon = 1$, unless $a^2 = 0$ and in this case $\varepsilon = 2$.

**Proof.** It is well known that there exist two invertible matrices $u$ and $v$ in $M_n(F)$ such that $uav = \sum_{i=1}^k E_{ii}$. We have, $(uav)x = 0$ (respectively, $x(uav) = 0$), for some $x \in M_n(F)$, if and only if the first $k$ rows (respectively, columns) of $x$ are zero. Hence $\dim_F \Ann_\ell(a) = \dim_F \Ann_\varepsilon(uav) = n(n-k)$ and $\dim_F \Ann_r(a) = \dim_F \Ann_\varepsilon(uav) = n(n-k)$. Thus the in-degree and the out-degree of $a$ in $Γ(M_n(F))$ are $|F|^{n(n-k)} - \varepsilon$, where $\varepsilon = 1$, unless $a^2 = 0$ and in this case $\varepsilon = 2$. Moreover, clearly

$$\dim_F \Ann_\ell(a) \cap \Ann_r(a) = \dim_F \Ann_\varepsilon(uav) \cap \Ann_r(uav) = (n-k)^2.$$  

So $|\Ann_\ell(a) \cup \Ann_r(a)| = |F|^{n(n-k)} + |F|^{n(n-k)} - |F|^{(n-k)^2}$. Therefore the degree of $a$ in $\overline{Γ}(M_n(F))$ equals $2|F|^{n(n-k)} - |F|^{(n-k)^2} - \varepsilon$, where $\varepsilon = 1$, unless $a^2 = 0$ and in this case $\varepsilon = 2$. \qed

Next we show that for a finite semisimple ring which is not a field, the ring $R$ is determined by the graph $\overline{Γ}(R)$.

**Remark 15.** Now we want to calculate the maximum degree and the minimum degree of $\overline{Γ}(M_n(F))$, for any finite field $F$ and any $n \geq 2$. By Lemma 14, the rank of any vertex with the maximum degree in $\overline{Γ}(M_n(F))$ is 1. Since clearly there is a matrix with rank 1 whose square is not zero, the maximum degree of $\overline{Γ}(M_n(F))$ is $2|F|^{n(n-1)} - |F|^{(n-1)^2} - 1$. Now let $a$ be a vertex with the minimum degree in $\overline{Γ}(M_n(F))$. By Lemma 14, we have rank $a = n-1$. On the other hand we know that the rank of any matrix $x \in M_n(F)$ with $x^2 = 0$, is not more than $n/2$. Hence if $n \geq 3$, then we conclude that $a^2 \neq 0$ and by Lemma 14 the degree of $a$ is $2|F|^n - |F| - 1$. Suppose that $n = 2$. Obviously by Lemma 14 the degree of any non-zero nilpotent matrix $x \in M_2(F)$ as a vertex of $\overline{Γ}(M_2(F))$ is $2|F|^2 - |F| - 2$. So in this case we have $a^2 = 0$ and the degree of $a$ is $2|F|^n - |F| - 2$.

**Theorem 16.** Let $R$ and $S$ be two finite semisimple rings which are not fields. If $\overline{Γ}(R) \simeq \overline{Γ}(S)$, then $R \simeq S$.

**Proof.** By the Wedderburn–Artin Theorem, we can write $R = M_{n_1}(F_1) \times \cdots \times M_{n_r}(F_r)$ and $S = M_{m_1}(K_1) \times \cdots \times M_{m_s}(K_s)$, where $F_1, \ldots, F_r$ and $K_1, \ldots, K_s$ are finite fields and $n_1, \ldots, n_r$ and $m_1, \ldots, m_s$ are natural numbers. Clearly we may assume that $r \geq s$. Indeed each vertex in $\overline{Γ}(R)$ with minimum degree has exactly one non-unit component. Without loss of generality, suppose that $a = (a_1, u_2, \ldots, u_r)$ is a vertex with minimum degree in $\overline{Γ}(R)$, where $a_1$ is a zero-divisor and $u_i$’s are units. Also we may assume that $a' = (a'_1, u'_2, \ldots, u'_r)$ is the vertex of $\overline{Γ}(S)$ corresponding to $a$, where $a'_1$ is a zero-divisor and $u'_i$’s are units. We claim that $n_1 = m_1$ and $F_1 \simeq K_1$.

If $n_1 = 1$, then $a_1 = 0$ and the set of all neighbors of $a$, that is $\{(u_1, 0, \ldots, 0) | u_1 \in F_i^\ast\}$, is independent. Suppose that $m_1 \geq 2$. Clearly the set of all neighbors of $a'$ is $\{(z_1, 0, \ldots, 0) | z_1 \in (\Ann_\ell(a'_1) \cup \Ann_r(a'_1))^\ast\}$. We know that there are two invertible matrices $p$ and $q$ such that the $m_1$th row of $pa'_1$ and the $m_1$th column of $a'_1q$
are 0. Assume that the \( i \)th column of \( a'_i \) is non-zero. Now for an index \( j \neq i \), we have \( E_{im_1}(pa'_i) = 0 \) and \( (a'_i')qE_{mj} = 0 \). Since \( a'_i(E_{im_1}p) \neq 0 \), we conclude that \( E_{im_1}p \neq qE_{mj} \). But \( (qE_{mj}')(E_{im_1}p) = 0 \), so obviously we can find two neighbors of \( a' \) which are adjacent. This contradiction implies that \( m_1 = 1 \). Since the degree of \( a \) is \( |F_1| - 1 \) and the degree of \( a' \) is \( |K_1| - 1 \), we conclude that \( |F_1| = |K_1| \) and \( F_1 \cong K_1 \).

So assume that \( n_1 \geq 2 \) and \( m_1 \geq 2 \). First suppose that \( r = s = 1 \). If \( n_1 = 2 \), then by Remark 15, the difference between the maximum degree and the minimum degree of \( \overline{T}(R) \) is 1. Since \( \overline{T}(R) \cong \overline{F}(S) \), by Remark 15 we conclude that \( m_1 = 1 \). Now by considering the maximum degree of two graphs \( \overline{T}(R) \) and \( \overline{F}(S) \), we find that \( 2|F_1|^2 - |F_1| - 1 = 2|K_1|^2 - |K_1| - 1 \). This implies that \( |F_1| = |K_1| \) and so \( F_1 \cong K_1 \). Thus we can assume that \( n_1 \geq 3 \) and \( m_1 \geq 3 \). By Remark 15 we have \( a'_i \neq 0 \) and so in this case the degree of \( a \) is \( 2|F_1|^{|n_1|} - |F_1| - 1 \). Similarly we find that the degree of \( a' \) is \( 2|K_1|^{|m_1|} - |K_1| - 1 \). Thus

\[
2|F_1|^{|n_1|} - |F_1| = 2|K_1|^{|m_1|} - |K_1|.
\]

Moreover, by Remark 15 the maximum degree of the vertices of \( \overline{T}(R) \) is \( 2|F_1|^{|n_1(n_1-1)|} - |F_1|^{|n_1(-1)|^2} - 1 \) and the maximum degree of the vertices of \( \overline{F}(S) \) is \( 2|K_1|^{|m_1(m_1-1)|} - |K_1|^{|m_1-1|^2} - 1 \). This yields that \( 2|F_1|^{|n_1(n_1-1)|} - |F_1|^{|n_1-1|^2} = 2|K_1|^{|m_1(m_1-1)|} - |K_1|^{|m_1-1|^2} \).

Now since \( |F_1| \) and \( |K_1| \) are prime powers, using Eq. (\(*\)), it is not hard to see that \( n_1 = m_1 \) and \( |F_1| = |K_1| \) and so \( F_1 \cong K_1 \).

Next suppose that \( r \geq 2 \). Since the set of all neighbors of \( a \) is \( \{(z_1, 0, \ldots, 0) \mid z_1 \in (\text{Ann}_t(a_1) \cup \text{Ann}_x(a_1))^\ast \} \), the degree of \( a \) is \( 2|F_1|^{|n_1|} - |F_1| - 1 \). Clearly in this case (\(*\)) holds. Assume that \( A \) is the set of all vertices with the minimum degree of \( \overline{T}(R) \) that have at least one common neighbor with \( a \). Since each pair of the vertices in \( \overline{T}(M_{n_1}(F_1)) \) has at least one common neighbor [12, Example 2.8], \( A \) is the set of all vertices whose first components are vertices with the minimum degree of \( \overline{T}(M_{n_1}(F_1)) \) and their other components are units. Let \( B \) be the set of all vertices of \( \overline{T}(R) \) that are adjacent to at least one vertex of \( A \). Since clearly every vertex of \( \overline{T}(M_{n_1}(F_1)) \) is adjacent to a vertex with the minimum degree in \( \overline{T}(M_{n_1}(F_1)) \), we obtain that \( B \) is the set of all vertices with the property that each vertex has a non-zero zero-divisor as its first component and all its other components are 0. Moreover, the vertices with the minimum degree among all vertices adjacent to all vertices of \( B \) have the form \( (0, v_2, \ldots, v_r) \), where \( v_j \)'s are units. Let \( C \) denote the set of these vertices. Since \( C \) is not empty and \( \overline{T}(R) \cong \overline{F}(S) \), we have \( s \geq 2 \). Moreover, the degree of each vertex of \( C \) is \( |F_1|^{|n_1|^2} - 1 \) and by the fact \( \overline{T}(R) \cong \overline{F}(S) \), we conclude that \( |F_1|^{|n_1|^2} - 1 = |K_1|^{|m_1|^2} - 1 \). Again since \( |F_1| \) and \( |K_1| \) are prime powers, by (\(*\)) we have \( n_1 = m_1 \) and \( |F_1| = |K_1| \). So \( F_1 \cong K_1 \) and the claim is proved.

Now let \( D \) be the set of all vertices not in \( C \) and adjacent to all vertices of \( B \). Indeed, \( D \) is the set of all vertices with the property that each vertex has zero as its first component and at least one of its other components is zero-divisor. Assume that \( R_1 = M_{n_2}(F_2) \times \cdots \times M_{n_r}(F_r) \) and \( S_1 = M_{m_2}(K_2) \times \cdots \times M_{m_r}(K_r) \). Now it is easily checked that the induced subgraph on \( D \) is isomorphic to \( \overline{T}(R_1) \). Again since \( \overline{T}(R) \cong \overline{F}(S) \), we conclude that \( \overline{T}(R_1) \cong \overline{T}(S_1) \). If \( R_1 \) is a field, then \( S_1 \) is also a field. In this case, the set \( C \) has \( |F_1|-1 \) vertices and by \( \overline{T}(R) \cong \overline{T}(S) \), we obtain \( R_1 \cong S_1 \). Otherwise, by induction on \( r \), we have \( R_1 \cong S_1 \), and this completes the proof. □
In [2] it has been proved that for any finite commutative ring $R$ with identity, with finitely many exceptions, if $\overline{\Gamma}(R)$ is isomorphic to the zero-divisor graph of a reduced ring $S$, then $R \simeq S$. Now we generalize it to any arbitrary ring.

**Theorem 17.** Let $R$ and $S$ be two finite rings which are not fields. If $S$ is reduced and $\overline{\Gamma}(R) \simeq \overline{\Gamma}(S)$, then $R \simeq S$, except in the following cases.

(i) $S \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and either $R$ is isomorphic to $\mathbb{Z}_3[x]/(x^2)$ or the null ring of order 3;
(ii) $S \simeq \mathbb{Z}_6$ and $R$ is isomorphic to one of the rings $\mathfrak{R}$, $\mathfrak{R}^{op}$, $\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x, x^2 - 2)$ or $2\mathbb{Z}_8 = \{0, 2, 4, 6\}$;
(iii) $S \simeq \mathbb{Z}_2 \times \mathbb{F}_{p^n}$ and $R$ is isomorphic to the direct product of the null ring of order 2 and $\mathbb{F}_{q^m}$, where $p$ and $q$ are primes and $p^n = 2q^m - 1$.

**Proof.** Since $S$ is reduced, by the Wedderburn–Artin Theorem, there are finite fields $F_1, \ldots, F_s$ such that $S \simeq F_1 \times \cdots \times F_s$. Clearly for any $x \in D(S)^*$, there exists $x' \in D(S)^*$ such that $xx' = 0$ and $x + x'$ is a unit. This yields that the vertices $x$ and $x'$ in $\overline{\Gamma}(S)$ have no common neighbor. If $R$ is a reduced ring, then $R$ is a semisimple ring and Theorem 16 implies that $R \simeq S$. So assume that there exists $a \in D(R)^*$ such that $a^2 = 0$. Since $\overline{\Gamma}(R) \simeq \overline{\Gamma}(S)$, there exists a vertex $b$ adjacent to $a$ such that $a$ and $b$ have no common neighbor.

First suppose that $ab = 0$ and $ba \neq 0$. Assume that $b^j = 0$ and $b^{j-1} \neq 0$ for some integer $j$. Since $ba \neq 0$, then $a + b^{j-1} \notin \{0, b\}$. So the vertex $a + b^{j-1}$ is adjacent to both $a$ and $b$, a contradiction. Hence $b$ is not nilpotent. Since $R$ is finite, there exist two integers $m > n$ such that $b^m = b^n$. Assume that $n$ is the smallest possible positive integer with this property. If $n \geq 2$, then $b^{m-1} = b^{n-1}$ is a common neighbor of $a$ and $b$, a contradiction. So $n = 1$ and hence $e = b^{m-1}$ is a non-trivial idempotent. Since $b$ and $e$ have the same neighbors, $a$ and $e$ have no common neighbor. Furthermore since $ea \in \text{Ann}_r(\langle a, b \rangle)$, we have $ea = a$. Now noting that $ae = 0$, we find that $a \in eR(1 - e) \subseteq \text{Ann}_r(\langle a, e \rangle)$ and thus $aR(1 - e) = \{0\}$. This implies that $1 - e \in \text{Ann}_r(a) \cap \text{Ann}_r(e)$. Since $a$ and $e$ have no common neighbor, $(1 - e)Re = \{0\}$. This yields that

$$R = eRe \oplus eR(1 - e) \quad (*)$$

and so $a$ is adjacent to all vertices of $\overline{\Gamma}(R)$. Hence by [4, Theorem 2.5], $\overline{\Gamma}(S)$ and therefore $\overline{\Gamma}(R)$ is a star graph. We have $ab \neq ba$, so $R$ is a non-commutative ring. Now Theorem 9 and the Wedderburn’s Little Theorem [8, p. 203] yield that $\overline{\Gamma}(R)$ has at most three vertices. By $(*)$, $e$ is a left identity which is a zero-divisor and hence $R$ has no identity. Also by Lemma 2, we conclude that $|R| \leq 4$. Thus we have $R = \{0, e, a, e + a\}$ and it is not hard to verify that $R \simeq \mathfrak{R}$. So $\overline{\Gamma}(R)$ is a star graph with three vertices. By [3, Example 2.1(a)], since $S$ is reduced, we have $S \simeq \mathbb{Z}_6$. Similarly, if $ba = 0$ and $ab \neq 0$, then $R \simeq \mathfrak{R}^{op}$ and again $S \simeq \mathbb{Z}_6$.

Next suppose that $ab = ba = 0$. Assume that $b^k = 0$ and $b^{k-1} \neq 0$, for some integer $k$. If $k = 2$, then $\langle a - b, b - a \rangle \subseteq \text{Ann}_r(\langle a, b \rangle)$ and so we conclude that $a - b = b$ and $b - a = a$. Since $\overline{\Gamma}(R)$ is a connected graph [9, Theorem 3.2] and $a$ and $b$ have no common neighbor,
if $\overline{\Gamma}(R)$ has more than two vertices, then there is a vertex $x$, adjacent to exactly one of the vertices $a$ and $b$, and $x \neq a, b$. But $a = 2b$ and $b = 2a$, a contradiction. Thus $\overline{\Gamma}(R)$ has exactly two vertices and by Remark 1, we have $|R| \leq 9$. If $|R| = 3$, then $R$ is the null ring of order 3. If $|R| \geq 4$, then noting that $|\mathcal{D}(R)| = 3$, by Lemma 2, $R$ has identity. Since $3a = 0$ and $\mathcal{D}(R)^2 = \{0\}$, we obtain that $|R| = 9$. Clearly, the subring generated by $a$ and 1 is equal to $R$ and so $R$ is commutative. Hence by [3, Example 2.1(a)], $R$ is isomorphic to $\mathbb{Z}_9$ or $\mathbb{Z}_3[x]/(x^2)$ and $S \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

So assume that $k \geq 3$. Since $b^{k-1}a = b^{k-1}b = 0$, we have $a = b^{k-1}$. This implies that $a$ is the only vertex adjacent to $b$. Now we claim that for any two vertices $u, v$ in $\overline{\Gamma}(S)$, if $v$ is adjacent to $u$ and all other neighbors of $u$, then $u$ is a vertex of degree 1. To prove this, suppose that there exists a vertex $w \neq v$ adjacent to $u$. The vertex $v - w$ is adjacent to $u$, so by assumption, $v$ is adjacent to both $w$ and $v - w$ and this implies that $v^2 = 0$. But since $S$ is a reduced ring, we reach a contradiction. This proves the claim. Since $\overline{\Gamma}(R) \simeq \overline{\Gamma}(S)$, the above argument shows that for any two vertices $x, y$ in $\overline{\Gamma}(R)$, if $y$ is adjacent to $x$ and all other neighbors of $x$, then $x$ is a vertex of degree 1. Note that $a = b^{k-1}$ is adjacent to $b^{k-1} + b^{k-2}$ and its all other neighbors. Thus by the above argument $a = b^{k-1}$ is the only vertex adjacent to $b^{k-1} + b^{k-2}$. We have $b^2(b^{k-1} + b^{k-2}) = 0$. Now if $k \geq 4$, then since $ab = 0$ and $k$ is the smallest integer which $b^k = 0$, $a \neq b^2$ and so we conclude that $b^{k-1} + b^{k-2} = b^2$. Multiplying this equation by $b^2$ yields that $k = 4$. But we have $b^3 + b^2 = b^2$, a contradiction. Therefore $k = 3$. Suppose $az = 0$, for some $z \in R \setminus \{0, a, b\}$. Since $a$ and $b$ have no common neighbors, $bz \neq 0$. On the other hand $a(bz) = b(bz) = 0$. Therefore, since $b^2 \neq 0$, we have $bz = a = b^2$. Hence $b(z - b) = 0$ and since $a(z - b) = 0$, we conclude that $z - b = a$. Thus $\text{Ann}_r(a) = \{0, a, b, a + b\}$. By a similar argument we obtain that $\text{Ann}_r(a) = \{0, a, b, a + b\}$. Since $bR$ is an additive subgroup of $\text{Ann}_r(a)$, $|bR| = 2$ or 4.

Furthermore since $a = b^2$, we have $\text{Ann}_r(b) = \{0, a\}$. If $|bR| = 2$, then $|R| = 4$ and so $R = \{0, a, b, a + b\}$. Clearly in this case we have $R \simeq 2\mathbb{Z}_8$. So assume that $|bR| = 4$. This implies that $|R| = 8$. Now since $a = b^2$, $b^3 = 0$ and $a$ and $b$ have no common neighbor, it is easy to see that $a$ is the only vertex adjacent to $a + b$. So $a$ is a vertex with degree 2 which is adjacent to two vertices of degree 1. Now since $\overline{\Gamma}(R)$ is a connected graph [9, Theorem 3.2], we conclude that $\overline{\Gamma}(R)$ is a star graph with three vertices. Since $|\mathcal{D}(R)| = 4$, by Lemma 2, $R$ has identity. If $R$ is non-commutative, then by [7] $R$ is isomorphic to an upper triangular matrix ring over $\mathbb{Z}_2$ and therefore $R$ has five zero-divisors, a contradiction. Thus $R$ is a commutative ring with identity. Hence by [3, Example 2.1(a)], in this case $R$ is isomorphic to one of the rings $\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, or $\mathbb{Z}_4[x]/(2x, x^2 - 2)$ and $S \simeq \mathbb{Z}_6$.

Now suppose that $b$ is not nilpotent. Since $R$ is finite, there exist integers $m > n$ which $b^m = b^n$. Assume that $n$ is the smallest possible positive integer with this property. If $n \geq 2$, then since $b^{m-1} - b^{n-1} \in \text{Ann}_r(\{a, b\})$ and $b^2 \neq 0$, we conclude that $b^{m-1} = b^{n-1} = a$. Thus $a$ is adjacent to $b^{n-1}$ and all neighbors of $b^{n-1}$. This implies that $a$ is the only vertex adjacent to $b^{n-1}$. But $b^{n-1}$ is also adjacent to $b^{m-n+1} - b$. Thus $b^{m-n+1} - b = a = b^{n-1} - b^{n-1}$. So $b^{m-n+2} = b^2$ and by the minimality of $n$, we obtain that $n = 2$. Now we claim that $a$ and $b^2$ have no common neighbor. Assume that there is a vertex $y \neq a$ in $\overline{\Gamma}(R)$ such that $b^2y = 0$. Since $a$ is the only vertex adjacent to $b$, $by \neq 0$ and $yb \neq 0$. Also since $a(by) = 0$ and $a$ and $b$ have no common neighbor and $b^2 \neq 0$, we conclude that $by = a$ and so $ay = (b^{m-1} - b)y = -a \neq 0$. On the other hand $b(yb) = ab = 0$. Again, since $(yb)a = 0$ and $a$ and $b$ have no common neighbor and $b^2 \neq 0$, we have $yb = a$. This
implies that \( yb^2 = ab = 0 \) and hence \( ya = y(b^{m-1} - b) = -a \neq 0 \). Also if there is a vertex \( y' \neq a \) such that \( y'b^2 = 0 \), then by a similar argument we find that the both \( ay' \) and \( y'a \) are non-zero. This proves the claim. Now let \( g = b^{m-1} \), if \( n = 1 \); and \( g = b^2m^{-4} \), if \( n = 2 \). Clearly \( g \) is a non-trivial idempotent. As \( b^n g = gb^n = b^n \), it is easily checked that \( \operatorname{Ann}_R(g) = \operatorname{Ann}_R(b^n) \) and \( \operatorname{Ann}_R(g) = \operatorname{Ann}_R(b^n) \). So we conclude that \( a \) is the only vertex adjacent to \( g \). Therefore, noting that \( ag = ga = 0 \), we obtain \( gR(1 - g) = (1 - g)Rg = \{0\} \) and \( (1 - g)R(1 - g) = \{0, a\} \). Hence \( R = gRg \oplus \{0, a\} \) and \( a \) is adjacent to all vertices of \( \Gamma(R) \). Thus by [4, Theorem 2.5], \( \Gamma(S) \) and therefore \( \Gamma(R) \) is a star graph. So by Theorem 9 and noting that \( S \) is reduced, we find \( S \cong \mathbb{Z}_2 \times \mathbb{F}_{p^n} \) and \( R \) is isomorphic to the direct product of the null ring of order 2 and \( \mathbb{F}_{q^m} \), where \( n, m \) are natural numbers and \( p, q \) are prime numbers. Now by considering the number of the vertices of \( \Gamma(R) \) and \( \Gamma(S) \), we find that \( p^n = 2q^m - 1 \) and so case (iii) occurs and the proof is complete. \( \square \)

Now by combining the previous theorem, Theorems 9 and 12, we are able to determine all finite rings \( R \) for which the graph \( \Gamma(R) \) is bipartite.

**Corollary 18.** Two rings \( R \) and \( R^{\text{op}} \) are the only finite non-commutative rings whose \( \Gamma(R) \) and \( \Gamma(R^{\text{op}}) \) are bipartite.

**Remark 19.** We would like to determine all rings whose zero-divisor graphs have at most four vertices. First assume that \( \Gamma(R) \) has at most three vertices. Corollary 4 characterizes all rings whose zero-divisor graphs have exactly one vertex. Since \( \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2) \) and \( \Gamma(\mathbb{Z}_6) \) are star graphs with two and three vertices, respectively, so Theorem 17 determines all rings whose zero-divisor graphs are a star with at most three vertices. Thus it is enough to assume that \( \Gamma(R) \) is the complete graph with three vertices. By Theorem 5, we have \( D(R)^2 = \{0\} \). If \( R = D(R) \), then \( R \) is a null ring of order 4; otherwise using Lemma 2, \( R \) is a local ring. By [3, Example 2.1(a)], if \( R \) is a commutative ring with identity such that \( \Gamma(R) \) is a complete graph with three vertices, then \( R \) is isomorphic to one of the rings \( \mathbb{F}_4[x]/(x^2), \mathbb{Z}_4[x]/(x^2 + x + 1), \mathbb{Z}_4[x]/(2, x)^2 \), or \( \mathbb{Z}_4[x, y]/(x, y)^2 \). Hence we should determine all non-commutative local rings \( R \) with \( |D(R)| = 4 \). Using Remark 1, we have \( |R| \leq 16 \). Since \( R \) has identity and \( D(R) \) is a nilpotent ideal of order 4, \( |R| \) is a power of 2. So \( |R| = 8 \) or 16. If \( |R| = 8 \), then since \( D(R)^2 = \{0\} \), the subring generated by \( D(R) \) is equal to \( R \) which contradicts the non-commutativity of \( R \). Therefore \( |R| = 16 \). Since \( R \) is a local ring, \( D(R) \) is the Jacobson radical of \( R \) and \( R/\overline{D(R)} = \mathbb{F}_4 \). Let \( \mathcal{F} \) be a subset with four elements of \( R \) such that \( 0 \in \mathcal{F} \) and \( R/\overline{D(R)} = \{u + \overline{D(R)} \mid u \in \mathcal{F}\} \). If \( z \) is an element of \( D(R)^* \), then it is easy to see that \( R = \{u + vz \mid u, v \in \mathcal{F}\} \). This implies that for each \( t \in \mathcal{F} \), there exists an element \( \sigma(t) \in \mathcal{F} \) such that \( zt = \sigma(t)z \). We show that \( \sigma \) induces a non-trivial automorphism of \( R/\overline{D(R)} = \mathbb{F}_4 \). For two elements \( t_1, t_2 \in \mathcal{F} \), we have \( \sigma(t_1 + t_2)z = z(t_1 + t_2) = (\sigma(t_1) + \sigma(t_2))z \). Thus \( \sigma(t_1 + t_2) - (\sigma(t_1) + \sigma(t_2)) \in D(R) \) and so \( \sigma((t_1 + \overline{D(R)}) + (t_2 + \overline{D(R)})) = \sigma(t_1 + t_2 + \overline{D(R)}) = (\sigma(t_1) + \overline{D(R)}) + (\sigma(t_2) + \overline{D(R)}) \). Similarly one can see that \( \sigma \) is a multiplicative homomorphism. Also if \( \sigma(t) \in D(R) \) for some \( t \in \mathcal{F} \), then \( \sigma(t) = 0 \) and thus \( t = 0 \). This implies that \( \sigma \) is injective and therefore it is an automorphism of \( R/\overline{D(R)} \), as desired. Since \( R \) is a non-commutative ring and \( \mathbb{F}_4 \) has
exactly two automorphisms, we conclude that \( \sigma(t) = t^2 \) for any \( t \in \mathbb{F} \). Now consider the map \( \phi : R \to \frac{R}{D(R)} \) defined by

\[
\phi(u + vz) = \begin{bmatrix} u + D(R) & v + D(R) \\ 0 + D(R) & u^2 + D(R) \end{bmatrix}.
\]

It is not hard to see that \( R \) is isomorphic to \( \mathbb{F}_4 \).

Now suppose that \( \Gamma(R) \) has four vertices. By Theorem 3, if \( R \) is not a reduced ring, then either \( R \) is isomorphic to one of the rings \( \mathbb{Z}_{25} \), \( \mathbb{Z}_5[x]/(x^2) \) or the null ring of order 5. So assume that \( R \) is reduced. If \( R \) is a direct product of three or more fields, then \( |D(R)| \geq 7 \).

Thus \( R \) is a direct product of two finite fields. Now since \( \Gamma(R) \) has four vertices, it is easily checked that \( R \) is isomorphic to one of the rings \( \mathbb{Z}_2 \times \mathbb{F}_4 \) or \( \mathbb{Z}_3 \times \mathbb{Z}_3 \). Therefore we have determined all rings whose zero-divisor graphs have at most four vertices.

Now we want to prove that if \( R \) is a full matrix ring over a finite field, then \( R \) is determined by its zero-divisor graph.

**Theorem 20.** Let \( F \) be a finite field and \( n \geq 2 \). If \( R \) is a ring such that \( \Gamma(R) \cong \Gamma(M_n(F)) \), then \( R \cong M_n(F) \).

**Proof.** Suppose that \( \Omega \) is an arbitrary clique which is maximal among all cliques containing the vertices of the maximum in-degree in \( \Gamma(M_n(F)) \). By Lemma 14, any element of \( \Omega \) has rank 1 and its square is non-zero, so \( \omega^2 = \text{trace}(\omega)\omega \neq 0 \), for any \( \omega \in \Omega \). Hence, noting that the elements of \( \Omega \) are pairwise commuting, \( \Omega \) is simultaneously diagonalizable.

So there is an invertible matrix \( h \) such that for each \( \omega \in \Omega \), \( h\omega h^{-1} = \lambda E_{ii} \), for some \( i \), \( 1 \leq i \leq n \), and \( \lambda \in F^* \), where \( i \) and \( \lambda \) depend on \( \omega \). Since the product of any two matrices of \( \Omega \) is zero, by the maximality of \( \Omega \), there are elements \( \lambda_1, \ldots, \lambda_n \in F^* \) such that \( h\Omega h^{-1} = \{ \lambda_1 E_{11}, \ldots, \lambda_n E_{nn} \} \). Moreover since \( \Gamma(R) \cong \Gamma(M_n(F)) \), every clique \( X \) in \( \Gamma(R) \) which is maximal among all cliques containing the vertices of maximum in-degree has the following properties:

(i) For any vertex \( a \in X \), there is a vertex \( u \notin X \) such that \( u \to x \) is an edge of \( \Gamma(R) \), for each \( x \in X \setminus \{a\} \), and the directed edge \( u \to a \) is not in \( \Gamma(R) \);
(ii) For any vertices \( x \in X \) and \( x' \notin X \), if \( (X \setminus \{x\}) \cup \{x'\} \) is a clique of \( \Gamma(R) \), then \( u \to x \) is an edge if and only if \( u \to x' \) is an edge, for any vertex \( u \neq x, x' \);
(iii) There is no vertex \( u \) such that \( x \to u \) is an edge of \( \Gamma(R) \), for all \( x \in X \), and there is no vertex \( v \) such that \( v \to x \) is an edge of \( \Gamma(R) \), for all \( x \in X \).

Now we claim that there is a clique which is maximal among all cliques containing the vertices of maximum in-degree of \( \Gamma(R) \) whose elements are idempotent. Let \( A \) be an arbitrary clique which is maximal among all cliques containing the vertices of maximum
in-degree in $\Gamma(R)$. We show that $A$ has no nilpotent element. Assume that there is a nilpotent element $a \in A$ and suppose that $m$ is the smallest positive integer such that $a^m = 0$. By the maximality of $A$, $a^{m-1} \in A$. Suppose that $a^{m-1} \neq a$. For any vertex $u \neq a^{m-1}$ of $\Gamma(R)$, if $u \to a$ is an edge, then $u \to a^{m-1}$ is also an edge, which contradicts (i). Thus $a^{m-1} = a$ and so $a^2 = 0$. Assume that $b$ is an element of $A \setminus \{a\}$. By (i), we have $a + b \neq 0$.

Now for any vertex $u \neq a + b$, if $u \to x$ is an edge of $\Gamma(R)$, for each $x \in A \setminus \{a + b\}$, then $u \to a + b$ is also an edge. Hence (i) implies that $a + b \notin A$. On the other hand $(A \setminus \{b\}) \cup \{a + b\}$ is a clique of $\Gamma(R)$. Therefore by (ii), $u \to a + b$ is an edge if and only if $u \to b$ is an edge, for any vertex $u \notin \{a + b, b\}$. This implies that if $u \to b$ is an edge, then $u \to a$ is an edge, for each vertex $u \neq a$, which contradicts (i). So none of the elements of $A$ is nilpotent. Now since $\Gamma(R)$ is a finite graph, $R$ is finite and hence for any $t \in A$ there is a positive integer $m(t)$ such that $t^{m(t)}$ is a non-trivial idempotent, see [11, p. 55]. By the maximality of in-degree of $t$, $\text{Ann}_t(t) = \text{Ann}_t(t^{m(t)})$ and thus $t^{m(t)}$ is a vertex with the maximum in-degree of $\Gamma(R)$. Thus $\{t^{m(t)} | t \in A\} = \{e_{11}, \ldots, e_{nn}\}$ is a clique which is maximal among all cliques containing the vertices of the maximum in-degree in $\Gamma(R)$ whose elements are idempotents, so the claim is proved.

Suppose that there is $z \in D(R)^*$ such that $(e_{11} + \cdots + e_{nn})z = 0$. Multiplying this equation by $e_{ii}$, we have $e_{ii}z = 0$, for any $i$, $1 \leq i \leq n$. Thus $e_{ii} \to z$ is an edge in $\Gamma(R)$, for any $i$, which contradicts (iii). Similarly, we observe that $\text{Ann}_e(e_{11} + \cdots + e_{nn}) = \{0\}$. Since $e_{11} + \cdots + e_{nn}$ is an idempotent element which is not a zero-divisor, $1 = e_{11} + \cdots + e_{nn}$ is the identity of $R$.

To complete the proof, we need other properties of the graph $\Gamma(M_n(F))$. Let $\Omega$ be an arbitrary clique which is maximal among all cliques containing the vertices of maximum in-degree of $\Gamma(M_n(F))$ and without loss of generality suppose that $\Omega = \{\lambda_1E_{11}, \ldots, \lambda_nE_{nn}\}$, where $\lambda_i$’s are contained in $F^*$. For any $i \neq j$, $1 \leq i, j \leq n$, assume that $W_{ij}$ is the set of all vertices $w$ such that $w \to \lambda_rE_{rr}$ and $\lambda_sE_{ss} \to w$ are edges in $\Gamma(M_n(F))$, for any $1 \leq r, s \leq n, r \neq j$, and $s \neq i$. It is not hard to see that $W_{ij} = \{\lambda_1E_{ij} | \lambda \in F^*\}$. Now, for any $i \neq j$, $1 \leq i, j \leq n$, suppose that $V_{ij}$’s are the sets with the above properties associated to the clique $\{e_{11}, \ldots, e_{nn}\}$ in $\Gamma(R)$. Since $\Gamma(R) \simeq \Gamma(M_n(F))$, $V_{ij}$’s have all of the properties of $W_{ij}$’s. Note that there is no a directed edge from $e_{ii}$ to a vertex of $V_{ij}$, and also there is no a directed edge from a vertex of $V_{ij}$ to $e_{jj}$, for any $i, j$. Furthermore for any $v_{ij} \in V_{ij}$ and $v_{kl} \in V_{kl}$, $v_{ij} \to v_{kl}$ is an edge of $\Gamma(R)$ if and only if $j \neq k$. Moreover, we show that for any distinct indices $i, j, k$ we have $V_{ij}V_{jk} \subseteq V_{ik}$. To see this, let $v_{ij} \in V_{ij}$ and $v_{jk} \in V_{jk}$ be two arbitrary elements. We know that $v_{ij}v_{jk} \neq 0$. For any $1 \leq r, s \leq n, r \neq k, s \neq i, j$, by the definitions of $V_{ij}$ and $V_{jk}$, we have $v_{jk}e_{rr} = 0$ and $e_{ss}v_{ij} = 0$. This implies that $v_{ij}v_{jk} \to e_{rr}$ and $e_{ss} \to v_{ij}v_{jk}$ are edges in $\Gamma(R)$. Thus by the definition, $v_{ij}v_{jk} \in V_{ik}$. Note that with a similar proof we find that for any $i \neq j$, the difference of two distinct elements of $V_{ij}$ is contained in $V_{ij}$.

We claim that for any $j$, $2 \leq j \leq n$, if $v_{ij}$ is a vertex in $V_{ij}$, then $e_{11}v_{ij} = v_{ij}$. By the definition, we know that for any $v_{ij} \in V_{ij}$, $e_{11}v_{ij} \in V_{ij}$. Also, if $e_{11}v_{ij} = e_{11}v_{ij}'$, for some $v_{ij}' \neq v_{ij}'$ in $V_{ij}$, then $e_{11}(v_{ij} - v_{ij}') = 0$, which is a contradiction, since $v_{ij} - v_{ij}' \in V_{ij}$. Therefore by the finiteness of $V_{ij}$, for any vertex $v_{ij} \in V_{ij}$, there is a vertex $\bar{v}_{ij}$ such that $e_{11}\bar{v}_{ij} = v_{ij}$. Thus $e_{11}v_{ij} = e_{11}(e_{11}\bar{v}_{ij}) = e_{11}\bar{v}_{ij} = v_{ij}$ and so the claim is established. For any $j$, $2 \leq j \leq n$, fix a vertex $e_{ij} \in V_{ij}$. Now with a similar proof to the above, we show that for any $i \neq j$, $2 \leq i \leq n$ and $1 \leq j \leq n$, there is a unique vertex $e_{ij} \in V_{ij}$ such
that $e_1e_{ij} = e_{1j}$. We know that for any $v_{ij} \in V_{ij}$, $e_{1i}v_{ij} \in V_{1i}$, if $e_{1i}v_{ij} = e_{1i}v_{ij}'$, for some $v_{ij} \neq v_{ij}'$ in $V_{ij}$, then $e_{1i}(v_{ij} - v_{ij}') = 0$, which is a contradiction, since $v_{ij} - v_{ij}' \in V_{ij}$. But $V_{ij}$ is a finite set, so there is a unique vertex $e_{ij} \in V_{ij}$ such that $e_{1i}e_{ij} = e_{1j}$.

Finally, we prove that $|e_{ij}|$ is a set of elements of $R$ with the properties that $e_{11} + \cdots + e_{nn} = 1$ and $e_{ij}e_{kl} = \delta_{jk}e_{il}$, for all indices $i$, $j$, $k$, $l$. For this, it is enough to show that $e_{ij}e_{jk} = e_{ik}$, for any indices $i$, $j$, $k$. As in the previous paragraph, there exists a vertex $e_{i,j} \in V_{ij}$ such that $e_{ij}e_{jk} = e_{ik}$. Thus $e_{jk} = e_{1i}e_{ik} = e_{1i}(e_{ij}e_{jk}) = e_{ij}e_{jk}$. Hence, by the uniqueness of $e_{jk}$, we conclude that $e_{jk} = e_{ik}$ and so $e_{ij}e_{jk} = e_{ik}$.

Now by [10, Proposition 1.1.3], we have $R \simeq M_n(S)$, for some finite ring $S$. To complete the proof, we prove that $S$ is a field. To get a contradiction, let $yy' = 0$, for some $y, y' \in D(S)^n$. Clearly, the in-degree of the vertex $y'E_{11}$ is more than $|S|^{n(n-1)} - 1$ and by Lemma 14, the maximum in-degree in $\Gamma(M_n(F))$ is $|F|^{n(n-1)} - 1$, so we have $|F|^{n(n-1)} - 1 > |S|^{n(n-1)} - 1$. This implies that $|F| > |S|$. Furthermore we know that the in-degree of the vertex $I - E_{11}$ is equal to $|S|^n - 1$ and by Lemma 14, the minimum in-degree in $\Gamma(M_n(F))$ is $|F|^n - 2$. This follows that $|S|^n - 1 \geq |F|^n - 2$ and so $0 < |F|^n - |S|^n \leq 1$, which is a contradiction. Hence by the Wedderburn’s Little Theorem [8, p. 203], $S$ is a field. Now by Lemma 14, the maximum in-degree of all vertices of the graphs $\Gamma(M_n(S))$ and $\Gamma(M_n(F))$ are $|F|^{n(n-1)} - 1$ and $|S|^{n(n-1)} - 1$, respectively. This yields that $|S| = |F|$ and so $S \simeq F$. Therefore $R \simeq M_n(F)$ and the proof is complete. $\square$

Now we want to generalize the previous theorem.

**Theorem 21.** Let $R$ be a ring and $S$ be a finite semisimple ring which is not a field. If $\Gamma(R) \simeq \Gamma(S)$, then $R \simeq S$, unless $R$ and $S$ are isomorphic to one of the commutative rings given in Theorem 17.

**Proof.** By the Wedderburn–Artin Theorem, we may write $S = M_{n_1}(F_1) \times \cdots \times M_{n_s}(F_s)$, where $F_i$'s are finite fields, $n_1, \ldots, n_s, s$ are natural numbers. Note that if either all $n_i$’s are equal 1 or $s = 1$, then by Theorems 17 and 20, the assertion is proved. So suppose that at least one of $n_i$’s and $s$ is more than 1. Without loss of generality, assume that $n_1 \geq 2$. Let $x$ be the vertex of $\Gamma(R)$ corresponding to the vertex $y = (I, 0, \ldots, 0)$ in $\Gamma(S)$. Since all edges incident to $y$ in $\Gamma(R)$ are multiple, $x$ has this property too. We show that $x$ is not nilpotent. To get a contradiction, assume that $x^k = 0$, where $k$ is the smallest positive integer with this property. We know that there is no vertex in $\overline{\Gamma}(S)$ adjacent to $y$ and all other neighbors of $y$. If $x^{k-1} \neq x$, then $x^{k-1}$ is adjacent to $x$ and all other neighbors of $x$ in $\overline{\Gamma}(R)$, a contradiction. Thus $x^{k-1} = x$ and so $x^2 = 0$. Moreover, since every vertex in $Rx \setminus \{0, x\}$ is adjacent to $x$ and all other neighbors of $x$, we have $Rx \subseteq \{0, x\}$. Since there is no a vertex adjacent to all other vertices of $\overline{\Gamma}(S)$, this property holds for $\overline{\Gamma}(R)$. Hence $Rx = \{0, x\}$ and so $|R : \text{Ann}_\ell(x)| = 2$. Since $x^2 = 0$, we conclude that the maximum in-degree of the vertices of $\Gamma(R)$ and the in-degree of $x$ differ at most 1. This is a contradiction, since $y$ clearly does not have this property as a vertex of $\Gamma(S)$. This shows that $x$ is not nilpotent.

Now since $R$ is finite, there is a natural number $m$ such that $e = x^m$ is a non-trivial idempotent, see [11, p. 55]. For some $z \in R$, if $ez = 0$, then $x^{(m-1)z} = 0$ and since all edges incident to $x$ in $\Gamma(R)$ are multiple, we have $(x^{m-1}z)x = 0$. By repeating this
procedure, we obtain that \(ze = zx^m = 0\). Thus we have \(\text{Ann}_r(e) \subseteq \text{Ann}_r(e)\). The converse is similarly proved and we find that \(\text{Ann}_r(e) = \text{Ann}_r(e)\). This shows that all edges incident to \(e\) in \(\Gamma(R)\) are multiple. Hence \(eR(1 - e) = (1 - e)Re = \{0\}\) and so

\[
R = eRe \oplus (1 - e)R(1 - e).
\]

(*)

Assume that \(g\) is the vertex in \(\Gamma(S)\) corresponding to \(e\). Since \(e\) is adjacent to all neighbors of \(x\) in \(\overline{\Gamma}(R)\), \(g\) is adjacent to all neighbors of \(y\) in \(\overline{\Gamma}(S)\). This yields that the \(j\)th component of \(g\) is zero, for each \(j \geq 2\). Also it is straightforward to see that for any matrix \(a \in M_{n_1}(F_1)\), the sets \(\text{Ann}_r(a) \setminus \text{Ann}_r(a)\) and \(\text{Ann}_r(a) \setminus \text{Ann}_r(a)\) are non-empty. Now since all edges incident to \(g\) in \(\Gamma(S)\) are multiple, we conclude that \(g = (g_1, 0, \ldots, 0)\), for some unit \(g_1 \in M_{n_1}(F_1)\). Let \(A_g\) be the set of all vertices in \(\overline{\Gamma}(S)\), adjacent to all neighbors of \(g\) such that their degrees are more than the degree of \(g\). Clearly the first component of every vertex of \(A_g\) is a non-zero zero-divisor and its other components are 0. Also let \(B_g\) be the set of all vertices in \(\overline{\Gamma}(S)\) adjacent to \(g\) and adjacent to at least one of the neighbors of \(g\). Hence the first component of any vertex of \(B_g\) is 0 and at least one of its other components is a zero-divisor. Now if the induced subgraphs of \(\Gamma(S)\) on \(A_g\) and \(B_g\) are denoted by \(\Gamma_{A_g}\) and \(\Gamma_{B_g}\), respectively, then we have \(\Gamma_{A_g} \simeq \Gamma(M_{n_1}(F_1))\) and \(\Gamma_{B_g} \simeq \Gamma(M_{n_2}(F_2) \times \cdots \times M_{n_1}(F_1))\).

Assume that \(A_e\) and \(B_e\) denote the sets of the vertices in \(\overline{\Gamma}(R)\) corresponding to \(A_g\) and \(B_g\), respectively. By (\(*\)), clearly \((1 - e)R(1 - e)^*\) is the set of all neighbors of \(e\) in \(\overline{\Gamma}(R)\). Suppose that \(a = a_1 + a_2\) is an arbitrary element of \(A_e\), where \(a_1 \in eRe\) and \(a_2 \in (1 - e)R(1 - e)\). So, by the definition of \(A_e\), \(a_1 + a_2\) is adjacent to all vertices of \((1 - e)R(1 - e)^*\) in \(\overline{\Gamma}(R)\) and therefore for any \(x \in R\) we have either \(a_2 x = 0\) or \(x a_2 = 0\). Since \(\overline{\Gamma}(R)\) has no vertex adjacent to all other vertices of the graph, we conclude that \(a_2 = 0\). Moreover since the degree of \(a\) is more than the degree of \(e\) in \(\overline{\Gamma}(R)\), we find that \(a = a_1\) is a non-zero zero-divisor in the ring \(eRe\). So clearly we have \(\Gamma_{A_e} \simeq \Gamma(eRe)\), where \(\Gamma_{A_e}\) is the induced subgraph of \(\Gamma(R)\) on \(A_e\). Suppose that \(b = b_1 + b_2\) is an arbitrary element of \(B_e\), where \(b_1 \in eRe\) and \(b_2 \in (1 - e)R(1 - e)\). Since \(b\) is adjacent to \(e\) in \(\overline{\Gamma}(R)\), we have \(b_1 = 0\). Also since at least one of the neighbors of \(b\) in \(\overline{\Gamma}(R)\) is contained in \((1 - e)R(1 - e)^*\), we obtain that \(b = b_2\) is a non-zero zero-divisor in the ring \((1 - e)R(1 - e)^*\). Thus clearly we have \(\Gamma_{B_e} \simeq \Gamma((1 - e)R(1 - e))\), where \(\Gamma_{B_e}\) is the induced subgraph of \(\Gamma(R)\) on \(B_e\).

Now since \(\Gamma(R) \simeq \Gamma(S)\), we conclude that \(\Gamma(eRe) \simeq \Gamma(M_{n_1}(F_1))\) and so by Theorem 20, we obtain \(eRe \simeq M_{n_1}(F_1)\). Furthermore, we have \(\Gamma((1 - e)R(1 - e)) \simeq \Gamma(M_{n_2}(F_2) \times \cdots \times M_{n_1}(F_1))\). By repeating this procedure \(l\) times, where \(l = |\{n_i | n_i \geq 2\}|\), we find a semisimple ring \(T\) such that \(R \simeq T \times R_1\) and \(S \simeq T \times S_1\), where \(S_1\) is a reduced ring and \(\Gamma(R_1) \simeq \Gamma(S_1)\). Since \(\Gamma(R) \simeq \Gamma(S)\), we have \(|D(R)| = |D(S)|\).

Hence we find

\[
|T||D(S_1)| + |D(T)||R_1| - |D(T)||D(R_1)|
= |T||D(S_1)| + |D(T)||S_1| - |D(T)||D(S_1)|.
\]
Therefore since $|\mathcal{D}(R_1)| = |\mathcal{D}(S_1)|$, we conclude that $|R_1| = |S_1|$. Now it is easy to see that for all exceptions given in Theorem 17, $|R_1| \neq |S_1|$. Hence by Theorem 17, $R_1 \cong S_1$ and the proof is complete. □

Example 22. The assumption that $S$ is semisimple is necessary in the previous theorem. To see this consider the following example.

$$
\mathcal{R} = \left\{ \begin{bmatrix} 2a & b \\ 0 & 0 \end{bmatrix} \mid a, b \in \mathbb{Z}_4 \right\} \quad \text{and} \quad \mathcal{S} = \left\{ \begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} \mid a, b, c, d \in \mathbb{Z}_2 \right\}.
$$

It is easy to verify that $\Gamma(\mathcal{R}) \simeq \Gamma(\mathcal{S})$. However, $\mathcal{R}$ and $\mathcal{S}$ have different characteristics and so they are not isomorphic.

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