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A Necessary and Sufficient Condition for a Finite-Dimensional Drinfel'd Double to Be a Ribbon Hopf Algebra

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In this paper we find a necessary and sufficient condition for the Drinfel'd double $(D(A), \mathscr{R})$ of a finite-dimensional Hopf algebra A with antipode s over a field k to have a ribbon element [15]. The condition has a remarkable connection with the formula for the fourth power of the antipode

$$s^4(a) = g(\alpha \rightharpoonup a \leftharpoonup \alpha^{-1})\, g^{-1}$$

given by [14, Proposition 6], where g and α are uniquely defined grouplike elements of A and A^*, respectively. Our main result is Theorem 3, which states that $(D(A), \mathscr{R})$ has a ribbon element if and only if

$$s^2(a) = l(\beta \rightharpoonup a \leftharpoonup \beta^{-1})\, l^{-1}$$

for all $a \in A$, where l and β are grouplike elements of A and A^*, respectively, which satisfy $l^2 = g$ and $\beta^2 = \alpha$. The grouplike elements g and α play a central role in this paper. By virtue of Theorem 3, the class of finite-dimensional doubles which are known to have a ribbon element is extended considerably. Our work here is based on a thorough scrutiny of the relationship between grouplike and ribbon elements.

The topological motivation for this paper is supported by the fact that ribbon Hopf algebras (Hopf algebras with a distinguished ribbon element) can be used to construct invariants of framed links embedded in three-dimensional space [15]. In the case of finite-dimensional ribbon Hopf

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algebras, the same structure can be used to produce invariants of three-dimensional manifolds [2]. These three-dimensional manifolds are represented by surgery on framed links, and their invariants are special cases of invariants for the links. In the case of the quantum group $SL(2)_q$ these invariants have been intensively investigated by Reshetikhin and Turaev [16], Kirby and Melvin [6], Lickorish [10], and others. We take our work on ribbon elements to be a starting point for an investigation of link and three-manifold invariants for other quantum groups. These directions will be explored in sequels to this paper.

In Section 1 we discuss background material needed for this paper, especially material related to quasitriangular Hopf algebras, and we recall the definition of ribbon element and ribbon Hopf algebra [15, p. 7]. In Section 2 we give a topological motivation for the definition of ribbon element. We study the connection between grouplike elements and ribbon elements in a finite-dimensional quasitriangular Hopf algebra (A, R) in Section 3, based on ideas and results of [2, 13, 15]. The grouplike elements g and α are seen to play a special role in the theory of ribbon Hopf algebras. We show in Proposition 2 that (A, R) has a ribbon element whenever the orders of g, α, and s^2 are odd. In Proposition 3 we describe a necessary and sufficient condition for (A, R) to possess a ribbon element when A is unimodular, thereby setting the stage for the proof of Theorem 3.

The odd-dimensional case is particularly interesting. It is not hard to see that (A, R) has at most one ribbon element when $\dim A$ is odd. Hennings showed that the existence of a ribbon element in the odd-dimensional case is equivalent to s^2 having odd order [2, Proposition 8]. Let $G(A)$ be the set of grouplike elements of A. It is easy to see (Theorem 2) that Hennings' hypothesis $\dim A$ odd can be replaced by $G(A)$ has odd order. This raises an interesting and relevant problem in the realm of finite-dimensional Hopf algebras: the implications that $|G(A)|$ is odd has for $\dim A$ and for the order of s^2.

Let A be a finite-dimensional Hopf algebra with antipode s over any field k. In Section 4 we prove Theorem 3 and explore some of its implications. We show in Proposition 5 that $(D(A), \mathscr{R})$ has a ribbon element whenever g, α, and s^2 have odd order. Suppose that A is ponted and odd-dimensional, and the characteristic of k is not two. By Taft's and Wilson's calculations [19], it follows that s^2 has odd order. Thus $(D(A), \mathscr{R})$ has a ribbon element, either by [2, Corollary 9] or by Proposition 5. Our Proposition 6 gives more general conditions under which $(D(A), \mathscr{R})$ has a ribbon element in the pointed case.

We end the section with a discussion of Taft's n^2-dimensional (pointed) Hopf algebras A_n defined and studied in [18]. Hennings observed that $(D(A_n), \mathscr{R})$ has a ribbon element when n is odd [2]. As a consequence of Proposition 7, the double $(D(A_n), \mathscr{R})$ does not when n is even, thereby providing a natural infinite family of finite-dimensional quasitriangular Hopf algebras which are not ribbon Hopf algebras. As doubles go, $(D(A_n), \mathcal{R})$ is relatively simple to describe an analyse. Throughout this paper k is a field.

1. PRELIMINARIES

In this paper we study the relationship between the grouplike elements and the ribbon elements in a finite-dimensional quasitriangular Hopf algebra over a field. The nature of square roots of certain grouplike elements turns out to be important. Some of our most interesting results involve square roots of odd order, and in their proofs we ultimately appeal to one of three simple facts about square roots of an element a of a group G: a has at most one square root in G of odd order, if a has odd order then a has a unique square root in G of odd order, and when G has odd order every element of G has a unique square root in G.

Suppose that C is a coalgebra over k. The "opposite" algebra C^{cop} is C as a vector space with comultiplication defined by $\Delta^{cop}(c) = \sum c_{(2)} \otimes c_{(1)}$ for $c \in C$, where $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$. A non-zero element $c \in C$ is said to be *grouplike* if $\Delta c = c \otimes c$. We let $G(C)$ denote the set of grouplike elements of C. Observe that $\varepsilon(c) = 1$ for $c \in G(C)$. By [17, Proposition 3.2.1] the set $G(C)$ is linearly independent. Thus $G(C)$ is finite whenever C is finite-dimensional. If A is a finite-dimensional algebra over k, then $G(A^*) = \mathrm{Alg}_k(A, k)$ is the set of k-algebra homomorphisms from A to k. If A is any bialgebra over k, then $G(A)$ is a semigroup under multiplication. If A is a Hopf algebra with antipode s over k, then $G(A)$ is a group, as $s(a) \in G(A)$ and is an inverse of $a \in G(A)$. We use the following important aspect of the relationship between $G(A)$ and A in this paper.

LEMMA 1. *Suppose that A is a finite-dimensional Hopf algebra over a field k. Then the order of $G(A)$ divides* $\dim A$. *In particular $G(A)$ has odd order if* $\dim A$ *is odd.*

Proof. The span B of $G(A)$ is a sub-Hopf algebra of A. Since $G(A)$ is linearly independent, the order of $G(A)$ is $\dim B$. By [11, Theorem 7] any finite-dimensional Hopf algebra is a free module over its sub-Hopf algebras. Therefore $\dim B \mid \dim A$, and the lemma follows.

Suppose that A is a bialgebra over k. The left and right A-module actions defined on A^* by

$$\langle a \rightharpoonup p, b \rangle = \langle p, ba \rangle = \langle p \leftharpoonup b, a \rangle$$

respectively for $a, b \in A$ and $p \in A^*$ give A^* an A-bimodule structure. Likewise the left and a right A^*-modules actions defined on A by

$$p \rightharpoonup a = \sum a_{(1)} \langle p, a_{(2)} \rangle, \qquad a \leftharpoonup p = \sum \langle p, a_{(1)} \rangle a_{(2)}$$

respectively for $p \in A^*$ and $a \in A$ give A an A^*-bimodule structure.

Now suppose that A is a finite-dimensional Hopf algebra with antipode s over k. Generally s is an algebra and a coalgebra antiendomorphism [17, Proposition 4.0.1]. Let $\Lambda \in A$ be a non-zero left integral for A, and let $\lambda \in A^*$ be a non-zero right integral for A^*. The left integrals for A form a one-dimensional ideal of A. Hence there is a unique $\alpha \in G(A^*)$ such that $\Lambda a = \alpha(a)\Lambda$ for all $a \in A$. Likewise there is a unique $g \in G(A^{**}) = G(A)$ such that $p\lambda = \langle p, g \rangle \lambda$ for all $p \in A^*$. We call g the *distinguished grouplike element of A* and we call α the *distinguished grouplike element of A^**. These grouplike elements play a fundamental role in the structure of A. For example, by [14, Proposition 6]

$$s^4(a) = g(\alpha \rightharpoonup a \leftharpoonup \alpha^{-1}) g^{-1} \tag{1}$$

for all $a \in A$. A is said to be *unimodular* if the one-dimensional ideal of left integrals for A is the one-dimensional ideal of right integrals for A. Thus A is unimodular if and only if $\alpha = \varepsilon$, and A^* is unimodular if and only if $g = 1$. For a discussion of integrals, the reader is referred to [17, Chap. 5].

For $a \in G(A)$ and $\eta \in G(A^*)$, the maps $\imath(a), \jmath(\eta): A \to A$ defined by $\imath(a)(b) = aba^{-1}$ and $\jmath(\eta)(b) = \eta \rightharpoonup b \leftharpoonup \eta^{-1}$ for $b \in A$ are Hopf algebra automorphisms. Regard $\mathrm{Aut}_{\mathrm{Hopf}}(A)$ as a group under composition. It is easy to see that $\imath: G(A) \to \mathrm{Aut}_{\mathrm{Hopf}}(A)$ and $\jmath: G(A^*) \to \mathrm{Aut}_{\mathrm{Hopf}}(A)$ are group homomorphisms. Since $G(A)$ and $G(A^*)$ are finite, it follows that $\imath(a)$ and $\jmath(\eta)$ have finite order. It is an easy exercise to show that $\imath(a)$ and $\jmath(\eta)$ commute and that both of these operators commute with s^2.

Our next result, which is used to prove Theorem 3, concerns an analog of (1) for s^2.

PROPOSITION 1. *Suppose that A is a finite-dimensional Hopf algebra with antipode s over a field k. Assume that the distinguished grouplike elements g and α of A and A^*, respectively, have odd order. Let $l \in G(A)$ and $\beta \in G(A^*)$ be the unique square roots of g and α, respectively, of odd order. Then the following are equivalent:*

(a) *s^2 has odd order.*

(b) *$s^2(a) = l(\beta \rightharpoonup a \leftharpoonup \beta^{-1}) l^{-1}$ for all $a \in A$.*

Proof. Consider the composite of commuting operators $t = \imath(l) \circ \jmath(\beta)$. Since $l^2 = g$ and $\beta^2 = \alpha$, we have that $t^2 = \imath(l)^2 \circ \jmath(\beta^2) = \imath(g) \circ \jmath(\alpha)$. Thus $t^2 = s^4$ by (1). Since l and β have odd order, it follows that $\imath(l)$ and $\jmath(\beta)$ have odd order as well. Thus t has odd order since $\imath(l)$ and $\jmath(\beta)$ commute. To prove the proposition we need only show that $s^2 = t$ when s^2 has odd order.

Suppose that s^2 has odd order. Then $s^2, t \in \mathrm{Aut}_{\mathrm{Hopf}}(A)$ are elements of odd order whose squares are equal. Therefore $s^2 = t$, and the proof is complete.

We note that the hypothesis of the proposition is satisfied when $\dim A$ is odd by virtue of Lemma 1.

Now assume that (A, R) is a finite-dimensional quasitriangular Hopf algebra with antipode s over k, and write $R = \sum R^{(1)} \otimes R^{(2)} \in A \otimes A$. For $\eta \in G(A^*)$ we define $g_\eta = \sum R^{(1)} \eta(R^{(2)})$. Since $(\Delta^{\mathrm{cop}}(a))\, R = R(\Delta(a))$ for $a \in A$, we conclude that g_η commutes with all $a \in G(A)$. Denote the center of a group G by $Z(G)$. Using [13, Proposition 3], we now have that

$$g_\eta \in Z(G(A)), \qquad g_\eta\, g_\rho = g_{\eta\rho} \qquad \text{for all} \quad \eta, \rho \in G(A^*). \tag{2}$$

Set

$$u = \sum s(R^{(2)}) R^{(1)}, \qquad h = g_\alpha\, g^{-1} \in G(A), \qquad c = us(u). \tag{3}$$

Since R is invertible it follows that u is as well. Also $e(u) = 1$; c is called the Casimir element of (A, R). It is well known [1A, Proposition 2.1] that

$$s^2(a) = uau^{-1} \qquad \text{for all} \quad a \in A. \tag{4}$$

Since s is an algebra anti-endomorphism, (4) implies that c is in the center of A. By [13, Theorem 2] we have

$$c = u^2 h. \tag{5}$$

Since c is central, (4) and (5) imply that

$$s^4(a) = h^{-1} a h \qquad \text{for all} \quad a \in A. \tag{6}$$

The grouplike element $h = g_\alpha\, g^{-1}$ plays the primary role in our study of ribbon Hopf algebras. It is Eq. (5) that links the Casimir element, the element u, and the distinguished grouplike elements of A and A^*. These connections ultimately lead to the determination of which finite-dimensional doubles have a ribbon element.

We say that $v \in A$ is a *quasi-ribbon element* of (A, R) if the following conditions are satisfied:

(R.1) $v^2 = c$,

(R.2) $s(v) = v$,

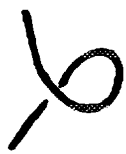
(R.3) $\varepsilon(v) = 1$, and

(R.4) $\Delta(v) = \mathcal{U}(v \otimes v)$, where $\mathcal{U} = (R_{21} R_{12})^{-1}$.

By definition $R_{12} = R = \sum R^{(1)} \otimes R^{(2)}$ and $R_{21} = \sum R^{(2)} \otimes R^{(1)}$. Drinfel'd observed that u satisfies this last condition [15, (3.1.13)]. A quasi-ribbon element in the center of A is called a *ribbon element*, and in this case (A, R, v) is called a *ribbon Hopf algebra* [15, p. 7]. The reader is referred to [15] for a detailed discussion of ribbon Hopf algebras and their relationship to links and three-manifolds.

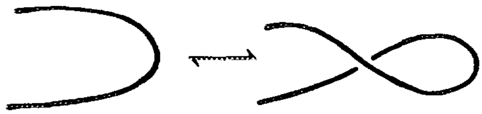
2. Topology of the Ribbon Elements

The purpose of this section is to outline the meaning of the ribbon elements from the topological point of view. A *knot* is an embedding of a cirle into three-dimensional space. A *link* is an embedding of a collection of disjoint circle into three-dimensional space. Invariants of knots and links are functions that can be computed from a convenient presentation of the link that depends only on the topology type of the link. Thus the values of the function on two presentations that can be deformed to one another are necessarily the same. It turns out that in order to obtain invariants of knots and links from quantum groups, it is most convenient to allow these functions to vary on presentations containing a curl:

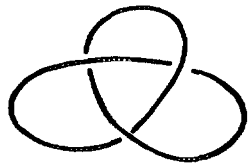


Thus the diagrams

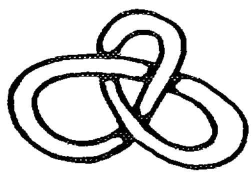
could receive different values. Without this leeway it would be very difficult to construct good invariants. One way to interpret the idea of variance under



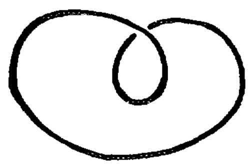
is to interpret the single strand diagrams as shorthand for the embedding of a twisted band. Thus



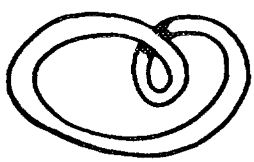
can mean



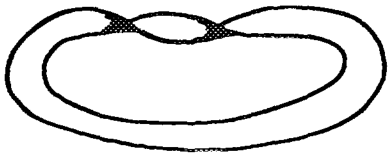
where this diagram represents the embedding of a knotted strip (circle $x[0, \varepsilon]$) in three-space. Similarly,



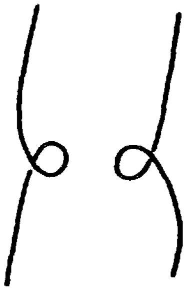
can be taken as shorthand for the ribbon



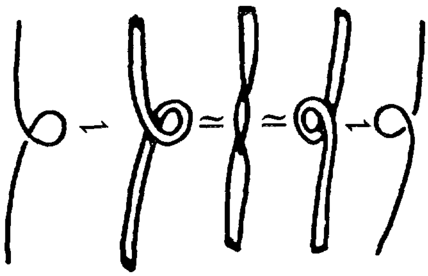
Note that this ribbon deforms the twisted ribbon shown below:



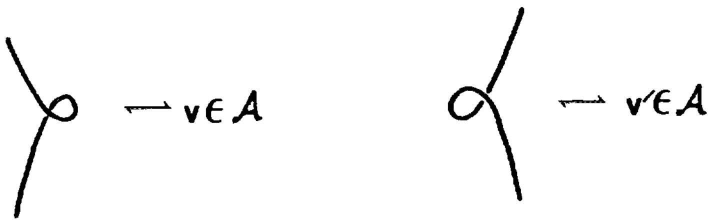
In particular we see that the following diagrams



deform to the *same* twisted ribbon:



This means that if we assign an element of the quantum group $\mathscr{A}$ to an elementary diagram such as



then we must demand that $v = v'$ for our construction to be well defined on embeddings of strips or ribbons. It turns out that invariants of these ribbons are the most appropriate constructions to manufacture from quantum groups, and that the condition $v = v'$ leads to the algebraic definition of ribbon element. See [2, 5, 15] for details of how this algebra comes about.

3. Ribbon Hopf Algebras

Here we reformulate some of the material found in [2, 15] and derive results based on [13] and Lemma 1 to highlight the role $G(A)$ plays in the study of ribbon and quasi-ribbon elements. Throughout this section (A, R) is a finite-dimensional quasitriangular Hopf algebra with antipode s over a field k. We let g and α be the distinguished grouplike elements of A and A^*, respectively.

We show that (A, R) has a ribbon element when g, α, and s^2 have odd order. We find a necessary and sufficient condition for (A, R) to have a ribbon element when A is unimodular. This result is applied to the double in Section 4 to prove Theorem 3, the main result of our paper. Suppose that both A and A^* are unimodular. Then $s^4 = I$. We show that (A, R) has at least one quasi-ribbon element and discuss the implications that $s^2 = I$ has for the existence of ribbon elements.

An odd-dimensional quasitriangular Hopf algebra over k always has a unique quasi-ribbon element. Whether or not it must be a ribbon element is an intriguing open question.

The starting point of our discussion is the one–one correspondence between quasi-ribbon (respectively, ribbon) elements of (A, R) and certain grouplike elements of A.

LEMMA 2. *Suppose that (A, R) is a finite-dimensional quasitriangular Hopf algebra over a field k and that $v \in A$ is a quesi-ribbon element of A. Let h and u be as in* (3) *and set $l = u^{-1}v$. Then*:

(a) $l^2 = h$ *and*

(b) $l \in G(A)$.

Proof. By (R.2) it follows that $s^2(v) = v$. Thus u and v commute by (4). By (R.1) and (5) we have $v^2 = u^2 h$. Thus $l^2 = h$ since u and v commute. Since $l \neq 0$ and both u and v satisfy (R.4), we can make the calculation $\Delta(l) = (\Delta(u))^{-1}(\Delta(v)) = (\mathcal{U}(u \otimes u))^{-1}(\mathcal{U}(v \otimes v)) = l \otimes l$, which establishes the fact the $l \in G(A)$.

THEOREM 1. *Suppose that (A, R) is a finite-dimensional quasitriangular Hopf algebra over a field k, and let u and h be as in* (3). *Then*:

(a) *$l \mapsto ul$ defines a one–one correspondence*

$$\{l \in G(A) \mid l^2 = h\} \leftrightarrow \{\textit{quasi-ribbon elements of } (A, R)\}.$$

(b) *Suppose that $l \in G(A)$ satisfies $l^2 = h$. Then $v = ul$ is a ribbon element of (A, R) if and only if $s^2(a) = l^{-1}al$ for all $a \in A$.*

Proof. Let $l \in G(A)$ satisfy $l^2 = h$. In light of Lemma 2, to prove part (a) we need only show that $v = ul$ is a quasi-ribbon element of (A, R). Recall that u commutes with the grouplike elements of A. Thus $v = ul = lu$.

Using (5) we see tht $v^2 = u^2l^2 = u^2h = c$, so (R.1) holds for v. Note that (R.3) is immediate since $\varepsilon(u) = 1 = \varepsilon(l)$. Since u satisfies (R.4) and $l \in G(A)$, it follows that (R.4) holds for $v = ul$.

To show that (R.2) holds for v, we first note that $s(v) = s(ul) = s(l)\,s(u) = l^{-1}s(u)$. Now $l^{-1} = lh^{-1}$, which follows from the equation $l^2 = h$, and $hu = s(u)$, which follows from (5) since h and u commute. Therefore $s(v) = lu = v$, and (R.2) holds for v. This completes the proof of part (a). Since $v = ul$ is central if and only if $l^{-1}al = uau^{-1}$ for all $a \in A$, part (b) follows by part (a) and (4). This completes the proof of the theorem.

By part (b) of Theorem 1 and Lemma 1:

COROLLARY 1. *Suppose that (A, R, v) is a finite-dimensional ribbon Hopf algebra with antipode s over a field k. Then $s^{2\,|G(A)|} = I$.*

The bound $|G(A)|$ that Corollary 1 gives for the order of the square of the antipode s^2 of a ribbon Hopf algebra is an improvement on the general bound $2 \dim A$, which follows from Lemma 1 and (1).

By [15, (3.3.3)] the ribbon elements of (A, R) are in one–one correspondence with the subgroup $E(A)$ of $G(A)$ of grouplike elements of order one or two which are in the center of A. Note that $E(A)$ does not depend on R. By Lemma 1, the number of ribbon elements of (A, R) divides $\dim A$, and (A, R) has at most one ribbon element when $\dim A$ is odd. Interestingly enough, (A, R) has a unique quasi-ribbon element in the odd-dimensional case.

COROLLARY 2. *Suppose that (A, R) is a finite-dimensional quasi-triangular Hopf algebra. Let g and α be the distinguished grouplike elements of A and A^*, respectively, and let h be as in* (3). *Then*:

(a) *If h has odd order, or if g and α have odd order, then (A, R) has a quasi-ribbon element.*

(b) *If $G(A)$ has odd order, in particular if* $\dim A$ *is odd, then (A, R) has a unique quasi-ribbon element.*

Proof. Using (2) we see that the conditions of part (a) or part (b) imply that h has a square root in $G(A)$, which must be unique if $G(A)$ has odd order. Therefore the corollary follows by part (a) of Theorem 1.

We find sufficient conditions for (A, R) to have a ribbon element in terms of the orders of s^2 and the grouplike elements g, α, and h.

PROPOSITION 2. *Suppose that (A, R) is a finite-dimensional quasitriangular Hopf algebra with antipode s over a field k. Let g and α be the distinguished grouplike elements of A and A^*, respectively, and suppose that h is as in* (3). *Then if either*

(a) *h and s^2, or*

(b) *g, α, and s^2*

have odd order, then (A, R) has a ribbon element.

Proof. Condition (b) implies condition (a) by (2). Suppose that h and s^2 have odd order. Let $l \in (h)$ be the unique square root of h having odd order. Then $\imath(l^{-1})^2 = \imath(h^{-1})$ and $\imath(l^{-1})$ has odd order. Recall that $s^4 = \imath(h^{-1})$ by (6). Thus s^2, $\imath(l^{-1}) \in \mathrm{Aut}_{\mathrm{Hopf}}(A)$ are two elements of odd order whose squares are equal. Consequently $s^2 = \imath(l^{-1})$, and the proposition follows by part (b) of Theorem 1.

The following theorem generalizes a result of Hennings [2, Proposition 9] to some extent. We replace his hypothesis that $\dim A$ is odd with $G(A)$ has odd order. The uniqueness statement we add is a consequence of Corollary 2. Our proof is a minor reformulation of the argument he uses.

THEOREM 2. *Suppose that (A, R) is a finite-dimensional quasitriangular Hopf algebra with antipode s over a field k and assume that $G(A)$ has odd order. Then (A, R) has a ribbon element (which is necessarily unique) if and only if s^2 has odd order.*

Proof. If (A, R) has a ribbon element, then there exists an $x \in G(A)$ such that $s^2(a) = xax^{-1}$ for all $a \in A$ by Theorem 1. Since x has odd order it follows that s^2 does also.

Conversely, suppose that s^2 has odd order. Since h has odd order, it follows that (A, R) has a ribbon element by Proposition 2. This completes our proof.

Appropos of the hypothesis of Theorem 2, we note it can be the case that h, g, and α have odd order, s^2 has even order, and (A, R) has ribbon elements. See the example in Section 4.

When A is unimodular we note that $h = g_\alpha g^{-1} = g^{-1}$, since $\alpha = \varepsilon$ in this case. Thus when A is unimodular, the existence of ribbon (or quasi-ribbon) elements is determined by square roots of g. By Theorem 1:

PROPOSITION 3. *Suppose that* (A, R) *is a finite-dimensional quasi-triangular Hopf algebra with antipode* s *over a field* k. *Suppose further that* A *is unimodular and let* g *be the distinguished grouplike element of* A. *Then*:

(a) (A, R) *has a quasi-ribbon element if and only if* $l^2 = g$ *for some* $l \in G(A)$.

(b) (A, R) *has a ribbon element if and only if* $l^2 = g$ *for some* $l \in G(A)$ *which satisfies* $s^2(a) = lal^{-1}$ *for all* $a \in A$.

Now suppose that both A and A^* are unimodular. Then $s^4 = I$ by [7, Corollary 5.7], which also follows from (1). Whether or not $s^2 = I$ has interesting implications for u.

PROPOSITION 4. *Suppose that* (A, R) *is a finite-dimensional quasi-triangular Hopf algebra with antipode* s *over a field* k. *Suppose, further, that both* A *and* A^* *are unimodular. Let* u *be as in* (3). *Then*:

(a) u *is a quasi-ribbon element of* (A, R).

(b) u *is a ribbon element of* (A, R) *if and only if* $s^2 = I$.

Proof. $h = g_\alpha g^{-1} = 1$ in this case. Thus the result follows by Theorem 1 with $l = 1$.

4. WHEN $(D(A), \mathcal{R})$ HAS A RIBBON ELEMENT

In this section A is any finite-dimensional Hopf algebra with antipode s over a field k. We let g and α be the distinguished grouplike elements of A and A^*, respectively. We show that the Drinfel'd double $(D(A), \mathcal{R})$ of A has a ribbon element if and only if the operator defined by the right-hand side of the equation

$$s^4(a) = g(\alpha \rightharpoonup a \leftharpoonup \alpha^{-1})\, g^{-1}$$

for all $a \in A$ has a square root of the same form which is equal to s^2. We construct an example of an even-dimensional Hopf algebra with four quasi-ribbon elements, two of which are ribbon. This example sheds light on the scope of some of the results of this paper.

Assume that Taft's n^2-dimensional Hopf algebra A_n is defined over k. Hennings showed that $(D(A_n), \mathcal{R})$ has a ribbon element when n is odd [2]. We show that this condition is also necessary.

To begin, we describe the structure of $(D(A), \mathcal{R})$ to the extent needed for this section. Our treatment follows [12].

As a coalgebra $D(A) = A^{*\mathrm{cop}} \otimes A$. Let $S = s^*$ be the antipode of A^*. By [12, (13) and (14)] multiplication in $D(A)$ can be described by

$$(p \otimes a)(q \otimes b) = \sum p(a_{(1)} \rightharpoonup q \leftharpoonup s^{-1}(a_{(3)})) \otimes a_{(2)} b$$

$$= \sum pq_{(2)} \otimes (S^{-1}(q_{(1)}) \rightharpoonup a \leftharpoonup q_{(3)}) b$$

for $p, q \in A^*$ and $a, b \in A$. Observe that

$$p \otimes a = (p \otimes 1)(\varepsilon \otimes a),$$

and that the maps $A \to D(A)$ and $A^{*\mathrm{cop}} \to D(A)$ defined by $a \mapsto \varepsilon \otimes a$ and $p \mapsto p \otimes 1$, respectively, are one–one Hopf algebra maps. At this point it is easy to see that the square of the antipode of $D(A)$ is $S^{-2} \otimes s^2$.

We note that $\alpha \otimes g$ is the distinguished grouplike element of $D(A)$ by [12, Corollary 7], and that $D(A)$ is unimodular by part (a) of [12, Theorem 4] or [2, Proposition 7]. The following theorem is the main result of this paper.

THEOREM 3. *Suppose that A is a finite-dimensional Hopf algebra with antipode s over a field k. Let g and α be the distinguished grouplike elements of A and A^*, respectively. Then:*

(a) *$(D(A), \mathscr{R})$ has a quasi-ribbon element if and only if there are $l \in G(A)$ and $\beta \in G(A^*)$ such that $l^2 = g$ and $\beta^2 = \alpha$.*

(b) *$(D(A), \mathscr{R})$ has a ribbon element if and only if there are l and β as in part* (a) *such that*

$$s^2(a) = l(\beta \rightharpoonup a \leftharpoonup \beta^{-1}) l^{-1}$$

for all $a \in A$.

Proof. By [12, Proposition 9] there is an isomorphism of groups $G(A^*) \times G(A) \to G(D(A))$ given by $(\eta, a) \mapsto \eta \otimes a$. Therefore part (a) follows by part (a) of Proposition 3.

By part (b) of Proposition 3, we see that $(D(A), \mathscr{R})$ has a ribbon element if and only if there are l and β as in part (a) such that

$$S^{-2}(p) \otimes s^2(a) = (\beta \otimes l)(p \otimes a)(\beta \otimes l)^{-1} \tag{7}$$

for all $p \in A^*$ and $a \in A$. Let $l \in G(A)$ and $\beta \in G(A^*)$. Since $(\beta \otimes l)^{-1} = \beta^{-1} \otimes l^{-1}$, by the equations describing multiplication in $D(A)$ we have that

$$(\beta \otimes l)(p \otimes a)(\beta \otimes l)^{-1} = (\beta(l \rightharpoonup p \leftharpoonup l^{-1}) \otimes la)(\beta^{-1} \otimes l^{-1})$$

$$= \beta(l \rightharpoonup p \leftharpoonup l^{-1})\, \beta^{-1} \otimes (\beta \rightharpoonup la \leftharpoonup \beta^{-1}) l^{-1}$$

$$= \beta(l \rightharpoonup p \leftharpoonup l^{-1})\, \beta^{-1} \otimes (\beta \rightharpoonup lal^{-1} \leftharpoonup \beta^{-1}),$$

the last equation following since β, β^{-1}: $A \to k$ are algebra maps. Therefore (7) holds if and only if

$$S^{-2}(p) \otimes s^2(a) = \beta(l \rightharpoonup p \leftharpoonup l^{-1})\, \beta^{-1} \otimes (\beta \rightharpoonup lal^{-1} \leftharpoonup \beta^{-1}) \tag{8}$$

for all $p \in A^*$ and $a \in A$. Recall that $\beta \rightharpoonup lal^{-1} \leftharpoonup \beta^{-1} = l(\beta \rightharpoonup a \leftharpoonup \beta^{-1})l^{-1}$ for all $a \in A$; that is, $\imath(l)$ and $\jmath(\beta)$ commute. Setting $p = \varepsilon$, we see (8) implies that $s^2(a) = l(\beta \rightharpoonup a \leftharpoonup \beta^{-1})l^{-1}$ for all $a \in A$. We have shown that (8) implies $s^2 = \imath(l) \circ \jmath(\beta)$. The reader is left with the easy exercise of showing that $s^2 = \imath(l) \circ \jmath(\beta)$ implies (8). This completes our proof.

Let $\mathscr{S}$ be the antipode of $D(A)$ and set $\mathscr{U} = \sum \mathscr{S}(\mathscr{R}^{(2)})\, \mathscr{R}^{(1)}$. By Proposition 3 and the calculations is used in the proof of Theorem 3, we have that

$$(\beta, l) \mapsto \mathscr{U}(\beta^{-1} \otimes l^{-1})$$

defines a one–one correspondence between those pairs $(\beta, l) \in G(A^*) \times G(A)$ such that $\beta^2 = \alpha$ and $l^2 = g$ and the quasi-ribbon elements of $(D(A), \mathscr{R})$. The ribbon elements correspond to those pairs (β, l) which further satisfy $s^2(a) = l(\beta \rightharpoonup a \leftharpoonup \beta^{-1})l^{-1}$ for all $a \in A$.

As a consequence of Theorem 3 and Proposition 1:

PROPOSITION 5. *Suppose that A is a finite-dimensional Hopf algebra with antipode s over a field k. Let g and α be the distinguished grouplike elements of A and A^*, respectively. If g, α, and s^2 have odd order, then $(D(A), \mathscr{R})$ has a ribbon element.*

Quasi-ribbon elements may not be ribbon. It can be the case that g, α have odd order, s^2 has even order, and (A,R) has ribbon elements. Consider a special case of [14, Example 1], which is also described in [8].

EXAMPLE. Assume that the characteristic of k is not two, and let A be the k-algebra generated by a, x, y, subject to the relations

$$a^2 = 1, \qquad x^2 = y^2 = 0,$$

$$ax = -xa, \qquad ay = -ya, \qquad xy = -yx.$$

The $\{a^i x^j y^l \mid 0 \leqslant i, j, l \leqslant 1\}$ is a linear basis for A. Thus $\dim A = 8$. Note that

$$\Delta(a) = a \otimes a, \qquad \Delta(x) = x \otimes a + 1 \otimes x, \qquad \Delta(y) = y \otimes a + 1 \otimes y$$

determine a Hopf algebra structure on A.

Let $\Lambda = (1+a)xy$. Then Λ is a non-zero left and right integral for A. Thus A is unimodular. Observe that $s^2(x) = -x$ and $s^2(y) = -y$. Consequently $s^4 = I \neq s^2$, the latter being the case since the characteristic of k is not two. Thus s^2 had even order. Observe that $G(A) = (a)$ and $G(A^*) = (\eta)$, where $\eta: A \to k$ is the algebra homomorphism determined by $\eta(a) = -1$ and $\eta(x) = \eta(y) = 0$. Therefore $G(A) \simeq Z_2 \simeq G(A^*)$. We deduce that A^* is unimodular by (1). Thus $g = 1$ and $\alpha = \varepsilon$. This means that $D(A)$ and $D(A)^*$ are unimodular, and thus the distinguished grouplike elements of $D(A)$ and $D(A)^*$ trivially have odd order. The square of the antipode of $D(A)$ has order two. By the comments made after Theorem 3, it is easy to see that $(D(A), \mathscr{R})$ has four quasi-ribbon elements, two of which are ribbon.

Since $G(D(A)) = G(A^*) \times G(A)$ and the squares of the antipodes of A and $D(A)$ have the same order, we have as a corollary to Theorem 1 the following generalization of [2, Corollary 9]:

COROLLARY 3. *Suppose that A is a finite-dimensional Hopf algebra over a field k. Assume that $G(A)$ and $G(A^*)$ have odd order. Then $(D(A), \mathscr{R})$ has a ribbon element (which is necessarily unique) if and only if s^2 has odd order.*

Now suppose that A is pointed. Taft and Wilson obtained useful estimates for computing the order of s^2 in [19]. We use their calculations in studying the double of A.

Let $a \in G(A)$ and $x \in A$ satisfy $\Delta(x) = x \otimes a + 1 \otimes x$. Noting that $\varepsilon(x) = 0$, we compute $0 = xs(a) + 1s(x)$, from which $s(x) = -xa^{-1}$ follows. Therefore $s^2(x) = -s(xa^{-1})\,s(x) = -a(-xa^{-1})$, from which we conclude that $s^2(x) = axa^{-1}$.

Let e be the smallest positive integer such that $s^{2e}(x) = x$ for all such $x \in A$. Then e divides the order of $G(A)$, and consequently by Lemma 1 we see that $e \mid \dim A$. Note that $s^2(x) = x$ when $x = a - 1$. The integer e may be considerably smaller than the exponent of $G(A)$.

Now suppose that $z \in A$ satisfies $\Delta(z) = z \otimes b + c \otimes z$ for some $b, c \in G(A)$. Set $x = zc^{-1}$ and $a = bc^{-1}$. Since s^2 is an algebra endomorphism of A which satisfies $s^2(a) = a$, it follows that $s^{2e}(z) = s^{2e}(xc) = z$. By the calculations in [19], we conclude that $s^{2e} = I$ when the characteristic of k is 0 and that $s^{2ep^n} = I$ in the characteristic $p > 0$ case, where $n \leqslant \dim A$. Thus as a consequence of Proposition 5:

PROPOSITION 6. *Suppose that A is a finite-dimensional pointed Hopf algebra with anitpode s over a field k which is not of characteristic two. Let g and α be the distinguished grouplike elements of A and A^*, respectively, and let e be defined as above. Suppose that g, α have odd order and the integer e is odd (which is the case when* $\dim A$ *is odd). Then $(D(A), \mathscr{R})$ has ribbon element.*

We note that (A, R) has a ribbon element whenever A is pointed, g and α have odd-order, e is an odd integer, and the characteristic of k is not two by Proposition 2. However, there are finite-dimensional pointed Hopf algebras whose double is not pointed.

Suppose that A is a finite-dimensional pointed Hopf algebra over k. Let G be a finite group and $k[G]$ be the group algebra of G over k. Then $A \otimes k[G]$ is pointed [3, 2.3.13]. It is easy to see that the orders of the distinguished grouplike elements for A and A^*, respectively, and the value of e for A are the same for their counterparts in $A \otimes k[G]$. Therefore the hypothesis Proposition 6 may hold even though the exponent of $G(A)$ is even (and thus dim A is even).

We conclude this paper by applying Theorem 3 to Taft's n^2-dimensional Hopf algebra A_n as presented in [18]. The A_n's form an interesting class of pointed Hopf algebras from a combinatorial point of view. When n is odd, $(D(A_n), \mathscr{R})$ provides an invariant of three-manifolds [2]. Generally, the double of A_n is of interest in connection with knot theory.

Suppose that $n \geqslant 1$ and that $\omega \in k$ is a primitive nth root of unity. As an algebra, A_n is generated by a and x subject to the relations

$$a^n = 1, \qquad x^n = 0, \qquad xa = \omega ax.$$

The coalgebra structure of A_n is determined by

$$\Delta(a) = a \otimes a, \qquad \Delta(x) = x \otimes a + 1 \otimes x.$$

The set of monomials $\{a^i x^j \mid 0 \leqslant i, j < n\}$ is a linear basis for A over k. Consequently $\Lambda = (1 + a + \cdots + a^{n-1})x^{n-1}$ is a non-zero left integral for A. Therefore the distinguished grouplike element of A^* is the algebra homomorphism $\alpha\colon A \to k$ determined by $\alpha(a) = \omega^{-1}$ and $\alpha(x) = 0$. Observe that $G(A^*) = (\alpha)$, and thus has order n. It is not hard to see that $G(A) = (a)$, either by direct calculation, or by noting that A_n is generated as an algebra by the span of its three-dimensional subcoalgebra $C = sp(1, a, x)$ by applying [3, 2.3.13 and 2.3.9] along with part (a) of [17, Proposition 8.0.3]. By (1) it follows that $g = a$ is the distinguished grouplike element of A. This can also been seen directly with a little effort.

PROPOSITION 7. *Suppose that n is a positive integer and that A_n is defined over k. Then the following are equivalent:*

(a) *$(D(A_n), \mathscr{R})$ has unique ribbon element.*

(b) *$(D(A_n), \mathscr{R})$ has a ribbon element.*

(c) *$(D(A_n), \mathscr{R})$ has a quasi-ribbon element.*

(d) *n is odd.*

Proof. We need only show that part (c) implies part (d), and that part (d) implies part (a). If $(D(A_n), \mathscr{R})$ has a quasi-ribbon element, then the generator α of the cyclic group $G(A_n^*)$ must have a square root by part (a) of Theorem 3. Therefore n, which is the order of $G(A^*)$, must be odd. We have shown that part (c) implies part (d).

That part (d) implies part (a) is noted in [2]. One can also appeal to Theorem 3 directly to establish this implication. Let s be the antipode of A_n. Note that $s^2(a) = a$ and $s^2(x) = \omega^{-1}x$. Now assume that n is odd, and write $n = 2m - 1$. It is easy to see that Theorem 3 can applied with $l = a^m$ and $\beta = \alpha^m$.

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