C^k-Smoothness of Invariant Curves in a Global Saddle-Node Bifurcation

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The birth of C^k-smooth invariant curves from a saddle-node bifurcation in a family of C^k diffeomorphisms on a Banach manifold (possibly infinite dimensional) is constructed in the case that the fixed point is a stable node along hyperbolic directions, and has a smooth noncritical curve of homoclinic orbits. This ensures that the map restricted to the resulting curve is equivalent to a C^k map of the circle. In particular, for a C^2 family of diffeomorphisms the resulting curve is C^2, and the "Denjoy example" cannot occur. Included is a new smoothness result for the foliation transversal to the center subspace, for the finite and infinite dimensional cases. Specifically, C^k-smoothness with respect to all variables of invariant foliations of the center-stable and center-unstable manifolds of a partially hyperbolic fixed point is proved in all cases. © 1996 Academic Press, Inc.

1. Introduction

We consider the saddle-node bifurcation of a stable fixed point of a C^k-diffeomorphism with a one-dimensional invariant curve \( \Gamma \) of homoclinic orbits. Specifically, we will study the case where the fixed point has one simple eigenvalue equal to 1 and is a stable node along hyperbolic directions. This bifurcation can occur in the Poincaré map of a limit cycle with characteristic multiplier 1, such as in the coalescence of a stable limit cycle with a limit cycle of saddle type. Afraimovich and Shil'nikov considered this bifurcation [AS] and proved for sufficient smoothness of the diffeomorphism the existence of a Lipschitz (in fact C^1) invariant curve homeomorphic to a circle after the fixed point disappears (see also [NPT] and [AAIS]). We prove in this article C^k-smoothness of the invariant curve if the original family is C^k-smooth under general assumptions.

For this bifurcation the case \( k = 2 \) (or \( k = 1 \) with bounded variation of the derivative) is of special importance due to the Denjoy Theory [De, Ni] for diffeomorphisms of the circle, \( S^1 \). If it is known that the resulting

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invariant curve is $C^2$-smooth, then the restriction of the map to the curve is a $C^2$ map as well. This would imply that either trajectories on the invariant curve are everywhere dense or a “resonance picture” occurs and there exist stable and unstable periodic points on this curve. This behavior is completely determined by the so called “rotation number” of the map. In the case that the map is merely $C^1$ the picture can be much more complicated.

We note that in any case this bifurcation is “inaccessible.” That is, for values of the bifurcation parameter greater than the bifurcation value, the bifurcation surface cannot be approached along a path of structurally stable systems. This is because the rotation number of the invariant circle must approach zero, passing through infinitely many rational and irrational numbers as the parameter approaches the bifurcation value.

The main idea of the proof will be to use the contraction mapping theorem applied to appropriate function spaces on certain “fundamental regions.” The idea of fundamental region depends heavily on the existence of a stable foliation transversal to the center subspace of the fixed point, and this article includes a new smoothness result for the foliation, in the form of a change of variables, for the infinite dimensional case. A generalized version of this result is given in Section 7.2. This general result implies the existence of $C^k$-smooth invariant foliations of the center-stable and center-unstable manifolds of a partially hyperbolic fixed point. As with other invariant manifold applications, the final degree of differentiability must be obtained separately. Following Chow and Lu [CL], we will employ a Lemma of D. Henry to accomplish this.

2. Main Results

Consider the following hypotheses.

**Hypothesis 1.** $T_\mu$ is a one-parameter family of $C^k$-diffeomorphisms of a $C^k$-manifold $M$, modeled on a Banach space $X$, which is Lipschitz continuous in $\mu$ in the $C^k$ topology, $\mu \in [-\mu_0, \mu_0]$. If $\dim(M) = n$ is finite, then we require $n \geq 2$.

**Hypothesis 2.** For $\mu = 0$, $T_\mu$ has a nonhyperbolic fixed point $\bar{0}$ which is a uniformly stable node along hyperbolic directions, that is, $DT_0(\bar{0})$ has spectrum with modulus uniformly less than one, with the exception of one eigenvalue with is equal to one.

With these assumptions we have the following change of variables theorem.
Theorem 1. With assumptions Hyp. 1, 2, on some neighborhood $U$ of 0 there are $C^k$-smooth coordinates $(x, z)$, $z \in \mathbb{R}$, $x \in X$, where $X = X_1 \oplus \mathbb{R}$, for which $T_\mu$ has the local form $T^\mu$ given by:

$$
\begin{align*}
\tilde{x} &= \Phi_1(x, z, \mu) = Ax + \tilde{f}(x, z, \mu)x \\
\tilde{z} &= \Phi_2(z, \mu) = z + g(z, \mu),
\end{align*}
$$

where the spectrum of $A$ satisfies

$$|\sigma(A)| < 1, \quad (2)$$

and where $f = \tilde{f} \cdot x$ and $g$ are $C^k$-smooth in $(x, z)$, and are Lipschitz continuous in $\mu$. The functions $f, g$ and their first derivatives vanish for $(x, z, \mu) = (0, 0, 0)$ and $f$ is $C^{k-1}$. Further, all the partial derivatives of $f$ with respect to $z$ up to order $k$ vanish identically for $x = 0$ and so for $j \leq k - 1$

$$
\frac{d^j f}{dz^j}(x, z, \mu) = \tilde{f}_j(x, z, \mu)x, \quad (3)
$$

for some $C^{k-j-1}$-smooth function $\tilde{f}_j$ and further

$$
\lim_{|x| \to 0} \frac{d^k f}{dz^k}(x, z, \mu) = 0. \quad (4)
$$

This form represents a stable foliation transversal to the center subspace, the leaves of the foliation being the planes $\{(x, z): z = z_0\}$. This is a special case of a more general result found in Section 7.2. We note that a stronger form may be obtained in the finite case and with the restriction of $C^s$-smoothness of the diffeomorphism $[Ta]$. 

Proof. See Section 7.1

At this point we make additional assumptions.

Hypothesis 3. The unstable set $S_u^0$ of 0 for $\mu = 0$ intersects the stable set $S_s^0$ in a 1-dimensional manifold $I_0$ of orbits which approach 0 along the non-hyperbolic direction in forward and backward time.

By the stable set $S_s^0$ we mean the set of all points whose forward orbits limit to 0. The unstable set $S_u^0$ is the set of all points whose backward orbits limit to 0. It is known that these sets are in fact $C^k$-manifolds with boundary [NPT].

Hypothesis 4. The unstable set $S_u^0$ for $\mu = 0$ is nowhere tangent to any leaf, $S_{z_0} = \{(x, z_0)\} \cap U$, of the stable foliation.
Hypothesis 5. For $\mu > 0$, $T_\mu$ has no fixed point in a neighborhood of $0$.

From Hypothesis 5 we can assume without loss of generality that $g$ in (1) has the following form

$$g(z, \mu) = z^2 + O(|z^3|),$$

with $z(0) = 0$, $z'(0) \geq 0$, $\gamma(0) > 0$, and $\pi(\mu) > 0$ for $\mu > 0$. We now state the main result.

Theorem 2. Under hypotheses Hypotheses 1–5 there exists $\mu_0 > 0$ such that for each $\mu$, with $0 < \mu < \mu_0$, $T_\mu$ has a unique invariant curve $\Gamma_\mu$ in a neighborhood of $\Gamma_0$. Moreover, each $\Gamma_\mu$ is $C^1$-smoothly embedded in $M$, attracts nearby points, and $\Gamma_\mu$ limits to $\Gamma_0$ in the $C^1$ sense as $\mu \to 0$.

3. SOME SMOOTHNESS LEMMAS

We will need the following two useful lemmas.

Lemma 1. Let $X$ and $Y$ be Banach spaces and $U$ an open subset of $X$. Then a closed bounded ball in $C^{j,1}(U, Y)$, ($j = 1, 2, 3, ...$), is a closed bounded subset in $C^0(U, Y)$.

Here $C^{j,1}$ is the space of $j$th continuously differentiable functions with Lipschitz $j$th derivative. The norms $\| \cdot \|_{j,1}$ will be the usual norms on those spaces.

Lemma 2 (D. Henry). Let $X$ and $Y$ be Banach spaces and $U$ be an open subset of $X$. Assume that $h: U \to Y$ is locally Lipschitz continuous. Then $h$ is continuously differentiable if and only if for every $x_0 \in U$,

$$\|h(x + \Delta) - h(x) - h(x_0 + \Delta) + h(x_0)\| = o(\Delta)$$

as $(x, \Delta) \to (x_0, 0)$.

For proofs of Lemmas 1 and 2 see [CL].

In addition, we use the next two lemmas, which we prove in Sections 6.1 and 6.2.

Lemma 3. Let $X$ be a Banach space and let $X^*$ be the dual space of $X$. Suppose $h \in C^{k-1}(X, X^*) \cap C^k(\bar{X} \setminus \{0\}, X^*)$, and that $h(x) x \in C^k(X, R)$. Then $h(x) g(x)$ is $C^k$ for any $g \in C^0(X, X)$ such that $g(x) = O(|x|)$.
Lemma 4. Let $X$ be a Banach space and let $Y = X^*$ (the dual space of $X$), and consider the equations

$$
\begin{align*}
\dot{x} &= Cx + N_1(x + y) \\
\dot{y} &= Ay + N_2(x + y)
\end{align*}
$$

where $x \in X$, $y \in Y$, $N_1 \in C^k$, $k \geq 1$ and $N_2$ has the property that $N_2(x + y)x$ is in $C^k(X, \mathbb{R})$. Also assume that $N_1$ and $N_2$, along with their first derivatives vanish at the origin, and

$$
|\sigma(C^{-1})| \leq 1 \\
|\sigma(A)| \leq 1.
$$

Then the unique local center-unstable manifold given as the graph of $h: X \to Y$ has the property that $h(x) x \in C^k(X, \mathbb{R})$.

Remark. Note that the equivalent result holds in the case of center-stable manifolds as well, since we can take the inverse of the diffeomorphism.

4. Fundamental Regions, Functions Spaces and Maps

Let $U$ be a small neighborhood of 0. Fix $a < 0$ and $\alpha > 0$ such that \{(x, z): x = 0, a \leq z \leq \alpha\} \in U and let

$$
\begin{align*}
b_\mu &= a + g(a, \mu) \\
\beta_\mu &= \alpha + g(\alpha, \mu).
\end{align*}
$$

We will suppress the $\mu$ except where dependence on it is emphasized. Denote the intervals $I^+ = \{z: a \leq z \leq b\}$ and $I^- = \{z: \alpha \leq z \leq \beta\}$ and define “fundamental regions,” to be the rectangular regions

$$
\begin{align*}
D^+ &= \{(x, z): |x| \leq |x_0|, z \in I^+\}, \\
D^- &= \{(x, z): |x| \leq |x_0|, z \in I^-\},
\end{align*}
$$

with $|x_0|$ sufficiently small that $D^+, D^- \in U$. Next we will define function spaces on $I^+$ and $I^-$. We define boundary conditions on the functions in such a way that the graph of a function in the space could be included in an invariant curve under the local diffeomorphism $\Phi_\mu$ in (1). Define
$H_{\alpha, e} = \{ \eta \in C^{0,1}: (\eta(z), \psi(z)) \in D^{-}, \|\eta\|_{0,1} \leq \varepsilon, \Phi_{\mu}(\eta(z), z) = (\eta(\beta), \beta) \},$

$H_{\alpha, e}^{-} = \{ \psi \in C^{0,1}: (\psi(z), \psi(z)) \in D^{-}, \|\psi\|_{0,1} \leq L + 1, \Phi_{\mu}(\psi(a), a) = (\psi(b), b) \},$

$H_{w, e} = H_{a, e}^{-} \cap \{ \eta \in C^{1,1}: \|\eta\|_{1,1} \leq \varepsilon, \eta'(\beta) = D_{\beta} \Phi_{\mu}(\eta(z), z)|_{z = a} \cdot (F_{\mu}(a, \mu))^{-1} \},$

$H_{w, e}^{-} = H_{a, e}^{-} \cap \{ \psi \in C^{1,1}: \|\psi\|_{1,1} \leq L + 1, \psi'(b) = D_{b} \Phi_{\mu}(\psi(z), z, \mu)|_{z = a} \cdot (F_{\mu}(a, \mu))^{-1} \}.$

Further, define iteratively

$H_{w, e}^{-} = H_{w, e}^{-} \cap \{ \eta \in C^{k,1}: \|\eta\|_{k,1} \leq \varepsilon, \eta^{(k)}(\beta) = D_{\beta} \Phi_{\mu}(\eta(z), z)|_{z = a} \cdot (F_{\mu}(a, \mu))^{-1} \},$

$H_{w, e}^{-} = H_{w, e}^{-} \cap \{ \psi \in C^{k,1}: \|\psi\|_{k,1} \leq L + 1, \psi^{(k)}(b) = D_{b} \Phi_{\mu}(\psi(z), z, \mu)|_{z = a} \cdot (F_{\mu}(a, \mu))^{-1} \}.$

We note from Lemma 1 that $H_{w, e}^+ \cap H_{w, e}^-$ are closed, bounded subsets of $C^{0}[\alpha, \beta]$ and $C^{0}[a, b]$, respectively, under the usual uniform topology.

Now we define maps $F_{\eta}$ on $H_{w, e}^{-}$ and $H_{w, e}^-$, respectively. First consider $\psi \in H_{w, e}^{-}$. Inside $U$, the z coordinate of the diffeomorphism is governed by the second equation of (1). Because of this, and the definitions of $I^+$ and $I^-$, for each $z_1 \in I^-$ and $\mu > 0$, $\exists! z_0 \in I^+/\{a\} \text{ and } n = m(z_0, \mu)$ such that $\Phi_{\mu}(z_0, \mu) = z_1$. Denote by $\eta(z_1)$ the function given by

$(\eta(z_1), z_1) = \Phi_{\mu}^{m(z)}(\psi(z_0), z_0, \mu),$

and define $F_{\eta}(\psi) = \eta$.

Now consider $S^* \cap D^-$. By the assumptions, there exists an integer $n$ such that $T_{m}^n(S^* \cap I^-) \cap D^-$ is not empty. For $\mu \geq 0$ sufficiently small, points near $S^*$ must be mapped by $T_{m}^n$ to points near $T_{m}^n(S^* \cap I^-)$. Again by the definition of $D^-$ and equation (1), each of these points must pass through $D^+$ on some iteration of $T_{m}^n$. In other words, for each point $Q = (x_1, z_1)$ near $S^* \cap D^-$, there is a unique $m = m(Q, \mu)$ such that $T_{m}^n(Q) \in D^+ \setminus \{z = \beta\}$. Now fix $\eta \in H_{w, e}^-$. By the preceding argument, for each $P = (\eta(z_1), z_1)$, $\exists! m = m(P, \mu)$ such that $T_{m}^n(P) \in D^+ \setminus \{z = \beta\}$. The endpoint conditions on $\eta$ guarantee that this determines a curve in $D^+$ which is the graph of a $C$ function $\psi$ on $I^+$. Define $F_{\eta}(\psi) = \psi$. We have that

$(\psi(z_0), z_0) = T_{\mu}^{m(z_0)}(\eta(z_1), z_1).$
Note that, locally $\mathcal{F}_\mu$ is the finite composition of $C^k$-diffeomorphisms and so, locally it can be written as a $C^k$-diffeomorphism,

$$
x_0 = F(x_1, z_1, \mu),
$$
$$
z_0 = G(x_1, z_1, \mu),
$$
where $(x_1, z_1)$ are coordinates in $D^-$ and $(x_0, z_0)$ are coordinates in $D^-$. 

5. Proof of the Main Theorem

We proceed with the proof in several parts.

5.1. Global Map

**Proposition 1.** There exist positive constants $L$, $\varepsilon_0$, and $\mu_0$ such that $\mathcal{F}_\mu : H_{k-1, \varepsilon} \to H_{k-1, L}$ for all $0 < \mu < \mu_0$ and $0 < \varepsilon < \varepsilon_0$.

**Proof.** Since the map (7) is a diffeomorphism,

$$
\begin{bmatrix}
F_x & F_z \\
G_x & G_z
\end{bmatrix} \neq 0.
$$

For $\mu = 0$ consider the image, $\psi_0$, of the zero function on $I^-$, which we denote by $\eta_0$. From (7) we have

$$
F(0, z_1, 0) = \psi_0(G(0, z_1, 0)).
$$

Thus,

$$
F_z(0, z_1, 0) = D\psi_0(G(0, z_1, 0)) \cdot G_z(0, z_1, 0).
$$

From (8) we have that $G_z(0, z, 0) \neq 0$, and so for $\varepsilon$ and $\mu$ sufficiently small, we still have $G_z(x, z, \mu) \neq 0$. Further, we can bound $G_z$ uniformly away from 0, since $z$ is on a compact interval.

Fix $\eta \in H_{k-1, \varepsilon}^-$. The image $\psi$ of this function under $\mathcal{F}_\mu$ is given locally by

$$
\psi(z_0) = F(\eta(z_1), z_1, \mu)
$$
$$
z_0 = G(\eta(z_1), z_1, \mu).
$$

We can invert the second equation of (9) by the Implicit Function Theorem. Let $\tilde{G} = z_0 - G(\eta(z_1), z_1, \mu)$. By continuity there is at least one pair $(\tilde{z}_0, \tilde{z}_1)$ such that $\tilde{G}(\tilde{z}_0, \tilde{z}_1) = 0$. Also $\tilde{G}_z(\tilde{z}_0, \tilde{z}_1) \neq 0$ since $G_z$ is
bounded away from 0. Therefore \( \exists! \ z^*(z_0) \in C^{k-1,1} \) such that \( z_0 = G(\eta(z^*(z_0)), z^*(z_0), \mu) \), and further

\[
    z^*(z_0) = \frac{1}{G(\eta(z^*), z^*, \mu) \eta'(z^*) + G(\eta(z^*), z^*, \mu)}.
\] (10)

Additionally, we note that if \( \eta \in C^k \), then \( z^* \in C^k \) as well.

Let \( \eta_0 \) and \( \psi_0 \) be as above and let \( L = \|\psi_0\|_{k-1,1} \). Then for \( \varepsilon \) and \( \mu \) sufficiently small and \( \|\eta - \eta_0\|_{k-1,1} < \varepsilon \) we have

\[ \|\psi\|_{k-1,1} \leq L + 1 \] (11)

by the continuous dependence of \( T_\mu \) on \( \mu \) in the \( C^k \) topology.

Finally, the endpoint conditions on \( \psi \) and \( \psi' \) are satisfied by composition.

\[ ]

5.2. Local Map

**Proposition 2.** For \( \mu \) sufficiently small, \( \mathcal{F}_{loc}: H^+_{k-1,1,L} \rightarrow H^-_{k-1,1} \).

**Proof.** Fix \( \psi \in H^+_{k-1,1,L} \) and consider the sequence of functions:

\[ \{\eta_0 = \psi, \eta_1, \eta_2, ..., \eta_{n-1}, \eta_n = \eta\} \]
determined by \( \Phi_\mu(\eta, (z), z) = (\eta, (z), z) \).

From (1)

\[ \eta_{i+1}(z) = A\eta_i(z) + f(\eta_i(z), z, \mu) \]

\[ z = z + g(z, \mu). \] (12)

By (5), \( g \) is small for \( \mu \) sufficiently small, so we can invert the second equation of (12), in a neighborhood of 0, using the Implicit Function Theorem. Specifically, there is a unique \( z_*(z) \in C^k \) such that \( z = z_* + g(z_*, \mu) \) and

\[
    z_* = \frac{1}{1 + g(z_*, \mu)}.
\] (13)

Now from (12)

\[
    |\eta_{i+1}| \leq |A + \tilde{f}| |\eta_i|,
\]

but \( |\tilde{f}| \) can be made arbitrarily small by choosing \( \mu \) small enough. In particular we may choose

\[
    |A + \tilde{f}| \leq q < 1,
\]
so that
\[ \| \eta_i \|_0 \leq q^i \| \psi \|_0. \]  
(14)

Differentiating we find
\[ \eta_{i+1}' = (A + f_x(\eta_i(z_*), z_*, \mu)) z_\ast' \eta_i'(z_*) + f_x(\eta_i(z_*), z_*, \mu) z_\ast'. \]  
(15)

For \( \mu \) small
\[ |A + f_x| z_\ast'| \leq q < 1, \]
and from (3) there exists a constant \( Q_1 \) so that
\[ |f_x(\eta_i(z_*), z_*, \mu)| z_\ast'| \leq q Q_1 |\eta_i|. \]
Therefore we obtain
\[ \| \eta_{i+1}' \| \leq q |\eta_i'| + q Q_1 |\eta_i|. \]
and, after induction, employing (14)
\[ \| \eta_i' \| \leq q^i \| \psi' \|_0 + q Q_1 \| \psi \|_0. \]  
(16)

For \( 2 \leq j \leq k - 1 \), differentiating \( j \) times we find
\[
\eta_{i+1}'(z) = (A + f_x)(z_\ast')^j \eta_i^{(j)}(z_*) + \sum_{l=1}^{j-1} P_l(z_*) \eta_i^{(l)}(z_*) \eta_i^{(j-l)}(z_*)

+ \frac{d^j f}{dz^j} (\eta_i(z_*), z_*, \mu)(z_\ast')^j,
\]  
(17)

where \( P_l \) is a polynomial with constant coefficients in the variables \( D^* z_* \), \( D^j \eta \), and \( D^* \eta_i \) with \( 1 \leq \alpha, \beta \leq j-l+1 \) and \( 0 \leq \gamma \leq l \). For \( \mu \) small enough
\[ |A + f_x| z_\ast'|^j \leq q < 1, \]
and from (3) we obtain
\[ \left| \frac{d^j f}{dz^j} \right| z_\ast'|^j \leq \bar{Q}_j |\eta_i|. \]
Therefore
\[ \| \eta_{i+1}' \| \leq q |\eta_i'| + q Q_j |\eta_i|_{j-1}, \]
where \( Q_j \) is independent of \( i \). After induction, employing (14)
\[ \| \eta_i'^{\gamma} \| \leq q^\gamma \| \psi^{(\gamma)} \| + p_j(i) q Q_{\gamma_j} \| \psi \|_{j-1}, \]
where $p_j(i)$ is a $j$th degree polynomial in $i$. In particular,

$$\|\eta\|_{k-1} \leq p_{k-1}(i) q^i \|\psi\|_{k-1}. \quad (18)$$

To show that $\eta^{(k-1)}$ is a Lipschitz function, consider

$$|\eta^{(k-1)}_l(z) - \eta^{(k-1)}_l(\tilde{z})|$$

$$\leq |A| \left(|z^\ell_\sigma|^{k-1} \eta^{(k-1)}_l(z_\sigma) - (z^\ell_\sigma)^{k-1} \eta^{(k-1)}_l(z_\sigma)| + |f(\eta_l(z_\sigma), z_\sigma, \mu)| (z^\ell_\sigma)^{k-1} \eta^{(k-1)}_l(z_\sigma) - f(\eta_l(z_\tilde{\sigma}), \tilde{z}_\sigma, \mu)\right)$$

$$\times (z^\ell_\sigma)^{k-1} \eta^{(k-1)}_l(z_\sigma)$$

$$+ \sum_{l=0}^{k-2} |P_l(z_\sigma) \eta_l(z_\sigma) - P_l(z_\tilde{\sigma}) \eta_l(z_\tilde{\sigma})|$$

$$+ \left| \frac{d^{k-1}f}{dz^{k-1}}(\eta_l(z_\sigma), z_\sigma, \mu)(z^\ell_\sigma)^{k-1} - \frac{d^{k-1}f}{dz^{k-1}}(\eta_l(z_\tilde{\sigma}), \tilde{z}_\sigma, \mu)(z^\ell_\tilde{\sigma})^{k-1} \right|. $$

Noting that $P_l \eta^{(l)}$ is Lipschitz for $l \leq k - 2$, with Lipschitz constant proportional to $\|\eta_l\|_{k-1}$, we obtain

$$|\eta^{(k-1)}_l(z) - \eta^{(k-1)}_l(\tilde{z})|$$

$$\leq (|A| + f_\sigma |(|z^\ell_\sigma|^{k-1} Lip(\eta^{(k-1)}_l))| z - \tilde{z})$$

$$+ Lip((A + f_\sigma)(\eta_l(z_\sigma), z_\sigma, \mu)) (z^\ell_\sigma)^{k-1} \|\eta^{(k-1)}_l\| |z - \tilde{z}|$$

$$+ \sum_{l=0}^{k-2} \hat{Q}_l |\eta_l|_{k-1} |z^\ell_\sigma| |z - \tilde{z}|$$

$$+ \left| \frac{d^{k}f}{dz^{k}}(\eta_l(z_\sigma), z_\sigma, \mu) \right| |z^\ell_\sigma| |z - \tilde{z}|$$

$$\leq \left( q Lip(\eta^{(k-1)}) + Q_k \|\eta_l\|_{k-1} + \left| \frac{d^{k}f}{dz^{k}}(\eta_l(z_\sigma), z_\sigma, \mu) \right| |z^\ell_\sigma| \right) |z - \tilde{z}|.$$ 

Again for $\mu$ small

$$|A + f_\sigma| |z^\ell_\sigma|^{k} \leq q < 1.$$ 

Thus, $\eta^{(k-1)}_{l+1}$ is Lipschitz, with constant

$$Lip(\eta^{(k-1)}_{l+1}) \leq q Lip(\eta^{(k-1)}_l) + Q_k \|\eta_l\|_{k-1} + \left| \frac{d^{k}f}{dz^{k}}(\eta_l(z_\sigma), z_\sigma, \mu) \right| |z^\ell_\sigma|^{k}. \quad (19)$$
After induction it can be shown that
\[
\begin{align*}
\text{Lip}(\eta^k_i) & \leq q^k \text{Lip}(\psi^k) + p_d(i) Q_\psi \|\psi\|_{k-1} \\
& + \sum_{i=0}^k q^i \left| \frac{d^k f}{dz^k}(\eta_i, z_*, \mu) \right| |z_\ast|^k.
\end{align*}
\]
Using (14) and (4) choose \( i_0 \) so that
\[
\left| \frac{d^k f}{dz^k}(\eta_i) \right| |z_\ast|^k \leq \frac{\varepsilon}{2} (1-q).
\]
Then
\[
\begin{align*}
\sum_{i=0}^k q^i \left| \frac{d^k f}{dz^k}(\eta_i, z_*, \mu) \right| |z_\ast|^k \\
\leq \sum_{i=0}^{i-i_0} q^i \left| \frac{d^k f}{dz^k}(\eta_i, z_*, \mu) \right| |z_\ast|^k \\
+ \sum_{i=i_0+1}^k q^i \left| \frac{d^k f}{dz^k}(\eta_i-z_*, \mu) \right| |z_\ast|^k \\
\leq \sum_{i=0}^{i-i_0} q^i \frac{\varepsilon}{2} (1-q) + \sum_{i=i_0+1}^k q^i \left| \frac{d^k f}{dz^k}(\psi_i) \right| |z_\ast|^k \\
\leq \frac{\varepsilon}{2} + (i_0-1) q^{i_0+1} \left| \frac{d^k f}{dz^k}(\psi_0) \right| |z_\ast|^k,
\end{align*}
\]
which, with (14) and (18) implies that if \( \mu \) is sufficiently small, then \( \|\eta\|_{k-1,\varepsilon} < \varepsilon \), since \( n_\mu \) is large.

Finally, \( \eta \) satisfies the boundary conditions for \( H_{k-1,\varepsilon} \) by composition. Thus \( \eta \in H_{k-1,\varepsilon} \).

5.3. \( \mathcal{F}_{loc} \circ \mathcal{F}_{gl} \) Has a Fixed Point in \( H_{k-1,\varepsilon} \)

**Proposition 3.** For \( \mu \) sufficiently small, \( \mathcal{F}_{loc} \circ \mathcal{F}_{gl} \) is a contraction on \( C^0[\bar{\alpha}, \bar{\beta}) \).

**Proof.** Let \( \eta, \hat{\eta} \in C^0_\varepsilon \), \( \bar{\xi} = \mathcal{F}_{loc} \circ \mathcal{F}_{gl}(\eta) \), and \( \hat{\xi} = \mathcal{F}_{loc} \circ \mathcal{F}_{gl}(\hat{\eta}) \). By (11) and (14)
\[
\|\hat{\xi} - \bar{\xi}\|_\varepsilon < q^\mu(L_0+1)\|\eta - \hat{\eta}\|_\varepsilon,
\]
and for \( \mu \) small enough, \( q^\mu(L_0+1) < 1 \).
Now since $H_{k-1}$ is a closed subset of $C^0[\beta]$, and $H_{k-1}$ is invariant under the contraction $\mathcal{F}_{loc} \cdot \mathcal{F}_{gl}$, by Lemma 1, $\mathcal{F}_{loc} \cdot \mathcal{F}_{gl}$ has a fixed point $\eta_*$ in $H_{k-1}$.

We will now show that this fixed point is in fact a $C^k$ function.

5.4. $\eta_* \in C^k$

We will show that $\eta_* \in C^k$ using Lemma 2. Let $\psi = \mathcal{F}_{gl}(\eta_*)$. From (9) for $k \geq 2$

$$\psi^{(k-1)}(z) = \left[ F_\eta \eta_*^{(k-1)}(z) - \psi'(z) \frac{d^{k-1}z}{dz^{k-1}} \right] (z^*)^{k-1} + R(z)$$

$$= [F_\eta - (F_\eta + F_\xi) G_\eta] \eta_*^{(k-1)}(z^*) (z^*)^{k-1} + R(z)$$

where $R$ is a function of $D^x F$, $D^y \eta_*$ and $D^z z^*$ with $1 \leq x \leq k-1$, $1 \leq \beta \leq k-2$ and $1 \leq \gamma \leq k-2$. Thus $R$ is continuously differentiable as a function of $z$, and so, using Lemma 2 we have

$$\psi^{(k-1)}(z + A) - \psi^{(k-1)}(z) - \psi^{(k-1)}(z_0 + A) + \psi^{(k-1)}(z_0)$$

$$= [F_\eta - (F_\eta + F_\xi) G_\eta] \cdot [\eta_*^{(k-1)}(z + A) - \eta_*^{(k-1)}(z)]$$

$$- \eta_*^{(k-1)}(z_0 + A) + \eta_*^{(k-1)}(z_0)] \cdot (z^*(z_0))^{k-1} + o(|A|)$$

as $(z, A) \to (z_0, 0)$. Define

$$l \equiv (z_0 + A) - z_0$$

$$= G(\eta_*(z_0 + A), z_0 + A, \mu) - G(\eta_*(z_0), z_0, \mu)$$

$$= [G_\eta(\eta_*(z_0), z_0, \mu) \eta_*'(z_0) + G_\xi(\eta_*(z_0), z_0, \mu)] A + o(|A|)$$

$$= (z^*)^{-1} A + o(|A|),$$

so that

$$|z^*||l| \leq |A|(1 + o(1)). \tag{20}$$

Also

$$z + A - (z + l) = G(\eta_*(z + A), z + A, \mu) - G(\eta_*(z), z, \mu)$$

$$- G(\eta_*(z_0 + A), z_0 + A, \mu) + G(\eta_*(z_0), z_0, \mu)$$

$$= o(|A|),$$

as $(z, A) \to (z_0, 0)$. Thus by (20)

$$z + A - (z + l) = o(|l|), \tag{21}$$
as \((z, l) \to (z_0, 0)\). Define

\[
\tilde{\lambda}(h, x_0) = \limsup_{(z, A) \to (z_0, 0)} \frac{|h(x + A) - h(x) - h(x_0 + A) + h(x_0)|}{|A|}. \tag{22}
\]

Then

\[
\frac{1}{|h|} |\psi^{(k-1)}(z + l) - \psi^{(k-1)}(z) - \psi^{(k-1)}(z_0 + l) + \psi^{(k-1)}(z_0)| \\
\leq \frac{|A|}{|l|} \max |F_1(1 - G_x \eta^*_z) - F_2 G_x| |z^*_L|^{\delta} \tilde{\lambda}(\eta^{(k-1)}_z, z_0) + o(1),
\]

and so there is a constant \(L_1\) such that

\[
\tilde{\lambda}(\psi^{(k-1)}_z, z_0) \leq L_1 \tilde{\lambda}(\eta^{(k-1)}_z, z_0). \tag{23}
\]

Next let \(\eta_0 = \psi\) and let \(\{\eta_i\}\) be the sequence of functions determined by (12). From (17), noting that the last two terms are continuously differentiable,

\[
\eta^{(k-1)}_{i+1}(z + A) - \eta^{(k-1)}_{i+1}(z) - \eta^{(k-1)}_{i+1}(z_0 + A) + \eta^{(k-1)}_{i+1}(z_0) \\
= e_i(z_0)[\eta^{(k-1)}_{i}(z + A) - \eta^{(k-1)}_{i}(z) - \eta^{(k-1)}_{i}(z_0 + A) + \eta^{(k-1)}_{i}(z_0)] \\
\cdot (z^*_L(z_0))^{k-1} + o(|A|),
\]

as \((z, A) \to (z_0, 0)\), where

\[
e_i(z) = A + f_z(\eta_i(z), z, \mu).
\]

Define

\[
s = z_0 + A - z_0 \\
= A + g(z_0 + A, \mu) - g(z_0, \mu) \\
= (1 + g_z(z_0, \mu)) A + o(|A|).
\]

Then

\[
|z^*_L| |s| \leq |A|(1 + o(1)) \tag{24}
\]

Also

\[
(z + A) - (z + s) = g(z + A, \mu) - g(z, \mu) - g(z_0 + A, \mu) + g(z_0, \mu) \\
= o(|A|)
\]
as \((z, A) \mapsto (z_0, 0)\), which implies with (24) that
\[
\dot{z} + A - (\dot{z} - s) = o(|s|) \quad (25)
\]
as \((z, s) \mapsto (z_0, 0)\). Now using (24) and (25) in definition (22) and noting
that the functions \(\{c_i\}\) are uniformly bounded we have
\[
\dot{\lambda}(\eta^{(k-1)}_+, z_0) \leq \frac{|A|}{|s|} |c| \cdot |z_0|^k \cdot \dot{\lambda}(\eta^{(k-1)}_+, z_0) \\
\leq q \dot{\lambda}(\eta^{(k-1)}_+, z_0)
\]
where \(q < 1\) for \(\mu\) sufficiently small. Now employing (23)
\[
\dot{\lambda}(\eta^{(k-1)}_+, z_0) \leq q^{n_0} \dot{\lambda}(\psi^{(k-1)}(z), z_0) \\
\leq q^{n_0} \lambda \dot{\lambda}(\eta^{(k-1)}_+, z_0) \\
\leq c \dot{\lambda}(\eta^{(k-1)}_+, z_0)
\]
where \(c < 1\) for \(\mu\) sufficiently small. Thus we have
\[
\sup_{z \in I^+} \dot{\lambda}(\eta^{(k-1)}_+, z_0) \leq c \sup_{z \in I^+} \dot{\lambda}(\eta^{(k-1)}_+, z_0) < \infty,
\]
which implies
\[
\dot{\lambda}(\eta^{(k-1)}_+, z_0) = 0
\]
for all \(z_0 \in I^+\). Therefore by Lemma 2 we have \(\eta \in C^k\).  

5.5. Proof of the Theorem

Now let
\[
\Gamma_\mu = \sum_{j=0}^{n_\mu + m_\mu} A_\mu(S),
\]
where \(S = \{(\eta_\mu(z), z) : \alpha \leq z \leq \beta\}\). By the uniqueness of \(\eta\), \(\Gamma_\mu\) is the unique invariant curve in a neighborhood of \(\Gamma_0\). In the neighborhood \(U\) of the fixed point let \(\eta_\mu\) denote the function whose graph is \(\Gamma_\mu\) in the local coordinates.

To show that \(\Gamma_\mu \to \Gamma_0\) in the \(C^k\) norm, we first establish that \(\Gamma_0\) is \(C^k\). It is known that the local unstable set for the fixed point is a unique \(C^k\) manifold. Since \(\Gamma_0 \setminus \{0\}\) is just the image of the local unstable set of the fixed point, it is \(C^k\). To get smoothness at the fixed point, let \(\psi_0\) be as before and repeat the construction of sec. 5.2. For each \(\mu\) denote by \(n_0\) the integer such that \(0 \in \Phi_\mu^{(0)}(D^+)\). Since \(n_0 = \infty\) for \(\mu = 0\), equations (14), (16)
and (19) imply that \( \eta_0 \to 0, \eta_0^j \to 0, \eta_0^k \to 0 \), etc., as \( z \to 0 \). This implies that \( I_\omega \) is \( C^k \) at 0, with \( \eta_0^j(0) = 0 \), for \( 0 \leq j \leq k \) (See [Ru, p. 115]).

We note in section 4, that \( I_\omega \to I_0 \) as \( \mu \to 0 \) in an appropriate sense. By Proposition 2, given \( \varepsilon \) there exists \( \mu \) small enough that \( \| \eta_\omega \|_k < \varepsilon \). This gives us that \( I_\omega \to I_0 \) in the \( C^k \) norm on the interval \( I_\omega \). It follows that \( I_\omega \to I_0 \) pointwise in the \( C^k \) norm away from the fixed point, since any point other than the fixed point is contained in the image of a finite iteration of \( \eta_\omega \).

That is, \( I_\omega \to I_0 \) uniformly in the \( C^k \) sense outside of any neighborhood of 0.

Now consider the neighborhood \( \tilde{U} = U \setminus \{ x, z : z < a \} \) of the fixed point 0. Given \( \varepsilon > 0 \), we can choose \( \mu \) small enough that \( \| \eta_\omega(z) - \eta_0(z) \|_k < \varepsilon/2 \) for all \( z \in I^+ \) and so that, for appropriate choice of local charts in a neighborhood of \( I_\omega \), \( \| I_\omega - I_0 \|_{C^k} < \varepsilon/2 \) outside \( \tilde{U} \) for all \( \mu \leq \mu_0 \). We will now show that \( |\eta_\omega(z) - \eta_0(z)| < \varepsilon \) for all \( z \) in \( U \) and all \( \mu \leq \mu_k \) for some \( \mu_k \leq \mu_0 \). Choose \( \tilde{z} \) in \( \tilde{U} \cup D^+ \) and denote by \( z_\mu \) the preimage of \( \tilde{z} \) under (1). Consider

\[
|\eta_\omega(z) - \eta_0(\tilde{z})| 
\leq |A\eta_\mu(z_\mu) - A\eta_0(z_0)| 
+ |f(\eta_\mu(z_\mu), z_\mu, \mu) - f(\eta_0(z_0), z_0, 0)| 
\leq |A + f_x| |\eta_\mu(z_\mu) - \eta_0(z_0)| 
+ |A\eta_\mu(z_\mu) - A\eta_0(z_0)| 
+ |f(\eta_\mu(z_\mu), z_\mu, \mu) - f(\eta_0(z_0), z_0, 0)|. 
\]

By continuity of \( f \) with respect to \( \mu \) and the continuity of \( \eta_\mu \) with respect to \( z \), we can make the last two terms less than \( \varepsilon/2 \) by an appropriate choice of \( \mu_1 \). That is

\[
|\eta_\omega(z) - \eta_0(z)| \leq q |\eta_\mu(z_\mu) - \eta_0(z_0)| + \varepsilon/2 \leq \varepsilon. 
\]

Similarly, for \( \eta' \)

\[
|\eta'_\omega(z) - \eta'_0(\tilde{z})| 
\leq |(A + f_x(\eta_\mu(z_\mu), z_\mu, \mu)) z'_\mu, \eta'_0(z_\mu) - (A + f_x(\eta_0(z_0), z_0, 0)) z'_0, \eta'_0(z_0)| 
+ |f_x(\eta_\mu(z_\mu), z_\mu, \mu) z'_\mu - f_x(\eta_0(z_0), z_0, 0) z'_0| 
\leq |(A + f_x)| z'_0| |\eta_\mu(z_\mu) - \eta_0(z_0)| 
+ |(A + f_x)| z'_0| |\eta_\mu(z_\mu) - \eta_0(z_0)| 
+ |(A + f_x)| z'_0| |\eta_\mu(z_\mu) - \eta_0(z_0)| 
+ |(A + f_x)| z'_0| |\eta_\mu(z_\mu) - \eta_0(z_0)|. 
\]


Again with $\mu \leq \mu_2$, for some $\mu_2$ the last two terms can be made smaller than $\epsilon/2$, and we obtain
\[
|\eta^j_\mu(z) - \eta^j_0(z)| \leq q|\eta^j_\mu(z_0) - \eta^j_0(z_0)| + \epsilon/2
\]
\[
\leq \epsilon.
\]
Finally, for $2 \leq j \leq k$
\[
|\eta^j_\mu(z) - \eta^j_0(z)| = (A + f_\mu(\eta_\mu(z), z_\mu, \mu))(z_\mu)'/\eta^j_\mu(z_\mu)
\]
\[-(A + f_\mu(\eta_0(z_0), z_0, 0))(z_0)'/\eta^j_0(z_0)
\]
\[+ R_j(z_\mu, \mu) - R_j(z_0, 0),
\]
where $R_j(z_\mu, \mu)$ is a polynomial with constant coefficients in the variables $D^j f, D^j z_\mu \ast$ and $D^j \eta_\mu$ with $1 \leq \alpha \leq j, 0 \leq \beta \leq j$ and $0 \leq \gamma \leq j - 1$. Thus
\[
|\eta^j_\mu(z) - \eta^j_0(z)| \leq |A + f_\mu||z_\mu|'/|\eta^j_\mu(z_0) - \eta^j_0(z_0)|
\]
\[+ |(A + f_\mu)\eta^j_\mu(z_\mu)(z_\mu)' - (A + f_\mu)\eta^j_0(z_0)(z_0)'|]
\[+ |R_j(z_\mu, \mu) - R_j(z_0, 0)|.
\]
We can make the last two terms less than $\epsilon/2$ by requiring $\mu \leq \mu_2$ for some $\mu_2$, and so
\[
|\eta^j_\mu(z) - \eta^j_0(z)| \leq q|\eta^j_\mu(z_0) - \eta^j_0(z_0)| + \epsilon/2
\]
\[\leq \epsilon.
\]
Thus we obtain $C^k$ convergence of $\Gamma_\mu$ to $\Gamma_0$ in a neighborhood of 0, and so $C^k$ convergence on all of $\Gamma_0$.

The fact that $\Gamma_\mu$ is attracting, follows immediately from the proof that $\eta_*$ is an attracting fixed point of the functional map. 

6. Proofs of Smoothness Lemmas

6.1. Proof of Lemma 3

We begin by noting that the $k - 1$st derivative of $h(x)g(x)$ is given by
\[
D^{k-1}(h(x)g(x)) = (D^{k-1}f(x))g(x) + R(x)
\]
where $R(x)$ is continuously differentiable by assumption. Thus we need only to prove the Lemma in the case $k = 1$. In that case
\[ h(0) g(0) - h(y) g(y) = -h(y) g(y) \]
\[ = h(y) (g(0) - g(y)) \]
\[ = h(y) g'(0)(-y) + o(|y|). \]

But since \( h \) is continuous
\[ |(h(y) - h(0))(−y)| = o(|y|). \]

So \( hg \) is differentiable with \( D(h(0) g(0)) = h(0) g'(0). \) For \( x \neq 0 \)
\[ D(h(x) g(x)) = h'(x) g(x) + h(x) g'(x). \]

It is easy to show from the assumption \( h(x) x \in C^1 \) that \( h'(x) x \to 0 \) as \( x \to 0 \) and it follows from \( g(x) = o(|x|) \) that \( h'(x) g(x) \to 0 \) as well. This gives us
\[ \lim_{x \to 0} D(hg) = h(0) g'(0) = D(hg) \big|_{0}, \]

and so the derivative is continuous. \[ \blacksquare \]

6.2. Proof of Lemma 4

The diffeomorphism (6) is known to possess a unique \( C^{k-1} \) center-unstable manifold which is the graph of a function \( h: X \to Y \), with \( h(0) = 0 \) and \( Dh(0) = 0 \) [CL]. This \( h \) is the fixed point of a contraction map \( II \) on a bounded subset of the function space \( C^{k-1}(X, Y) \), given by

\[ \bar{x} = C x + N_1(x + h(x)) \]
\[ \bar{h}(x) = Ah(x) + N_2(x + h(x)). \] \hspace{1cm} (26)

In the proof of this fact it is shown that a bounded subset of the space \( C^{k-2,1} \) with norm less than \( b \) is invariant under \( II \). And so, given \( h_0 \in C^{k-2,1}(X, Y) \) such that \( \|Dh_0\|_{k-2,1} \leq b \), equations (26) define a sequence of functions \( \{h_i\} \) which converge uniformly to \( h \), and for which \( \|h_i\|_{k-2,1} \leq b \). It is clear from (26) that if we choose \( h_0 \in C^k(X, Y) \) then \( h_i \in C^k(X \setminus \{0\}, Y) \) and has the property that \( h_i(x) x \) is \( C^k(X, R) \) for all \( i \).

Differentiating \( k \) times

\[ D^k h_{i+1}(\bar{x}) [C + DN_1(x + h_i(x))(I + Dh_i(x))]^k \]
\[ = [A + DN_2(x + h_i(x)) - Dh_{i+1}(\bar{x}) DN_1(X + h_i(x))] D^k h_i(x) + R_{i+1}(x), \]
where $R_k$ is a polynomial with constant coefficients in the variables $D^1 h_i(x)$, $D^2 N_1$, and $D^3 N_2$, with $0 \leq x \leq k-1$ and $0 \leq \beta, \gamma \leq k$. If we let

$$E_i(x) = C + DN_1(x + h_i(x))(I + Dh_i(x)),$$

$$F_i(x) = A + DN_2(x + h_i(x)) - Dh_i(x) DN_1(x + h_i(x))$$

then

$$D^k h_{i+1}(\bar{x}) \bar{x} = F_i(x) D^k h_i(x) E^{-k}(x) \bar{x} + R_k(x) E^{-k}(x) \bar{x}.$$

Since $\{h_i\}$ are uniformly bounded in the $C^{k-1}$ norm, $R_k(x) E^{-k}(x) \bar{x}$ is uniformly bounded and

$$|D^k h_{i+1}(\bar{x}) \bar{x}| \leq q \|D^k h_i(x) \cdot x\| + Q_k,$$

where $q < 1$ for $\mu$ sufficiently small and where $Q_k$ is independent of $i$. After induction,

$$\|D^k h_i(x) \cdot x\|_0 \leq q^i \|D^k h_0(x) \cdot x\|_0 + \frac{Q}{1-q}.$$  

This implies that $\{h_i(x) \cdot x\}$ is bounded in the $C^k$-norm and in particular it is bounded in the $C^{k-1,1}$-norm. Since $\{h_i(x) \cdot x\}$ is tending to $\{h(x) \cdot x\}$ in the $C^0$-norm, by Lemma 1 we have that $\{h(x) \cdot x\} \in C^{k-1,1}$.

Next we show that $\{h(x) \cdot x\} \in C^k$, using Lemma 2. Differentiating (26)

$$D^{k-1} h(x)[C + DN_1(x + h(x))(I + Dh(x))]^{k-1}$$

$$= [A + DN_2(x + h(x)) - Dh(x) DN_1(x + h(x))] D^{k-1} h(x) + R_{k-1}(x).$$  

Let

$$E(x) = C + DN_1(x + h(x))(I + Dh(x))$$

$$F(x) = A + DN_2(x + h(x))$$

$$G(x) \bar{x} = (E^{-1}(x))^{k-1} (Cx + N_1(x + h(x))),$$

then

$$D^{k-1} h(x) \bar{x} = [F(x) D^{k-1} h(x) + R_{k-1}(x)] G(x) \bar{x}.$$

Note that $Cx + N_1(x + h(x)) = O(|x|)$, and so $R_{k-1}(x) G(x) \bar{x}$ is continuously differentiable by Lemma 3. Thus
\[ D^{k-1}h(x+A)(\bar{x}+\bar{A}) - D^{k-1}h(\bar{x}) \bar{x} \]
\[ - D^{k-1}h(x_0 + A)(x_0 + A) + D^{k-1}h(\bar{x}_0) \bar{x}_0 \]
\[ = F(x + A) D^{k-1}h(x + A) G(x + A)(x + A) - F(x) D^{k-1}h(x) G(x) x \]
\[ - F(x_0 + A) D^{k-1}h(x_0 + A) G(x_0 + A)(x_0 + A) \]
\[ + F(x_0) D^{k-1}h(x_0) G(x_0) x_0 + o(|A|) \]
\[ \rightarrow F(x_0)[D^{k-1}h(x + A) G(x_0)(x + A) - D^{k-1}h(x) G(x_0) x \]
\[ - D^{k-1}h(x_0 + A) G(x_0)(x_0 + A) + D^{k-1}h(x_0) G(x_0) x_0] + o(|A|), \]
as \((x, A) \rightarrow (x_0, 0)\).

Next, define
\[ l = (x_0 + A) - \bar{x}_0. \]

Since
\[ l = C A N_1(x_0 + A + h(x_0 + A)) - N_1(x_0 + h(x_0)) \]
\[ = (C + D N_1(x_0 + h(x_0))) A + o(|A|) \]
we have
\[ |(C + |N_1|^{-1})|l| \geq |A|(1 + o(1)). \] (27)

Also, using (26)
\[ |(x + A) - \bar{x} + l| \leq |N_1| |h(x + A) - h(x) - h(x_0 + A) + h(x_0)| + o(|A|) \]
\[ = o(|A|), \]
as \((x, A) \rightarrow (x_0, 0)\), since \(h \in C^{k-1,1}\). Thus using (27) we obtain
\[ |(x + A) - \bar{x} + l| = o(|l|) \] (28)
as \((x, l) \rightarrow (x_0, 0)\). Then using (27) and (28) in Definition (22) we have as
\((x, A) \rightarrow (x_0, 0)\)
\[ \frac{1}{|l|} |D^{k-1}h(\bar{x} + l)(\bar{x} + l) - D^{k-1}h(\bar{x}) \bar{x} - D^{k-1}h(\bar{x}_0 + l)(\bar{x}_0 + l) \]
\[ + D^{k-1}h(\bar{x}_0) \bar{x}_0| \]
\[ \leq \frac{|A|}{|l|} |F(x_0)||G(x_0)| \lambda(D^{k-1}h(x, x_0) + o(1) \]
\[ \leq (|A| + a(\delta))(|C^{-1}| + c(\delta)) \lambda(D^{k-1}h(x, x_0) + o(1) \]
where \( a(\delta) \) and \( c(\delta) \) can be made small by making \( \mu \) sufficiently small and using a cutoff function outside of a small enough neighborhood of \((0, 0)\). Thus

\[
\lambda(D^k-1h(x) x, x_0) \leq s\lambda(D^k-1h(x) x, x_0)
\]

where \( s < 1 \) and so

\[
\sup_{x_0 \in X} \lambda(D^k-1h(x) x, x_0) \leq s \sup_{x_0 \in X} \lambda(D^k-1h(x) x, x_0) \leq \infty.
\]

This implies that \( \lambda(D^k-1h(x) x, x_0) = 0 \) for every \( x_0 \in X_1 \). By Lemma 2, the function \( D^k-1h(x) x \) is continuously differentiable and this of course implies that \( h(x) x \in C^k \).

7. \( C^k \)-Smooth Changes of Variables

7.1. Proof of Theorem 1

From the Hypotheses 1-2, the family of diffeomorphisms \( T_\mu \) may be written locally as

\[
\begin{align*}
\bar{x} &= A x + f(x, z, \mu) \\
\bar{z} &= z + g(z, \mu) + g_1(x, z, \mu)
\end{align*}
\]

(29)

where \( |\sigma(A)| < 1 \), the functions \( f, g \) and \( g_1 \) are \( C^k \)-smooth in \((x, z)\) and, with their derivatives, vanish for \((x, z, \mu) = (0, 0, 0)\).

It is known that for each \( \mu \) there is a \( C^k \)-smooth function \( X(z, \mu) \) from \( \mathbb{R} \) to \( X_1 \) whose graph is a center manifold for (29). Under the change of variables

\[
\begin{align*}
x' &= x - X(z, \mu) \\
z' &= z
\end{align*}
\]

(30)

the system (29) takes the form

\[
\begin{align*}
\bar{x} &= A x + f(x + X(z, \mu), z, \mu) - f(X(z, \mu), z, \mu) \\
\bar{z} &= z + g(z, \mu) + g_2(x, z, \mu) x.
\end{align*}
\]

(31)

Here \( g_2 \) is a \( C^k-1 \) function and \( g_2 \cdot x \) is \( C^k \). If we let

\[
\begin{align*}
f_1(x, z, \mu) &= f(x + X(z, \mu), z, \mu) - f(X(z, \mu), z, \mu),
\end{align*}
\]

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then \( f_j(0, z, \mu) = 0 \). In fact, by computation, it can be seen that all the partial derivatives of \( f_j \) with respect to \( z \) up to order \( k \) vanish identically for \( x = 0 \). By Taylor’s Theorem, for \( j \leq k - 1 \), we may write
\[
f_j(x, z, \mu) = \tilde{f}_j(x, z, \mu) x
\]
where \( \tilde{f}_j \in C^{k-j-1} \), and further
\[
\lim_{|x| \to 0} \frac{d^k f_j}{dz^k}(x, z, \mu) = 0.
\]

Next we seek a change of variables of the form
\[
\eta = z + h(x, z, \mu) x,
\]
with \( h(x, z, \mu) \in \mathbb{X}^* \), which reduces the second equation of (31) to the form
\[
\tilde{\eta} = \eta + g(\eta, \mu).
\]
(See [Af]). We obtain from (31), (32), and (33) that such a function \( h \) must satisfy
\[
h(x, z) x + g(z + h(x, z)x) = g(z) + g_2(x, z) x + h(x, z)(Ax + \tilde{f} x).
\]
If we let
\[
g(x, z, h(x, z), \mu)x \equiv g(z + h(x, z)x) - g(z),
\]
then (34) is equivalent to
\[
h(\tilde{x}, \tilde{z})(A + \tilde{f})x = h(x, z) x - g_2(x, z, \mu) x + \tilde{g}(x, z, h) x
\]
where \( \tilde{g} \) is \( C^{k-1} \) and \( \tilde{g} \cdot x \in C^k \). The equation (35) is true if we can find an \( h \) which satisfies
\[
h(\tilde{x}, \tilde{z}) = (h(x, z) - g_2(x, z, \mu) + \tilde{g}(x, z, h))(A + f_j)^{-1}.
\]

Now this equation is equivalent to an equation
\[
h(\tilde{x}, \tilde{y}, \mu) = h(x, y, \mu) A^{-1} + F(x, z, h(x, y, \mu), \mu),
\]
where the linear part of \( F \) is zero for \( (0, 0, 0, 0) \). This is satisfied if the graph of \( h \) is the center-stable manifold \( h = h(x, z, \mu) \) of the diffeomorphism
\[
\begin{align*}
\dot{x} &= Ax + \tilde{f}(x, z, \mu) x \\
\dot{z} &= z + g(z, \mu) + g_1(x, z, \mu) x \\
\dot{h} &= hA^{-1} + F(x, z, h, \mu).
\end{align*}
\]

It is well known that these equations have a center-stable manifold. The first two equations are \(C^k\) and the last equation is seen to be \(C^{k-1}\). Thus \(h\) exists and is \(C^k\). Furthermore, the last equation is seen to be \(C^k\) for \(x \neq 0\) and has the property that \(F(x, z, h, \mu)x \in C^k\). Therefore, by Lemma 4, the function \(h\) has the property that \(h(x, z, \mu)x \in C^k\). Thus, for each \(\mu\), there is a \(C^k\) change of variables of the form (32) which puts the system in the form (33).

### 7.2. Generalized Change of Variables Theorem

Let \(X\) and \(Z\) be Banach spaces. Consider the family of diffeomorphisms \(T_\mu\) on \(X \oplus Z\) given by

\[
\begin{align*}
\dot{x} &= Ax + f(x, z, \mu) \\
\dot{z} &= Bz + g(z, \mu) + g_1(x, z, \mu)
\end{align*}
\]

where the functions \(f, g\) and \(g_1\) are \(C^k\)-smooth in \((x, z)\) and, with their first derivatives, vanish for \((x, z, \mu) = (0, 0, 0)\). Suppose

\[
|\sigma(A)| < a \\
|\sigma(B)| < b_2.
\]

As in Section 6.3, let \(X(z, \mu)\) be a function whose graph is a center manifold for (36). Provided that

\[
a < b_1,
\]

it is known that for each \(\mu\) the function \(X\) is \(C^k\) in \(z\). This condition is known as a “gap condition” or “rate condition” \([\text{Fe}]\) \([\text{HPS}]\). Under the change of variables (30) the system (36) takes the form

\[
\begin{align*}
\dot{x} &= Ax + \tilde{f}(x, z, \mu) x \\
\dot{z} &= Bz + g(z, \mu) + g_1(x, z, \mu)x.
\end{align*}
\]

Here \(\tilde{f}, g_2\) are \(C^{k-1}\) functions, \(\tilde{f} \cdot x, g_2 \cdot x\) are \(C^k\), and equations (3) and (4) are satisfied.

Next we seek a change of variables of the form

\[
\eta = z + h(x, z, \mu) x,
\]
with \( h(x, z, \mu) \in L(X, Z) \) (the space of bounded linear operators from \( X \) to \( Z \)), which reduces the second equation of (38) to the form

\[
\bar{\eta} = B\eta + g(\eta, \mu).
\]

We obtain from (38), (39), and (40)

\[
Bh(x, z)x + g(z + h(x, z)x) = g(z) + g_2(x, z)x + h(x, z)(Ax + \hat{f}x).
\]

If we let

\[
\hat{g}(x, z, h(x, z), \mu) \equiv g(z + h(x, z)x) - g(z)
\]

then (41) is equivalent to

\[
h(\bar{x}, \bar{z})(A + \hat{f})x = Bh(x, z)x - g_2(x, z, \mu)x + \hat{g}(x, z, \mu)x
\]

where \( \hat{g} \) is \( C^{k-1} \) and \( \hat{g} \cdot x \in C^k \). The equality is true if

\[
h(\bar{x}, \bar{z}) = (Bh(x, z) - g_2(x, z, \mu) + \hat{g}(x, z, \mu))(A + f_1)^{-1}.
\]

This equation is satisfied if the graph of \( h(x, z, \mu) \) is an invariant manifold of the diffeomorphism

\[
\bar{x} = Ax + \hat{f}(x, z, \mu)x
\]

\[
\bar{z} = Bz + g(z, \mu) + g_2(x, z, \mu)x
\]

\[
h = Bh A^{-1} + F(x, z, h, \mu).
\]

The first two equations are \( C^k \) and the last equation is seen to be \( C^{k-1} \).

Using the assumption that \( A^{-1} \) is onto, we have

\[
\|Bh A^{-1}\| = \sup_{\|x\|=1} \|Bh A^{-1}x\| \geq b_1 a^{-1} \|h\|.
\]

Therefore, the operator \( B \cdot A^{-1} : L(X, Z) \to L(X, Z) \) has spectrum bounded below by \( b_1 a^{-1} \). Thus, provided that \( b_1 a^{-1} > b_2^{k-1} \), then \( h \) exists and is \( C^{k-1} \). Furthermore, the last equation is seen to be \( C^k \) for \( x \neq 0 \) and has the property \( F(x, z, h, \mu)x \in C^k \).

Now we note that, with no essential changes in the proofs, Lemmas 3 and 4 may be replaced by the following more general results.

**Lemma 5.** Let \( X \) and \( Z \) be a Banach spaces and \( Y = L(X, Z) \). Suppose that \( h \in C^{k-1}(X, Y) \cap C^k(X \setminus \{0\}, Y) \), and that \( h(x)x \in C^k(X, Z) \). Then \( h(x)g(x) \in C^k \) for any \( g \in C^k(X, X) \) such that \( g(x) = O(|x|^\epsilon) \).
**Lemma 6.** Let $X$ and $Z$ be a Banach space and let $Y = L(X, Z)$ and consider the map

\[
\begin{align*}
\tilde{x} &= Cx + N_1(x + y) \\
\tilde{y} &= Ay + N_2(x + y)
\end{align*}
\]

where $x \in X$, $y \in Y$, $N_1 \in C^k$, $k \geq 1$ and $N_2$ has the property that $N_2(x + y)x$ is in $C^k(X, \mathbb{R})$. Also assume that $N_1$ and $N_2$, along with their first derivatives vanish at the origin, and

\[a < c^k\]

where

\[
|\sigma(A)| \leq a < 1 \quad |\sigma(C)| \geq c.
\]

Then the unique local center-unstable manifold given as the graph of $h: X \to Y$ has the property that $h(x)x \in C^k(X, Z)$.

Thus provided that

\[b_1a^{-1} > b_2^2\] (42)

then for each $\mu$ there is a $C^k$ change of variables of the form (39) which puts the system in the form (40), and we have proved the following theorem.

**Theorem 3.** Suppose conditions (37) and (42) hold. Then there are $C^k$-smooth coordinates $(x, z)$, for which (36) has the form

\[
\begin{align*}
\tilde{x} &= Ax + \bar{f}(x, z, \mu)x \\
\tilde{z} &= Bz + g(z, \mu)
\end{align*}
\]

where $f = \bar{f}$, $x$ and $g$ are $C^k$-smooth in $(x, z)$, and are Lipschitz continuous in $\mu$. The functions $f, g$ and their first derivatives vanish for $(x, z, \mu) = (0, 0, 0)$ and $\bar{f}$ is $C^{k-1}$. Further, equations (3) and (4) hold.

This implies existence of $C^k$-smooth foliations of the center-stable and center-unstable manifolds of a partially hyperbolic fixed point.
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References


