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GENERALIZED VON KÁRMÁN EQUATIONS

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ABSTRACT. – In a previous work, the first author has identified three-dimensional boundary conditions "of von Kármán's type" that lead, through a formal asymptotic analysis of the three-dimensional solution, to the classical von Kármán equations, when they are applied to the entire lateral face of a nonlinearly elastic plate.

In this paper, we consider the more general situation where only a portion of the lateral face is subjected to boundary conditions of von Kármán's type, while the remaining portion is subjected to boundary conditions of free edge. We then show that the asymptotic analysis of the three-dimensional solution still leads in this case to a two-dimensional boundary value problem that is analogous to, but is more general than, the von Kármán equations. In particular, it is remarkable that the boundary conditions for the Airy function can still be determined solely from the data. © 2001 Éditions scientifiques et médicales Elsevier SAS

RÉSUMÉ. – Dans un travail antérieur, le premier auteur a identifié des conditions aux limites tri-dimensionnelles "de von Kármán" qui, lorsqu'elles sont appliquées à la totalité de la face latérale d'une plaque non linéairement élastique, conduisent, au moyen d'une analyse asymptotique formelle de la solution tri-dimensionnelle, aux équations classiques de von Kármán.

Dans ce travail, on considère la situation plus générale où seule une partie de la face latérale est soumise aux conditions aux limites de von Kármán, la partie restante étant soumise à des conditions aux limites de bord libre. On établit alors que l'analyse asymptotique de la solution tri-dimensionnelle conduit encore dans ce cas à un problème aux limites bi-dimensionnel plus général que les équations de von Kármán, mais qui leur reste analogue. Il est en particulier remarquable que les conditions aux limites pour la fonction d'Airy puissent être encore déterminées à partir des seules données. © 2001 Éditions scientifiques et médicales Elsevier SAS

1. Outline

The notations not defined here are defined in Section 2. Consider a nonlinearly elastic plate, with reference configuration $\overline{\Omega}^{\varepsilon} = \overline{\omega} \times [-\varepsilon, \varepsilon]$, $\omega \subset \mathbb{R}^2$, made with a *St Venant–Kirchhoff material* with Lamé constants $\lambda > 0$ and $\mu > 0$, subjected to *body forces* in its interior, to *surface forces* on its upper and lower faces, and to "*von Kármán surface forces*" on a portion $\gamma_1 \times [-\varepsilon, \varepsilon]$ of its lateral face, where $\gamma_1 \subset \gamma = \partial\omega$ and *length* $\gamma_1 > 0$. Such von Kármán surface forces have been proposed by Ciarlet [5]. The remaining portion $(\gamma - \gamma_1) \times [-\varepsilon, \varepsilon]$ of the lateral face is *free*. The *unknown displacement field* $\mathbf{u}^{\varepsilon} = (u_i^{\varepsilon})$ then satisfies the following three-dimensional

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boundary value problem:

$$-\partial_j^\varepsilon\big(\sigma_{ij}^\varepsilon+\sigma_{kj}^\varepsilon\partial_k^\varepsilon u_i^\varepsilon\big)=f_i^\varepsilon\quad\text{in }\Omega^\varepsilon,$$

$$u_\alpha^\varepsilon\text{ independent of }x_3^\varepsilon\text{ and }u_3^\varepsilon=0\text{ on }\gamma_1\times[-\varepsilon,\varepsilon],$$

$$\frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\big(\sigma_{\alpha\beta}^\varepsilon+\sigma_{k\beta}^\varepsilon\partial_k^\varepsilon u_\alpha^\varepsilon\big)\nu_\beta\,\mathrm{d}x_3^\varepsilon=h_\alpha^\varepsilon\quad\text{on }\gamma_1,$$

$$\big(\sigma_{ij}^\varepsilon+\sigma_{kj}^\varepsilon\partial_k^\varepsilon u_i^\varepsilon\big)n_j^\varepsilon=0\quad\text{on }\gamma_2\times[-\varepsilon,\varepsilon],$$

$$\big(\sigma_{ij}^\varepsilon+\sigma_{kj}^\varepsilon\partial_k^\varepsilon u_i^\varepsilon\big)n_j^\varepsilon=g_i^\varepsilon\quad\text{on }\omega\times\{-\varepsilon,\varepsilon\},$$

where $\gamma_2=\gamma-\gamma_1$ and

$$\sigma_{ij}^\varepsilon=\lambda E_{pp}^\varepsilon\big(\mathbf{u}^\varepsilon\big)\delta_{ij}+2\mu E_{ij}^\varepsilon\big(\mathbf{u}^\varepsilon\big)\quad\text{and}\quad E_{ij}^\varepsilon\big(\mathbf{u}^\varepsilon\big)=\frac{1}{2}\big(\partial_i^\varepsilon u_j^\varepsilon+\partial_j^\varepsilon u_i^\varepsilon+\partial_i^\varepsilon u_m^\varepsilon\partial_j^\varepsilon u_m^\varepsilon\big).$$

As shown in Ciarlet [5], the classical *two-dimensional von Kármán equations* are obtained by applying the method of formal asymptotic expansions to the solution to this problem, *under the assumption that* $\gamma_2=\emptyset$. The purpose of this paper is to *consider the more general case where* length $\gamma_2>0$.

Following a by now well-established procedure (see, e.g., [7, Chaps. 4 and 5]), this more general problem is first put in *variational*, or *weak*, form and "*scaled*" over the *fixed* domain $\Omega=\omega\times\,]-1,1[$. It is then assumed that its solution $\mathbf{u}(\varepsilon):\overline{\Omega}\to\mathbb{R}^3$ admits a formal asymptotic expansion of the form:

$$\mathbf{u}(\varepsilon)=\mathbf{u}^0+\varepsilon\mathbf{u}^1+\varepsilon^2\mathbf{u}^2+\varepsilon^3\mathbf{u}^3+\varepsilon^4\mathbf{u}^4+\cdots.$$

It is first shown that the leading term $\mathbf{u}^0=(u_i^0)$ of this expansion is such that

$$u_\alpha^0=\zeta_\alpha-x_3\partial_\alpha\zeta_3\quad\text{and}\quad u_3^0=\zeta_3,$$

where the field $\boldsymbol{\zeta}=(\zeta_i)$ satisfies a *two-dimensional problem*, which may be expressed either as a *variational problem* (Theorem 3; the *existence* of a solution to the variational problem is established in Theorem 4) or as a *boundary value problem* (Theorem 5). The main result of this paper (Theorem 7) then consists in showing that, if the solution to this variational problem is smooth enough, it also satisfies a *boundary value problem that generalizes the well-known von Kármán equations* (a converse property also holds; cf. Theorem 8). More specifically, assume that ω is simply connected, that its boundary γ is smooth, and that $\zeta_\alpha\in H^3(\omega)$ and $\zeta_3\in H^4(\omega)$. Then there exists an *Airy function* $\phi\in H^4(\omega)$ that satisfies

$$\partial_{11}\phi=N_{22},\qquad\partial_{12}\phi=-N_{12},\qquad\partial_{22}\phi=N_{11}\quad\text{in }\omega,$$

where

$$N_{\alpha\beta}=\frac{4\lambda\mu}{\lambda+2\mu}E_{\sigma\sigma}^0(\boldsymbol{\zeta})\delta_{\alpha\beta}+4\mu E_{\alpha\beta}^0(\boldsymbol{\zeta}),$$

$$E_{\alpha\beta}^0(\boldsymbol{\zeta})=\frac{1}{2}(\partial_\alpha\zeta_\beta+\partial_\beta\zeta_\alpha+\partial_\alpha\zeta_3\partial_\beta\zeta_3).$$

In addition, the pair $(\zeta_3,\phi)\in H^4(\omega)\times H^4(\omega)$ satisfies the following *generalized von Kármán equations*:

$$\frac{8\mu(\lambda+\mu)}{3(\lambda+2\mu)}\Delta^2\zeta_3 = [\phi,\zeta_3] + p_3 \quad \text{in } \omega,$$

$$\Delta^2\phi = -\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}[\zeta_3,\zeta_3] \quad \text{in } \omega,$$

$$\zeta_3 = \partial_\nu\zeta_3 = 0 \quad \text{on } \gamma_1,$$

$$m_{\alpha\beta}\nu_\alpha\nu_\beta = 0 \quad \text{on } \gamma_2,$$

$$(\partial_\alpha m_{\alpha\beta})\nu_\beta + \partial_\tau(m_{\alpha\beta}\nu_\alpha\tau_\beta) = 0 \quad \text{on } \gamma_2,$$

$$\phi = \phi_0 \quad \text{and} \quad \partial_\nu\phi = \phi_1 \quad \text{on } \gamma,$$

where

$$m_{\alpha\beta} = -\frac{1}{3}\left\{\frac{4\lambda\mu}{\lambda+2\mu}\Delta\zeta_3\delta_{\alpha\beta} + 4\mu\partial_{\alpha\beta}\zeta_3\right\},$$

and ϕ_0 and ϕ_1 are known functions in terms of the given functions h_α^ε.

In particular then, the *boundary conditions on the Airy function* can still be determined from the sole knowledge of the data even if *length* $\gamma_2 > 0$. Furthermore, the pair (ζ_3, ϕ) satisfies a boundary value problem that generalizes the well-known *von Kármán equations*, corresponding to the case where $\gamma_2 = \emptyset$.

These results were announced in [9].

2. The three-dimensional problem

Greek indices, corresponding to the "horizontal" coordinates, vary in the set $\{1,2\}$, while Latin indices vary in the set $\{1,2,3\}$, the index 3 corresponding to the "vertical" coordinate. The summation convention with respect to repeated indices is systematically used. The notions needed below from *three-dimensional nonlinear elasticity* are detailed in, e.g., [6].

Let ω be a *domain* in $\mathbb{R}^2$, i.e., a bounded, open, and connected subset of $\mathbb{R}^2$ with a Lipschitz-continuous boundary γ, the set ω being locally situated on a same side with respect to γ. Let $\gamma = \gamma_1 \cup \gamma_2$ be a partition of γ such that *length* $\gamma_1 > 0$.

Consider a *nonlinearly elastic plate* with middle surface $\overline{\omega}$ and thickness $2\varepsilon > 0$, made with a *St Venant–Kirchhoff material* with Lamé constants $\lambda^\varepsilon > 0$ and $\mu^\varepsilon > 0$. In particular then, the material constituting the plate is *homogeneous* and *isotropic* and the *reference configuration* $\overline{\omega} \times [-\varepsilon, \varepsilon]$ of the plate is a *natural state*.

Remark. – Although the "simplest" among all nonlinearly elastic materials that satisfy these assumptions, St Venant–Kirchhoff materials admittedly suffer from severe mechanical and mathematical drawbacks. They can nevertheless be safely employed for justifying, as here, nonlinear plate theories by means of an asymptotic analysis of the three-dimensional solution, because the two-dimensional nonlinear equations that are eventually obtained as the outcome of the asymptotic analysis are essentially the same as those that are obtained when more satisfactory models of nonlinearly elastic materials are used at the onset, but then at the expense of increased technical difficulties. Compare for instance the analysis of Ciarlet and Destuynder [8] and that of Davet [12] or the analysis of Le Dret and Raoult [18] and that of Ben Belgacem [3].

The plate is subjected to *body forces* in its interior $\Omega^\varepsilon = \omega \times]-\varepsilon, \varepsilon[$, with density $(f_i^\varepsilon) \in \mathbf{L}^2(\Omega^\varepsilon)$; to *surface forces* on its upper and lower faces $\Gamma_+^\varepsilon = \omega \times \{\varepsilon\}$ and $\Gamma_-^\varepsilon = \omega \times \{-\varepsilon\}$, with density $(g_i^\varepsilon) \in \mathbf{L}^2(\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon)$; and finally, to *horizontal surface forces* on the portion $\gamma_1 \times [-\varepsilon, \varepsilon]$ of its lateral face, whose only the *resultant after integration across the thickness*, with density $(h_\alpha^\varepsilon) \in \mathbf{L}^2(\gamma_1)$, is known.

Let $x^\varepsilon = (x_i^\varepsilon)$ denote a generic point in the set $\overline{\Omega^\varepsilon}$, let $\partial_i^\varepsilon = \partial/\partial x_i^\varepsilon$, let (n_i^ε) denote the unit outer normal vector along the boundary of the set Ω^ε, and finally, let (ν_α) denote the horizontal unit outer normal vector along the boundary γ of the set ω. The unknown displacement field $\mathbf{u}^\varepsilon = (u_i^\varepsilon) : \overline{\Omega} \to \mathbb{R}^3$ then satisfies the following *three-dimensional boundary value problem*:

$$-\partial_j^\varepsilon\big(\sigma_{ij}^\varepsilon + \sigma_{kj}^\varepsilon \partial_k^\varepsilon u_i^\varepsilon\big) = f_i^\varepsilon \quad \text{in } \Omega^\varepsilon,$$

$$u_\alpha^\varepsilon \text{ independent of } x_3^\varepsilon \text{ and } u_3^\varepsilon = 0 \text{ on } \gamma_1 \times [-\varepsilon, \varepsilon],$$

$$\frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon} \big(\sigma_{\alpha\beta}^\varepsilon + \sigma_{k\beta}^\varepsilon \partial_k^\varepsilon u_\alpha^\varepsilon\big)\nu_\beta \,\mathrm{d}x_3^\varepsilon = h_\alpha^\varepsilon \quad \text{on } \gamma_1,$$

$$\big(\sigma_{ij}^\varepsilon + \sigma_{kj}^\varepsilon \partial_k^\varepsilon u_i^\varepsilon\big) n_j^\varepsilon = 0 \quad \text{on } \gamma_2 \times [-\varepsilon, \varepsilon],$$

$$\big(\sigma_{ij}^\varepsilon + \sigma_{kj}^\varepsilon \partial_k^\varepsilon u_i^\varepsilon\big) n_j^\varepsilon = g_i^\varepsilon \quad \text{on } \Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon,$$

where

$$\sigma_{ij}^\varepsilon = \lambda^\varepsilon E_{pp}^\varepsilon\big(\mathbf{u}^\varepsilon\big)\delta_{ij} + 2\mu^\varepsilon E_{ij}^\varepsilon\big(\mathbf{u}^\varepsilon\big) \quad \text{and} \quad E_{ij}^\varepsilon\big(\mathbf{u}^\varepsilon\big) = \frac{1}{2}\big(\partial_i^\varepsilon u_j^\varepsilon + \partial_j^\varepsilon u_i^\varepsilon + \partial_i^\varepsilon u_m^\varepsilon \partial_j^\varepsilon u_m^\varepsilon\big).$$

The *stresses* $\sigma_{ij}^\varepsilon : \overline{\Omega^\varepsilon} \to \mathbb{R}$ are the components of the *second Piola–Kirchhoff stress tensor* and the *strains* $E_{ij}^\varepsilon(\mathbf{u}^\varepsilon)$ are those of the *Green–St Venant strain tensor*. The partial differential equations in Ω^ε together with the boundary conditions involving the stresses σ_{ij}^ε form the *equilibrium equations*, while the relations between the stresses σ_{ij}^ε and the strains $E_{ij}(\mathbf{u}^\varepsilon)$ form the *constitutive equation*. The boundary conditions on $\gamma_1 \times [-\varepsilon, \varepsilon]$ together with those on γ_1 are of the form proposed by Ciarlet [5] for justifying, in the special case where $\gamma_1 = \gamma$, the well-known *von Kármán equations* through a formal asymptotic analysis of the three-dimensional solution, with the thickness as the "small" parameter.

Our objective consists in extending this asymptotic analysis to the more general case where *length* $\gamma_2 > 0$, i.e., where the plate is also subjected to a boundary condition of *free edge* on the portion $\gamma_2 \times [-\varepsilon, \varepsilon]$ of its lateral face.

3. The method of formal asymptotic expansions

Following a by now well-established procedure (see, e.g., [7, Chaps. 4 and 5]), we begin by re-writing the boundary value problem of Section 2 in the weak form of the *principle of virtual work*. To this end, we simply use the *Green formula*, which shows that any smooth enough solution $\mathbf{u}^\varepsilon = (u_i^\varepsilon)$ to the boundary value problem of Section 2 also satisfies the following *variational problem* $\mathcal{P}(\Omega^\varepsilon)$:

$$\mathbf{u}^\varepsilon \in \mathbf{V}\big(\Omega^\varepsilon\big) = \big\{\mathbf{v}^\varepsilon = \big(v_i^\varepsilon\big) \in \mathbf{W}^{1,4}\big(\Omega^\varepsilon\big);\ v_\alpha^\varepsilon \text{ independent of } x_3^\varepsilon \text{ and } v_3^\varepsilon = 0 \text{ on } \gamma_1 \times [-\varepsilon, \varepsilon]\big\},$$

$$\int_{\Omega^\varepsilon} \big(\sigma_{ij}^\varepsilon + \sigma_{kj}^\varepsilon \partial_k^\varepsilon u_i^\varepsilon\big)\partial_j^\varepsilon v_i^\varepsilon \,\mathrm{d}x^\varepsilon = \int_{\Omega^\varepsilon} f_i^\varepsilon v_i^\varepsilon \,\mathrm{d}x^\varepsilon + \int_{\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon} g_i^\varepsilon v_i^\varepsilon \,\mathrm{d}\Gamma^\varepsilon + \frac{1}{2}\int_{\gamma_1} \left\{\int_{-\varepsilon}^{\varepsilon} v_\alpha^\varepsilon \,\mathrm{d}x_3^\varepsilon\right\} h_\alpha^\varepsilon \,\mathrm{d}\gamma$$

for all $\mathbf{v}^\varepsilon \in \mathbf{V}(\Omega^\varepsilon)$, where

$$\sigma_{ij}^\varepsilon = \lambda^\varepsilon E_{pp}^\varepsilon\big(\mathbf{u}^\varepsilon\big)\delta_{ij} + 2\mu^\varepsilon E_{ij}^\varepsilon\big(\mathbf{u}^\varepsilon\big) \quad \text{and} \quad E_{ij}^\varepsilon\big(\mathbf{u}^\varepsilon\big) = \frac{1}{2}\big(\partial_i^\varepsilon u_j^\varepsilon + \partial_j^\varepsilon u_i^\varepsilon + \partial_i^\varepsilon u_m^\varepsilon \partial_j^\varepsilon u_m^\varepsilon\big).$$

Note that the boundary conditions on $\gamma_1 \times [-\varepsilon, \varepsilon]$ imposed on the fields $\mathbf{v}^\varepsilon \in \mathbf{V}(\Omega^\varepsilon)$ insure that the boundary conditions on γ_1 appearing in the boundary value problem are indeed recovered by means of Green formula. The regularity imposed on the elements in the space $\mathbf{V}(\Omega^\varepsilon)$ merely guarantees that all the integrals found in the variational problem $\mathcal{P}(\Omega^\varepsilon)$ make sense.

We next define an equivalent variational problem, but now posed over a domain Ω that is *independent of* ε. This transformation involves *ad hoc assumptions on the data* λ^ε, μ^ε, f_i^ε, g_i^ε, and h_α^ε, regarding their asymptotic behaviors as functions of ε, and *ad hoc scalings* on the *unknowns* u_i^ε and also on the *stresses* σ_{ij}^ε. That we also scale the stresses mean that we use the "*displacement-stress approach*" originally advocated by Ciarlet and Destuynder [8], then justified by Raoult [21] who showed its equivalence with the otherwise more natural, but substantially more delicate, "*displacement approach*" (see also [7, Sections 4.3 and 4.7]).

More specifically, let $\Omega = \omega \times]-1, 1[$, let $\Gamma_\pm = \omega \times \{\pm 1\}$, let $x = (x_i)$ denote a generic point in the set $\overline{\Omega}$, and let ∂/∂_i. We then define the *scaled displacements* $u_i(\varepsilon) : \overline{\Omega} \to \mathbb{R}$ and the *scaled stresses* $\sigma_{ij}(\varepsilon) : \Omega \to \mathbb{R}$ by letting:

$$u_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2 u_\alpha(\varepsilon)(x), \qquad u_3^\varepsilon(x^\varepsilon) = \varepsilon u_3(\varepsilon)(x),$$

$$\sigma_{\alpha\beta}^\varepsilon(x^\varepsilon) = \varepsilon^2 \sigma_{\alpha\beta}(\varepsilon)(x), \qquad \sigma_{\alpha 3}^\varepsilon(x^\varepsilon) = \varepsilon^3 \sigma_{\alpha 3}(\varepsilon)(x), \qquad \sigma_{33}^\varepsilon(x^\varepsilon) = \varepsilon^4 \sigma_{33}(\varepsilon)(x)$$

for all $x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon$, where $\pi^\varepsilon(x_1, x_2, x_3) = (x_1, x_2, \varepsilon x_3)$. We next *assume* that there exist *constants* $\lambda > 0$ and $\mu > 0$ and functions $f_i \in L^2(\Omega)$, $g_i \in L^2(\Gamma_+ \cup \Gamma_-)$, and $h_\alpha \in L^2(\gamma_1)$ that are all *independent of* ε, such that

$$\lambda^\varepsilon = \lambda \quad \text{and} \quad \mu^\varepsilon = \mu,$$

$$f_i^\varepsilon(x^\varepsilon) = \varepsilon^3 f_i(x) \quad \text{for all } x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon,$$

$$g_i^\varepsilon(x^\varepsilon) = \varepsilon^4 g_i(x) \quad \text{for all } x^\varepsilon = \pi^\varepsilon x \in \Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon,$$

$$h_\alpha^\varepsilon(y) = \varepsilon^2 h_\alpha(y) \quad \text{for all } y \in \gamma_1.$$

Remarks. – (1) The assumptions on the functions f_α^ε and g_α^ε will ultimately guarantee that the functions $N_{\alpha\beta}$ found in Theorem 5 satisfy $\partial_\alpha N_{\alpha\beta} = 0$ in ω. These relations in turn insure that an *Airy function* may be associated with the two-dimensional problem found at the outcome of the asymptotic analysis; cf. Theorem 7.

(2) The above scalings and assumptions on the data have been justified by Miara [20], who showed that they constitute the necessary preliminaries to any asymptotic analysis that lead to a *nonlinear Kirchhoff–Love plate theory*, such as that found here (naturally, the assumptions on the data may take a more general form, as $\lambda^\varepsilon = \varepsilon^t \lambda$, $f_i^\varepsilon(x^\varepsilon) = \varepsilon^{3+t} f_i(x)$, etc., with t any fixed real number).

Thanks to these scalings and assumptions on the data, problem $\mathcal{P}(\Omega^\varepsilon)$ now takes the form of a variational problem $\mathcal{P}(\varepsilon; \Omega)$ posed over the *fixed* domain Ω:

THEOREM 1. – *The scaled displacement field* $\mathbf{u}(\varepsilon) = (u_i(\varepsilon))$ *satisfies the following variational problem* $\mathcal{P}(\varepsilon; \Omega)$:

$$\mathbf{u}(\varepsilon) \in \mathbf{V}(\Omega) = \{\mathbf{v} = (v_i) \in \mathbf{W}^{1,4}(\Omega);\ v_\alpha \textit{ independent of } x_3 \textit{ and } v_3 = 0 \textit{ on } \gamma_1 \times [-1, 1]\},$$

$$\int_\Omega \sigma_{ij}(\varepsilon) \partial_j v_i \,\mathrm{d}x + \int_\Omega \sigma_{ij}(\varepsilon) \partial_i u_3(\varepsilon) \partial_j v_3 \,\mathrm{d}x + \varepsilon^2 \int_\Omega \sigma_{ij}(\varepsilon) \partial_i u_\alpha(\varepsilon) \partial_j v_\alpha \,\mathrm{d}x$$

$$= \int_\Omega f_3 v_3 \,\mathrm{d}x + \int_{\Gamma_+\cup\Gamma_-} g_3 v_3 \,\mathrm{d}\Gamma + \frac{1}{2}\int_{\gamma_1}\left\{\int_{-1}^{1} v_\alpha \,\mathrm{d}x_3\right\} h_\alpha \,\mathrm{d}\gamma$$
$$+ \varepsilon^2\left(\int_\Omega f_\alpha v_\alpha \,\mathrm{d}x + \int_{\Gamma_+\cup\Gamma_-} g_\alpha v_\alpha \,\mathrm{d}\Gamma\right)$$

for all $\mathbf{v}\in\mathbf{V}(\Omega)$, *the scaled displacements* $u_i(\varepsilon)$ *and the scaled stresses* $\sigma_{ij}(\varepsilon)$ *being related by*:

$$\frac{1}{2}\big(\partial_\alpha u_\beta(\varepsilon) + \partial_\beta u_\alpha(\varepsilon) + \partial_\alpha u_3(\varepsilon)\partial_\beta u_3(\varepsilon)\big) + \frac{\varepsilon^2}{2}\partial_\alpha u_\sigma(\varepsilon)\partial_\beta u_\sigma(\varepsilon)$$
$$= -\frac{\lambda}{2\mu(3\lambda+2\mu)}\big\{\sigma_{\tau\tau}(\varepsilon) + \varepsilon^2\sigma_{33}(\varepsilon)\big\}\delta_{\alpha\beta} + \frac{1}{2\mu}\sigma_{\alpha\beta}(\varepsilon),$$
$$\frac{1}{2}\big(\partial_\alpha u_3(\varepsilon) + \partial_3 u_\alpha(\varepsilon) + \partial_\alpha u_3(\varepsilon)\partial_3 u_3(\varepsilon)\big) + \frac{\varepsilon^2}{2}\partial_\alpha u_\sigma(\varepsilon)\partial_3 u_\sigma(\varepsilon) = \frac{\varepsilon^2}{2\mu}\sigma_{\alpha 3}(\varepsilon),$$
$$\partial_3 u_3(\varepsilon) + \frac{1}{2}\partial_3 u_3(\varepsilon)\partial_3 u_3(\varepsilon) + \frac{\varepsilon^2}{2}\partial_3 u_\sigma(\varepsilon)\partial_3 u_\sigma(\varepsilon)$$
$$= -\frac{\lambda}{2\mu(3\lambda+2\mu)}\big\{\varepsilon^2\sigma_{\tau\tau}(\varepsilon) + \varepsilon^4\sigma_{33}(\varepsilon)\big\} + \frac{\varepsilon^4}{2\mu}\sigma_{33}(\varepsilon). \qquad \square$$

The variational problem $\mathcal{P}(\varepsilon;\Omega)$ constitutes the point of departure of our asymptotic analysis, inasmuch as its specific dependence on the parameter ε makes it amenable to the *method of formal asymptotic expansions* (for details about this well-known method, see, e.g., [2, Chap. XIV, Section 14] or [7, Section 4.3]):

THEOREM 2. – *Assume that the scaled displacements and stresses can be written as formal asymptotic expansions of the form*:

$$\mathbf{u}(\varepsilon) = \mathbf{u}^0 + \varepsilon\mathbf{u}^1 + \cdots \quad \textit{and} \quad \sigma_{ij}(\varepsilon) = \sigma_{ij}^0 + \varepsilon\sigma_{ij}^1 + \cdots$$

and that the leading terms of these expansions satisfy

$$\mathbf{u}^0 = \big(u_i^0\big) \in \mathbf{V}(\Omega), \qquad \partial_3 u_3^0 \in \mathcal{C}^0(\overline{\Omega}), \qquad \sigma_{ij}^0 = \sigma_{ji}^0 \in L^2(\Omega).$$

Then the cancellation of the factors of ε^0 *in problem* $\mathcal{P}(\varepsilon;\Omega)$ *shows that the leading term* $\mathbf{u}^0$ *should satisfy the following "limit" problem* $\mathcal{P}_{KL}(\Omega)$:

$$\mathbf{u}^0 \in \mathbf{V}_{KL}(\Omega) = \big\{\mathbf{v} = (v_i) \in \mathbf{H}^1(\Omega);\ v_\alpha \text{ independent of } x_3 \text{ and } v_3 = 0$$
$$\text{on } \gamma_1\times[-1,1],\ \partial_i v_3 + \partial_3 v_i = 0 \text{ in } \Omega\big\},$$

$$\int_\Omega \sigma_{\alpha\beta}^0 \partial_\beta v_\alpha \,\mathrm{d}x + \int_\Omega \sigma_{\alpha\beta}^0 \partial_\alpha u_3^0 \partial_\beta v_3 \,\mathrm{d}x$$
$$= \int_\Omega f_3 v_3 \,\mathrm{d}x + \int_{\Gamma_+\cup\Gamma_-} g_3 v_3 \,\mathrm{d}\Gamma + \frac{1}{2}\int_{\gamma_1}\left\{\int_{-1}^{1} v_\alpha \,\mathrm{d}x_3\right\} h_\alpha \,\mathrm{d}\gamma$$

for all $\mathbf{v}\in\mathbf{V}_{KL}(\Omega)$, *where*

$$\sigma^0_{\alpha\beta} = \frac{2\lambda\mu}{\lambda+2\mu} E^0_{\sigma\sigma}(\mathbf{u}^0)\delta_{\alpha\beta} + 2\mu E^0_{\alpha\beta}(\mathbf{u}^0),$$
$$E^0_{\alpha\beta}(\mathbf{u}^0) = \frac{1}{2}\big(\partial_\alpha u^0_\beta + \partial_\beta u^0_\alpha + \partial_\alpha u^0_3 \partial_\beta u^0_3\big).$$

Proof. – The proof is analogous to that corresponding to a clamped plate (see [7, Theorem 4.7-2]) and for this reason is omitted. Suffice it to say that the above variational equations indeed make sense for vector fields $\mathbf{v} = (v_i) \in \mathbf{H}^1(\Omega)$ satisfying $\partial_i v_3 + \partial_3 v_i = 0$ in Ω, as these relations imply that their third component v_3 is in $H^2(\Omega)$; cf. [7, Theorem 1.4-4]. □

Problem $\mathcal{P}_{KL}(\Omega)$, like its two-dimensional counterpart $\mathcal{P}(\omega)$ studied in the next section, is called a "*limit*" *problem* to remind that, since it is satisfied by the *leading term* $\mathbf{u}^0$ of the formal asymptotic expansion of the scaled unknown $\mathbf{u}(\varepsilon)$, it formally corresponds to letting $\varepsilon = 0$. The subscript "KL" reminds that $\mathbf{u}^0$ is a (scaled) *Kirchhoff–Love displacement field* (cf. Thm. 3).

4. The limit two-dimensional "displacement" problem

We now show that *the three-dimensional limit problem* $\mathcal{P}_{KL}(\Omega)$ *found in Theorem* 2 *is in effect a two-dimensional problem* "*in disguise*", in that any solution $\mathbf{u}^0 = (u^0_i) : \overline{\Omega} \to \mathbb{R}^3$ to $\mathcal{P}_{KL}(\Omega)$ can be computed from a solution $\boldsymbol{\zeta} = (\zeta_i) : \overline{\omega} \to \mathbb{R}^3$ to a two-dimensional problem, denoted $\mathcal{P}(\omega)$ below. This problem is called a "*displacement*" *problem* to reflect that its unknown is the (scaled) *displacement field* of the middle surface $\overline{\omega}$ of the plate.

THEOREM 3. – (a) *Define the space* (∂_ν *denotes the outer normal derivative along* γ):

$$\mathbf{V}(\omega) = \big\{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega);\ \eta_3 = \partial_\nu \eta_3 = 0 \text{ on } \gamma_1\big\}.$$

Then there exists $\boldsymbol{\zeta} = (\zeta_i) \in \mathbf{V}(\omega)$ *such that the components of the leading term* $\mathbf{u}^0 = (u^0_i)$ *satisfying problem* $\mathcal{P}_{KL}(\Omega)$ *are of the form*

$$u^0_\alpha = \zeta_\alpha - x_3 \partial_\alpha \zeta_3 \quad \text{and} \quad u^0_3 = \zeta_3.$$

(b) *Let*

$$a_{\alpha\beta\sigma\tau} = \frac{4\lambda\mu}{\lambda+2\mu}\delta_{\alpha\beta}\delta_{\sigma\tau} + 2\mu(\delta_{\alpha\sigma}\delta_{\beta\tau} + \delta_{\alpha\tau}\delta_{\beta\sigma}),$$
$$E^0_{\alpha\beta}(\boldsymbol{\eta}) = \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha + \partial_\alpha \eta_3 \partial_\beta \eta_3),$$
$$p_3 = \int_{-1}^{1} f_3 \, \mathrm{d}x_3 + g_3(\cdot, 1) + g_3(\cdot, -1).$$

Then $\mathbf{u}^0 = (u^0_i)$ *satisfies* $\mathcal{P}_{KL}(\Omega)$ *if and only if* $\boldsymbol{\zeta} = (\zeta_i)$ *satisfies the following two-dimensional variational problem* $\mathcal{P}(\omega)$: $\boldsymbol{\zeta} \in \mathbf{V}(\omega)$ *and*

$$\int_\omega a_{\alpha\beta\sigma\tau}\partial_{\sigma\tau}\zeta_3 \partial_{\alpha\beta}\eta_3 \, \mathrm{d}\omega + \int_\omega a_{\alpha\beta\sigma\tau} E^0_{\sigma\tau}(\boldsymbol{\zeta})\partial_\alpha \zeta_3 \partial_\beta \eta_3 \, \mathrm{d}\omega + \int_\omega a_{\alpha\beta\sigma\tau} E^0_{\sigma\tau}(\boldsymbol{\zeta})\partial_\beta \eta_\alpha \, \mathrm{d}\omega$$
$$= \int_\omega p_3 \eta_3 \, \mathrm{d}\omega + \int_{\gamma_1} h_\alpha \eta_\alpha \, \mathrm{d}\gamma$$

for all $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}(\omega)$.

Proof. – It is known (see, e.g., [7, Theorem 1.4-4]) that $\mathbf{v} = (v_i) \in \mathbf{V}_{KL}(\Omega)$ if and only if there exists $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}(\omega)$ such that $v_\alpha = \eta_\alpha - x_3 \partial_\alpha \eta_3$ and $v_3 = \eta_3$. Thanks to this equivalence, the variational equations of problem $\mathcal{P}_{KL}(\Omega)$ are easily converted into those of problem $\mathcal{P}(\omega)$, and vice versa. □

We next establish the *existence* of a solution to problem $\mathcal{P}(\omega)$.

THEOREM 4. – *Let ω be a domain in $\mathbb{R}^2$, let γ_1 be a subset of its boundary that satisfies* length $\gamma_1 > 0$, *let $p_3 \in L^2(\omega)$ and $h_\alpha \in L^2(\gamma_1)$ be given functions, and let $\mathcal{P}(\omega)$ be the variational problem found in Theorem* 3.

(a) *A necessary condition for the existence of a solution to $\mathcal{P}(\omega)$ is that the functions h_α satisfy the compatibility conditions*:

$$\int_{\gamma_1} h_1 \,\mathrm{d}\gamma = \int_{\gamma_1} h_2 \,\mathrm{d}\gamma = \int_{\gamma_1} (x_1 h_2 - x_2 h_1) \,\mathrm{d}\gamma = 0.$$

(b) *If the necessary condition of* (a) *is satisfied and the norms $\|h_\alpha\|_{L^2(\gamma_1)}$ are small enough, $\mathcal{P}(\omega)$ has at least one solution.*

Proof. – To begin with, we specify some notations: First, given $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}(\omega)$, we let:

$$\boldsymbol{\eta}_H = (\eta_\alpha) \quad \text{and} \quad e_{\alpha\beta}(\boldsymbol{\eta}_H) = \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha).$$

We then define the space:

$$\begin{aligned}\mathbf{V}_H^0(\omega) &= \{\boldsymbol{\eta}_H = (\eta_\alpha) \in \mathbf{H}^1(\omega);\ e_{\alpha\beta}(\boldsymbol{\eta}_H) = 0 \text{ in } \omega\} \\ &= \{\boldsymbol{\eta}_H = (\eta_\alpha);\ \eta_1 = a_1 - bx_2, \eta_2 = a_2 + bx_1 \text{ with } a_1, a_2, b \in \mathbb{R}\}.\end{aligned}$$

Finally, we let $|\cdot|_{0,\omega}$ and $\|\cdot\|_{m,\omega}$ denote the norms in the spaces $L^2(\omega)$ and $H^m(\omega)$, or $\mathbf{L}^2(\omega)$ and $\mathbf{H}^m(\omega)$ (boldface letters mean that we consider spaces of vector-valued functions).

(i) We first note that, if the variational equations of problem $\mathcal{P}(\omega)$ are satisfied for $\boldsymbol{\eta} = (\eta_i) \in \mathbf{V}(\omega)$, they must also be satisfied by $\boldsymbol{\eta}' = (\eta_1 + a_1 - bx_2, \eta_2 + a_2 + bx_1, \eta_3)$ for any constants a_1, a_2, b, since $\boldsymbol{\eta}'$ is again in $\mathbf{V}(\omega)$. Hence *we must have* $\int_{\gamma_1} h_\alpha \eta_\alpha \,\mathrm{d}\gamma = 0$ *for all* $\boldsymbol{\eta}_H = (\eta_\alpha) \in \mathbf{V}_H^0(\omega)$, since the other terms in the variational equations are unaltered; or equivalently

$$\int_{\gamma_1} h_1 \,\mathrm{d}\gamma = \int_{\gamma_2} h_2 \,\mathrm{d}\gamma = \int_{\gamma_1} (x_1 h_2 - x_2 h_1) \,\mathrm{d}\gamma = 0.$$

Hence (a) is proved.

(ii) We next show that *solving problem $\mathcal{P}(\omega)$ is equivalent to finding the stationary points of an* ad hoc *functional over an* ad hoc *function space*.

To this end, we first define a function $J : \mathbf{V}(\omega) \to \mathbb{R}$ by letting:

$$J(\boldsymbol{\eta}) = \frac{1}{2} \int_\omega \left\{ \frac{1}{3} a_{\alpha\beta\sigma\tau} \partial_{\sigma\tau} \eta_3 \partial_{\alpha\beta} \eta_3 + a_{\alpha\beta\sigma\tau} E^0_{\sigma\tau}(\boldsymbol{\eta}) E^0_{\alpha\beta}(\boldsymbol{\eta}) \right\} \mathrm{d}\omega - \left(\int_\omega p_3 \eta_3 \,\mathrm{d}\omega + \int_{\gamma_1} h_\alpha \eta_\alpha \,\mathrm{d}\gamma \right),$$

for any $\boldsymbol{\eta} \in \mathbf{V}(\omega)$. Noting X/Y the quotient space of X by Y, we then define the space (here, $\mathbf{H}^1(\omega) = (H^1(\omega))^2$)

$$\tilde{\mathbf{V}}(\omega) = \{\mathbf{H}^1(\omega)/\mathbf{V}_H^0(\omega)\} \times V_3(\omega),$$

where

$$V_3(\omega) = \{\eta_3 \in H^2(\omega);\ \eta_3 = \partial_\nu \eta_3 = 0 \text{ on } \gamma_1\}.$$

Given any $\boldsymbol{\eta}_H = (\eta_\alpha) \in \mathbf{H}^1(\omega)$, let $\tilde{\boldsymbol{\eta}}_H \in \mathbf{H}^1(\omega)/\mathbf{V}_H^0(\omega)$ denote the equivalence class of $\boldsymbol{\eta}_H$, and let $\tilde{\boldsymbol{\eta}} = (\tilde{\boldsymbol{\eta}}_H, \eta_3)$, with $\tilde{\boldsymbol{\eta}}_H \in \mathbf{H}^1(\omega)/\mathbf{V}_H^0(\omega)$ and $\eta_3 \in V_3(\omega)$, denote a generic element in the space $\tilde{\mathbf{V}}(\omega)$.

If the necessary condition of (a) is satisfied, we have $J(\boldsymbol{\eta}_H, \eta_3) = J(\boldsymbol{\zeta}_H, \eta_3)$ for any $\boldsymbol{\zeta}_H \in \tilde{\boldsymbol{\eta}}_H$. Hence we can unambiguously define a *functional* $\tilde{J} : \tilde{\mathbf{V}}(\omega) \to \mathbb{R}$ by letting, for each $\tilde{\boldsymbol{\eta}} = (\tilde{\boldsymbol{\eta}}_H, \eta_3)$,

$$\tilde{J}(\tilde{\boldsymbol{\eta}}) = J(\boldsymbol{\zeta}_H, \eta_3) \quad \text{for any } \boldsymbol{\zeta}_H \in \tilde{\boldsymbol{\eta}}_H.$$

We then note that, as a sum of continuous multi-linear forms, the functional $\tilde{J}$ is differentiable (in fact, infinitely so) over the space $\tilde{\mathbf{V}}(\omega)$, equipped with its "natural" norm $\|\cdot\|_{\tilde{\mathbf{V}}(\omega)}$ defined for any $\tilde{\boldsymbol{\eta}} = (\tilde{\boldsymbol{\eta}}_H, \boldsymbol{\eta}_3)$ by

$$\|\tilde{\boldsymbol{\eta}}\|_{\tilde{\mathbf{V}}(\omega)} = \|\tilde{\boldsymbol{\eta}}_H\|_{\mathbf{H}^1(\omega)/\mathbf{V}_H^0(\omega)} + \|\eta_3\|_{2,\omega},$$

where

$$\|\tilde{\boldsymbol{\eta}}_H\|_{\mathbf{H}^1(\omega)/\mathbf{V}_H^0(\omega)} = \inf_{\boldsymbol{\zeta}_H \in \tilde{\boldsymbol{\eta}}_H} \|\boldsymbol{\zeta}_H\|_{\mathbf{H}^1(\omega)}.$$

For arbitrary elements $\tilde{\boldsymbol{\zeta}}$, $\tilde{\boldsymbol{\eta}} \in \tilde{\mathbf{V}}(\omega)$, the Gâteaux derivatives $\tilde{J}'(\tilde{\boldsymbol{\zeta}})(\tilde{\boldsymbol{\eta}})$ are obtained by computing the linear part with respect to $\tilde{\boldsymbol{\eta}}$ in the difference $\{\tilde{J}(\tilde{\boldsymbol{\zeta}} + \tilde{\boldsymbol{\eta}}) - \tilde{J}(\tilde{\boldsymbol{\zeta}})\}$. This gives

$$\begin{aligned}\tilde{J}'(\tilde{\boldsymbol{\zeta}})(\tilde{\boldsymbol{\eta}}) &= \int_\omega a_{\alpha\beta\sigma\tau} \partial_{\sigma\tau}\zeta_3 \partial_{\alpha\beta}\eta_3 \,\mathrm{d}\omega + \int_\omega a_{\alpha\beta\sigma\tau} E^0_{\sigma\tau}(\boldsymbol{\zeta}) \partial_\alpha \zeta_3 \partial_\beta \eta_3 \,\mathrm{d}\omega \\ &\quad + \int_\omega a_{\alpha\beta\sigma\tau} E^0_{\sigma\tau}(\boldsymbol{\zeta}) \partial_\beta \eta_\alpha \,\mathrm{d}\omega - \left(\int_\omega p_3 \eta_3 \,\mathrm{d}\omega + \int_{\gamma_1} h_\alpha \eta_\alpha \,\mathrm{d}\gamma \right).\end{aligned}$$

Hence $\boldsymbol{\zeta} \in \mathbf{V}(\omega)$ *satisfies problem* $\mathcal{P}(\omega)$ *if and only if* $\tilde{J}'(\tilde{\boldsymbol{\zeta}}) = 0$, i.e., *if and only if* $\tilde{\boldsymbol{\zeta}}$ is a stationary point of the functional $\tilde{J}$ over the space $\tilde{\mathbf{V}}(\omega)$.

(iii) *The functional* $\tilde{J}$ *is sequentially weakly lower semi-continuous over the space* $\tilde{\mathbf{V}}(\omega)$.

The *quadratic* part

$$\tilde{\boldsymbol{\eta}} = (\tilde{\boldsymbol{\eta}}_H, \eta_3) \to \frac{1}{6} \int_\omega a_{\alpha\beta\sigma\tau} \partial_{\sigma\tau}\eta_3 \partial_{\alpha\beta}\eta_3 \,\mathrm{d}\omega + \frac{1}{2} \int_\omega a_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\tilde{\boldsymbol{\eta}}_H) e_{\alpha\beta}(\tilde{\boldsymbol{\eta}}_H) \,\mathrm{d}\omega$$

of the functional $\tilde{J}$ is weakly lower semi-continuous, as a strongly continuous and convex function (the convexity is a consequence of the inequality $a_{\alpha\beta\sigma\tau} t_{\sigma\tau} t_{\alpha\beta} \geqslant 4\mu t_{\alpha\beta} t_{\alpha\beta}$, which holds for all symmetric matrices $(t_{\alpha\beta})$).

Let next $(\tilde{\boldsymbol{\zeta}}^k)_{k=0}^\infty$ be a weakly convergent sequence in $\tilde{\mathbf{V}}(\omega)$ and let $\tilde{\boldsymbol{\zeta}}$ denotes its weak limit. The *linear* part

$$L : \tilde{\boldsymbol{\eta}} \to -\left(\int_\omega p_3 \eta_3 \,\mathrm{d}\omega + \int_{\gamma_1} h_\alpha \eta_\alpha \,\mathrm{d}\gamma \right)$$

of the functional $\tilde{J}$ is such that

$$L(\tilde{\boldsymbol{\zeta}}^k) \to L(\tilde{\boldsymbol{\zeta}}) \quad \text{as } k \to \infty,$$

by definition of weak convergence, since L is strongly continuous.

To study to behavior of the *cubic* and *quartic* parts, we first observe that there exist $\boldsymbol{\eta}_H^k \in \tilde{\boldsymbol{\zeta}}_H^k$ and $\boldsymbol{\eta}_H \in \tilde{\boldsymbol{\zeta}}_H$ such that (weak convergence is denoted $\rightharpoonup$)

$$\boldsymbol{\eta}_H^k \rightharpoonup \boldsymbol{\eta}_H \quad \text{in } \mathbf{H}^1(\omega).$$

Then the weak convergences $e_{\alpha\beta}(\boldsymbol{\eta}_H^k) \rightharpoonup e_{\alpha\beta}(\boldsymbol{\eta}_H)$ in $L^2(\omega)$ together with the compact imbedding of $H^1(\omega)$ into $L^4(\omega)$ implies that

$$\int_\omega a_{\alpha\beta\sigma\tau} e_{\sigma\tau}\big(\boldsymbol{\eta}_H^k\big)\partial_\alpha \zeta_3^k \partial_\beta \zeta_3^k \,\mathrm{d}\omega \to \int_\omega a_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\boldsymbol{\eta})\partial_\alpha \zeta_3 \partial_\beta \zeta_3 \,\mathrm{d}\omega$$

as $k \to \infty$; the same compact imbedding implies that

$$\int_\omega a_{\alpha\beta\sigma\tau} \partial_\sigma \zeta_3^k \partial_\tau \zeta_3^k \partial_\alpha \zeta_3^k \partial_\beta \zeta_3^k \,\mathrm{d}\omega \to \int_\omega a_{\alpha\beta\sigma\tau} \partial_\sigma \zeta_3 \partial_\tau \zeta_3 \partial_\alpha \zeta_3 \partial_\beta \zeta_3 \,\mathrm{d}\omega$$

as $k \to \infty$.

(iv) *If the norms $\|h_\alpha\|_{L^2(\gamma_1)}$ are small enough, the functional $\tilde{J}$ is coercive on the space $\tilde{\mathbf{V}}(\omega)$.*

An inspection of the various terms found in the functional $\tilde{J}$ shows that:

$$\tilde{J}(\tilde{\boldsymbol{\eta}}) \geqslant \frac{2\mu}{3}|\eta_3|_{2,\omega}^2 + 2\mu \sum_{\alpha,\beta} \big|E_{\alpha\beta}^0(\tilde{\boldsymbol{\eta}})\big|_{0,\omega}^2 - c_1|\eta_3|_{0,\omega} - c_2\|\tilde{\boldsymbol{\eta}}_H\|_{\mathbf{H}^1(\omega)/\mathbf{V}_H^0(\omega)}$$

for all $\tilde{\boldsymbol{\eta}} = (\tilde{\boldsymbol{\eta}}_H, \eta_3) \in \tilde{\mathbf{V}}(\omega)$, where $|\eta_3|_{2,\omega}^2 = \sum_{\alpha,\beta} |\partial_{\alpha\beta}\eta_3|_{0,\omega}^2$, $c_1 = |p_3|_{0,\omega}$, and

$$c_2 = \chi \left\{ \sum_\alpha \|h_\alpha\|_{L^2(\gamma_1)}^2 \right\}^{1/2},$$

χ denoting the norm of the trace operator from $H^1(\omega)$ into $L^2(\gamma_1)$.

There exists a constant $c_3 > 0$ such that

$$\|\tilde{\boldsymbol{\eta}}_H\|_{\mathbf{H}^1(\omega)/\mathbf{V}_H^0(\omega)} \leqslant c_3 \sum_{\alpha,\beta} \big|e_{\alpha\beta}(\tilde{\boldsymbol{\eta}}_H)\big|_{0,\omega}$$

for all $\tilde{\boldsymbol{\eta}}_H \in \mathbf{H}^1(\omega)/\mathbf{V}_H^0(\omega)$ (this two-dimensional Korn inequality in the quotient space $\mathbf{H}^1(\omega)/\mathbf{V}_H^0(\omega)$ is established as in [14, Chap. 3, Theorem 3.4]). This inequality, combined with the definition of the functions $E_{\alpha\beta}^0(\tilde{\boldsymbol{\eta}})$ and with the continuous imbedding of $H^1(\omega)$ into $L^4(\omega)$, shows that there exists a constant c_4 such that:

$$c_3^{-1}\|\tilde{\boldsymbol{\eta}}_H\|_{\mathbf{H}^1(\omega)/\mathbf{V}_H^0(\omega)} \leqslant \sum_{\alpha,\beta} \big|E_{\alpha\beta}^0(\tilde{\boldsymbol{\eta}})\big|_{0,\omega} + \frac{1}{2}\sum_{\alpha,\beta} \|\partial_\alpha \eta_3\|_{L^4(\omega)} \|\partial_\beta \eta_3\|_{L^4(\omega)}$$
$$\leqslant \sum_{\alpha,\beta} \big|E_{\alpha\beta}^0(\tilde{\boldsymbol{\eta}})\big|_{0,\omega} + c_4\|\eta_3\|_{2,\omega}^2$$

for all $\tilde{\boldsymbol{\eta}} = (\tilde{\boldsymbol{\eta}}_H, \eta_3) \in \tilde{\mathbf{V}}(\omega)$. Since *length* $\gamma_1 > 0$, there exists a constant $c_5 > 0$ such that $c_5\|\eta_3\|_{2,\omega} \leqslant |\eta_3|_{2,\omega}$ for all $\eta_3 \in V_3(\omega)$.

Together, the previous inequalities thus give

$$\tilde{J}(\tilde{\boldsymbol{\eta}}) \geqslant \left(\frac{2\mu}{3}c_5^2 - c_2c_3c_4\right)\|\eta_3\|_{2,\omega}^2 - c_1\|\eta_3\|_{2,\omega} + 2\mu\sum_{\alpha,\beta}\left|E^0_{\alpha\beta}(\tilde{\boldsymbol{\eta}})\right|^2_{0,\omega} - c_2c_3\sum_{\alpha,\beta}\left|E^0_{\alpha\beta}(\tilde{\boldsymbol{\eta}})\right|_{0,\omega}$$

for all $\tilde{\boldsymbol{\eta}} = (\tilde{\boldsymbol{\eta}}_H, \eta_3) \in \tilde{\mathbf{V}}(\omega)$. Consequently, if $2\mu c_5^2 > 3c_2c_3c_4$, i.e., if the norms $\|h_\alpha\|_{L^2(\gamma_1)}$ are small enough, there exist constants $c_6 > 0$, $c_7 > 0$, and c_8 such that

$$\tilde{J}(\tilde{\boldsymbol{\eta}}) \geqslant c_6\|\eta_3\|_{2,\omega}^2 + c_7\sum_{\alpha,\beta}\left|E^0_{\alpha\beta}(\tilde{\boldsymbol{\eta}})\right|^2_{0,\omega} + c_8$$

for all $\tilde{\boldsymbol{\eta}} = (\tilde{\boldsymbol{\eta}}_H, \eta_3) \in \tilde{\mathbf{V}}(\omega)$. To conclude, we then simply observe that the relation

$$\|\tilde{\boldsymbol{\eta}}\|_{\tilde{\mathbf{V}}(\omega)} = \left(\|\tilde{\boldsymbol{\eta}}_H\|_{\mathbf{H}^1(\omega)/\mathbf{V}^0_H(\omega)} + \|\eta_3\|_{2,\omega}\right) \to +\infty$$

implies that $(\sum_{\alpha,\beta}|E_{\alpha\beta}(\tilde{\boldsymbol{\eta}})|^2_{0,\omega} + \|\eta_3\|^2_{2,\omega}) \to +\infty$, hence that $\tilde{J}(\tilde{\boldsymbol{\eta}}) \to +\infty$. □

We next write the *boundary value problem* that is, at least formally, equivalent to the variational problem $\mathcal{P}(\omega)$. In what follows, (ν_α) denotes the unit outer normal vector along γ, (τ_α) denotes the unit tangential vector defined by $\tau_1 = -\nu_2$, $\tau_2 = \nu_1$, and ∂_ν and ∂_τ denote the associated normal and tangential derivatives along γ.

THEOREM 5. – *Assume that the boundary γ is smooth enough. Then any smooth enough solution $\boldsymbol{\zeta} = (\zeta_i)$ of the variational problem $\mathcal{P}(\omega)$ found in Theorem* 3 *also satisfies the following boundary value problem*:

$$\begin{aligned}
\frac{8\mu(\lambda+\mu)}{3(\lambda+2\mu)}\Delta^2\zeta_3 - N_{\alpha\beta}\partial_{\alpha\beta}\zeta_3 = p_3 &\quad \textit{in } \omega,\\
\partial_\beta N_{\alpha\beta} = 0 &\quad \textit{in } \omega,\\
\zeta_3 = \partial_\nu\zeta_3 = 0 &\quad \textit{on } \gamma_1,\\
N_{\alpha\beta}\nu_\beta = h_\alpha &\quad \textit{on } \gamma_1,\\
m_{\alpha\beta}\nu_\alpha\nu_\beta = 0 &\quad \textit{on } \gamma_2,\\
(\partial_\alpha m_{\alpha\beta})\nu_\beta + \partial_\tau(m_{\alpha\beta}\nu_\alpha\tau_\beta) = 0 &\quad \textit{on } \gamma_2,\\
N_{\alpha\beta}\nu_\beta = 0 &\quad \textit{on } \gamma_2,
\end{aligned}$$

where

$$\begin{aligned}
m_{\alpha\beta} &= -\frac{1}{3}a_{\alpha\beta\sigma\tau}\partial_{\sigma\tau}\zeta_3 = -\frac{1}{3}\left\{\frac{4\lambda\mu}{\lambda+2\mu}\Delta\zeta_3\delta_{\alpha\beta} + 4\mu\partial_{\alpha\beta}\zeta_3\right\},\\
N_{\alpha\beta} &= a_{\alpha\beta\sigma\tau}E^0_{\sigma\tau}(\boldsymbol{\zeta}) = \frac{4\lambda\mu}{\lambda+2\mu}E^0_{\sigma\sigma}(\boldsymbol{\zeta})\delta_{\alpha\beta} + 4\mu E^0_{\alpha\beta}(\boldsymbol{\zeta}).
\end{aligned}$$

Proof. – The proof rests on the *Green formulas*

$$\begin{aligned}
-\int_\omega m_{\alpha\beta}\partial_{\alpha\beta}\eta_3\,\mathrm{d}\omega = -\int_\omega(\partial_{\alpha\beta}m_{\alpha\beta})\eta_3\,\mathrm{d}\omega + \int_\gamma\left\{(\partial_\alpha m_{\alpha\beta})\nu_\beta + \partial_\tau(m_{\alpha\beta}\nu_\alpha\tau_\beta)\right\}\eta_3\,\mathrm{d}\gamma\\
-\int_\gamma m_{\alpha\beta}\nu_\alpha\nu_\beta\partial_\nu\eta_3\,\mathrm{d}\gamma,
\end{aligned}$$

$$\int_\omega N_{\alpha\beta}\partial_\alpha\zeta_3\partial_\beta\eta_3\,\mathrm{d}\omega = -\int_\omega \bigl\{\partial_\beta(N_{\alpha\beta}\partial_\alpha\zeta_3)\bigr\}\eta_3\,\mathrm{d}\omega + \int_\gamma (N_{\alpha\beta}\partial_\alpha\zeta_3)\nu_\beta\eta_3\,\mathrm{d}\gamma,$$

$$\int_\omega N_{\alpha\beta}\partial_\beta\eta_\alpha\,\mathrm{d}\omega = -\int_\omega (\partial_\beta N_{\alpha\beta})\eta_\alpha\,\mathrm{d}\omega + \int_\gamma N_{\alpha\beta}\nu_\beta\eta_\alpha\,\mathrm{d}\gamma,$$

valid for all vector fields $\boldsymbol{\eta} = (\eta_i) \in H^1(\omega)\times H^1(\omega)\times H^2(\omega)$ and all functions $m_{\alpha\beta}\in H^2(\omega)$ and $N_{\alpha\beta}\in H^1(\omega)$). Also used are the relation

$$\partial_{\alpha\beta}m_{\alpha\beta} = \frac{8\mu(\lambda+\mu)}{3(\lambda+2\mu)}\Delta^2\zeta_3,$$

the relations $\partial_\beta N_{\alpha\beta} = 0$ in ω, which allow to replace $\partial_\beta(N_{\alpha\beta}\partial_\alpha\zeta_3)$ by $N_{\alpha\beta}\partial_{\alpha\beta}\zeta_3$ in the first partial differential equation in ω, and the relations $N_{\alpha\beta}\nu_\beta = 0$ on γ_2, which allow to cancel the term $(N_{\alpha\beta}\partial_\alpha\zeta_3)\nu_\beta$ otherwise appearing in the second boundary condition on γ_2. □

In order that this boundary value problem be expressed in terms of “physical” quantities, it remains to “de-scale” the unknowns: To this end we are naturally led, in view of the scalings made in Section 3, to define the “*limit*” *displacement field* $\boldsymbol{\zeta}^\varepsilon = (\zeta_i^\varepsilon):\overline{\omega}\to\mathbb{R}^3$ of the middle surface of the plate through the *de-scalings*:

$$\zeta_\alpha^\varepsilon = \varepsilon^2\zeta_\alpha \quad \text{and} \quad \zeta_3^\varepsilon = \varepsilon\zeta_3 \quad \text{in } \overline{\omega}.$$

Together with the assumptions on the data made in Section 3, these de-scalings lead to the following immediate corollary to Theorem 5 (naturally, the variational problem $\mathcal{P}(\omega)$ of Theorem 3 could be likewise de-scaled):

THEOREM 6. – *Assume that the boundary γ is smooth enough and that $\boldsymbol{\zeta} = (\zeta_i)$ is a smooth enough solution of problem $\mathcal{P}(\omega)$. Then the corresponding de-scaled limit displacement field $\boldsymbol{\zeta}^\varepsilon = (\zeta_i^\varepsilon)$ satisfies the following boundary value problem*:

$$\begin{aligned}
\frac{8\mu^\varepsilon(\lambda^\varepsilon+\mu^\varepsilon)}{3(\lambda^\varepsilon+2\mu^\varepsilon)}\varepsilon^3\Delta^2\zeta_3^\varepsilon - N_{\alpha\beta}^\varepsilon\partial_{\alpha\beta}\zeta_3^\varepsilon = p_3^\varepsilon &\quad \text{in } \omega,\\
\partial_\beta N_{\alpha\beta}^\varepsilon = 0 &\quad \text{in } \omega,\\
\zeta_3^\varepsilon = \partial_\nu\zeta_3^\varepsilon = 0 &\quad \text{on } \gamma_1,\\
N_{\alpha\beta}^\varepsilon\nu_\beta = h_\alpha^\varepsilon &\quad \text{on } \gamma_1,\\
m_{\alpha\beta}^\varepsilon\nu_\alpha\nu_\beta = 0 &\quad \text{on } \gamma_2,\\
(\partial_\alpha m_{\alpha\beta}^\varepsilon)\nu_\beta + \partial_\tau(m_{\alpha\beta}^\varepsilon\nu_\alpha\tau_\beta) = 0 &\quad \text{on } \gamma_2,\\
N_{\alpha\beta}^\varepsilon\nu_\beta = 0 &\quad \text{on } \gamma_2,
\end{aligned}$$

where

$$\begin{aligned}
m_{\alpha\beta}^\varepsilon &= -\frac{\varepsilon^3}{3}\left\{\frac{4\lambda^\varepsilon\mu^\varepsilon}{\lambda^\varepsilon+2\mu^\varepsilon}\Delta\zeta_3^\varepsilon\delta_{\alpha\beta} + 4\mu^\varepsilon\partial_{\alpha\beta}\zeta_3^\varepsilon\right\},\\
N_{\alpha\beta}^\varepsilon &= \varepsilon\left\{\frac{4\lambda^\varepsilon\mu^\varepsilon}{\lambda^\varepsilon+2\mu^\varepsilon}E_{\sigma\sigma}^0\bigl(\boldsymbol{\zeta}^\varepsilon\bigr)\delta_{\alpha\beta} + 4\mu^\varepsilon E_{\alpha\beta}^0\bigl(\boldsymbol{\zeta}^\varepsilon\bigr)\right\},\\
E_{\alpha\beta}^0(\boldsymbol{\zeta}^\varepsilon) &= \frac{1}{2}\bigl(\partial_\alpha\zeta_\beta^\varepsilon + \partial_\beta\zeta_\alpha^\varepsilon + \partial_\alpha\zeta_3^\varepsilon\partial_\beta\zeta_3^\varepsilon\bigr),
\end{aligned}$$

$$p_3^\varepsilon = \int_{-\varepsilon}^{\varepsilon} f_3^\varepsilon \, \mathrm{d}x_3^\varepsilon + g_3^\varepsilon(\cdot, \varepsilon) + g_3^\varepsilon(\cdot, -\varepsilon).$$

The partial differential equations in ω found in Theorem 6 show that the limit two-dimensional equations justified here belong to the *nonlinear Kirchhoff–Love plate theory*, like those of a *nonlinearly elastic clamped plate* justified by a similar method by Ciarlet and Destuynder [8]. We recall that a nonlinear Kirchhoff–Love plate theory is essentially characterized by the scalings that are made at the onset of the asymptotic analysis, of order two and one with respect to the horizontal and vertical components of the displacement, respectively. These scalings eventually produce semilinear partial differential equations of the fourth order with respect to the vertical component ζ_3^ε and of the second order with respect to the horizontal components ζ_α^ε, which reduce to those of the *linear Kirchhoff–Love plate theory* (see, e.g., [7, Section 1.7]) when only the linear terms with respect to the unknowns are retained. The same scalings also produce a limit displacement field across the thickness of the plate that is a *Kirchhoff–Love displacement field*, i.e., that is of the form $((\zeta_\alpha^\varepsilon - x_3^\varepsilon \partial_\alpha \zeta_3^\varepsilon), \zeta_3^\varepsilon)$.

For further comments about the nonlinear Kirchhoff–Love theory, see in particular [7, Section 4.9]. For its relation and difference with other "limit" two-dimensional nonlinear theories for planar elastic bodies, see in particular [15], where the crucial influence of the scalings in this respect is particularly well highlighted.

Remark. – The coefficient $\frac{8\mu^\varepsilon(\lambda^\varepsilon+\mu^\varepsilon)}{3(\lambda^\varepsilon+2\mu^\varepsilon)}$ factorizing $\Delta^2\zeta_3^\varepsilon$ in the first partial differential equation is the *flexural rigidity* of the plate.

5. Equivalence of the limit two-dimensional displacement problem with generalized von Kármán equations

Under the crucial assumption that the domain ω is simply connected, we now establish (in two stages; cf. Theorems 7 and 8) the *equivalence, within the class of smooth solutions, of the two-dimensional "displacement" boundary value problem found in Section* 4 *with a two-dimensional problem that generalizes the well-known von Kármán equations*. While the unknowns in the former problem are the three components ζ_i^ε of the limit displacement field $\boldsymbol{\zeta}^\varepsilon$ of the middle surface of the plate, there are only two unknowns in the latter, one being the vertical component ζ_3^ε of the displacement field $\boldsymbol{\zeta}^\varepsilon$ of the middle surface of the plate and the other being an *Airy function* ϕ^ε, from the knowledge of which the horizontal components ζ_α^ε can be determined.

Without loss of generality, we henceforth assume that the origin 0 belongs to the boundary γ of ω.

THEOREM 7. – *Assume that the domain ω is simply connected and that its boundary γ is smooth enough. Let there be given a solution (ζ_i^ε) of the boundary value problem found in Theorem* 6 *with the regularity*

$$\zeta_\alpha^\varepsilon \in H^3(\omega) \quad \textit{and} \quad \zeta_3^\varepsilon \in H^4(\omega).$$

Then the functions $\tilde{h}_\alpha^\varepsilon : \gamma \to \mathbb{R}$ defined by $\tilde{h}_\alpha^\varepsilon = h_\alpha^\varepsilon$ on γ_1 and by $\tilde{h}_\alpha^\varepsilon = 0$ on γ_2 necessarily belong to the space $H^{3/2}(\gamma)$ and they necessarily satisfy the compatibility relations:

$$\int_\gamma \tilde{h}_1^\varepsilon \, \mathrm{d}\gamma = \int_\gamma \tilde{h}_2^\varepsilon \, \mathrm{d}\gamma = \int_\gamma \left(x_1\tilde{h}_2^\varepsilon - x_2\tilde{h}_1^\varepsilon\right) \mathrm{d}\gamma = 0.$$

In addition, there exists an **Airy function** $\phi^\varepsilon \in H^4(\omega)$*, uniquely determined by the requirements that* $\phi^\varepsilon(0) = \partial_\alpha \phi^\varepsilon(0) = 0$*, such that*

$$N_{11}^\varepsilon = \varepsilon \partial_{22}\phi^\varepsilon, \qquad N_{12}^\varepsilon = -\varepsilon \partial_{12}\phi^\varepsilon, \qquad N_{22}^\varepsilon = \varepsilon \partial_{11}\phi^\varepsilon \quad \text{in } \omega.$$

Finally, the pair $(\zeta_3^\varepsilon, \phi^\varepsilon) \in H^4(\omega) \times H^4(\omega)$ *satisfies the following* **generalized von Kármán equations**:

$$\begin{aligned}
\frac{8\mu^\varepsilon(\lambda^\varepsilon + \mu^\varepsilon)}{3(\lambda^\varepsilon + 2\mu^\varepsilon)}\varepsilon^3 \Delta^2 \zeta_3^\varepsilon &= \varepsilon\big[\phi^\varepsilon, \zeta_3^\varepsilon\big] + p_3^\varepsilon \quad \text{in } \omega,\\
\Delta^2\phi^\varepsilon &= -\frac{\mu^\varepsilon(3\lambda^\varepsilon + 2\mu^\varepsilon)}{\lambda^\varepsilon + \mu^\varepsilon}\big[\zeta_3^\varepsilon, \zeta_3^\varepsilon\big] \quad \text{in } \omega,\\
\zeta_3^\varepsilon = \partial_\nu \zeta_3^\varepsilon &= 0 \quad \text{on } \gamma_1,\\
m_{\alpha\beta}^\varepsilon \nu_\alpha \nu_\beta &= 0 \quad \text{on } \gamma_2,\\
\big(\partial_\alpha m_{\alpha\beta}^\varepsilon\big)\nu_\beta + \partial_\tau\big(m_{\alpha\beta}^\varepsilon \nu_\alpha \tau_\beta\big) &= 0 \quad \text{on } \gamma_2,\\
\phi^\varepsilon = \phi_0^\varepsilon \quad \text{and} \quad \partial_\nu \phi^\varepsilon &= \phi_1^\varepsilon \quad \text{on } \gamma,
\end{aligned}$$

where $m_{\alpha\beta}^\varepsilon$, $N_{\alpha\beta}^\varepsilon$*, and* p_3^ε *are defined as in Theorem* 6*, and*

$$[\eta, \chi] = \partial_{11}\eta\partial_{22}\chi + \partial_{22}\eta\partial_{11}\chi - 2\partial_{12}\eta\partial_{12}\chi,$$

$$\phi_0^\varepsilon(y) = -y_1 \int_{\gamma(y)} \tilde{h}_2^\varepsilon \,\mathrm{d}\gamma + y_2 \int_{\gamma(y)} \tilde{h}_1^\varepsilon \,\mathrm{d}\gamma + \int_{\gamma(y)} \big(x_1\tilde{h}_2^\varepsilon - x_2\tilde{h}_1^\varepsilon\big)\,\mathrm{d}\gamma, \quad y \in \gamma,$$

$$\phi_1^\varepsilon(y) = -\nu_1(y) \int_{\gamma(y)} \tilde{h}_2^\varepsilon \,\mathrm{d}\gamma + \nu_2(y) \int_{\gamma(y)} \tilde{h}_1^\varepsilon \,\mathrm{d}\gamma, \quad y \in \gamma,$$

where $\gamma(y)$ *denotes the oriented arc joining* 0 *to* y *along* γ.

Proof. – For convenience, the proof is given in terms of "scaled" unknowns ζ_i and ϕ defined by $\zeta_3^\varepsilon = \varepsilon\zeta_3$, $\zeta_\alpha^\varepsilon = \varepsilon^2\zeta_\alpha$, and $\phi^\varepsilon = \varepsilon^2\phi$ and in terms of "scaled" data λ, μ, p_3, and h_α defined as in Sections 3 and 4.

(i) The assumed regularity on the functions ζ_i imply that $N_{\alpha\beta} \in H^2(\omega)$ and $N_{\alpha\beta}\nu_\beta = \tilde{h}_\alpha$ on the *entire* boundary γ. Hence the functions $\tilde{h}_\alpha$ belong to the space $H^{3/2}(\gamma)$. Besides, they satisfy the announced compatibility relations, as these are simply a re-statement of part (a) in Theorem 4.

(ii) Since the domain ω is simply connected, the equation $\partial_\beta N_{\alpha\beta} = 0$ in ω imply that there exist distributions $\psi_\alpha \in \mathcal{D}'(\omega)$, unique up to the addition of constants, such that (see [23, Theorem VI, p. 59]):

$$N_{1\alpha} = \partial_2\psi_\alpha \quad \text{and} \quad N_{2\alpha} = -\partial_1\psi_\alpha.$$

Since the equation $N_{12} = N_{21}$ in ω implies that $\partial_\alpha\psi_\alpha = 0$, there likewise exists a distribution $\phi \in \mathcal{D}'(\omega)$, unique up to the addition of polynomials of degree $\leqslant 1$, such that

$$\psi_1 = \partial_2\phi \quad \text{and} \quad \psi_2 = -\partial_1\phi,$$

hence such that

$$N_{11} = \partial_{22}\phi, \qquad N_{12} = -\partial_{12}\phi, \qquad N_{22} = \partial_{11}\phi.$$

As shown by Amrouche and Girault [1], a domain ω is a *Nikodym set* in the sense of Deny and Lions [13, p. 328], i.e., any distribution $T \in \mathcal{D}'(\omega)$ such that $\partial_\alpha T \in L^2(\omega)$ is in $L^2(\omega)$. Consequently, the assumed regularities $N_{\alpha\beta} \in H^2(\omega)$ imply that $\phi \in H^4(\omega)$. Clearly, ϕ is then uniquely defined if we impose that $\phi(0) = \partial_\alpha \phi(0) = 0$.

(iii) The relations just established between the functions $\partial_{\alpha\beta}\phi$ and $N_{\alpha\beta}$ show that

$$\partial_\tau(\partial_2\phi) = \nu_1\partial_{22}\phi - \nu_2\partial_{21}\phi = N_{1\beta}\nu_\beta = \tilde{h}_1,$$
$$-\partial_\tau(\partial_1\phi) = -\nu_1\partial_{12}\phi + \nu_2\partial_{11}\phi = N_{2\beta}\nu_\beta = \tilde{h}_2,$$

along the boundary γ. For any $y \in \gamma$, we thus have

$$\partial_1\phi(y) = -\int_{\gamma(y)} \tilde{h}_2 \,\mathrm{d}\gamma \quad \text{and} \quad \partial_2\phi(y) = \int_{\gamma(y)} \tilde{h}_1 \,\mathrm{d}\gamma,$$

so that

$$\partial_\nu\phi(y) = -\nu_1(y) \int_{\gamma(y)} \tilde{h}_2 \,\mathrm{d}\gamma + \nu_2(y) \int_{\gamma(y)} \tilde{h}_1 \,\mathrm{d}\gamma,$$
$$\partial_\tau\phi(y) = -\tau_1(y) \int_{\gamma(y)} \tilde{h}_2 \,\mathrm{d}\gamma + \tau_2(y) \int_{\gamma(y)} \tilde{h}_1 \,\mathrm{d}\gamma.$$

Hence

$$\phi = \phi_0 \quad \text{and} \quad \partial_\nu\phi = \phi_1 \quad \text{on } \gamma,$$

where the functions ϕ_0 and ϕ_1 are of the form given in the theorem. Note in passing that these boundary conditions provide another means of deriving the compatibility conditions that must be satisfied by the functions $\tilde{h}_\alpha$.

(iv) The expression of the functions $N_{\alpha\alpha}$ in terms of the functions ζ_i show that

$$\Delta^2\phi = \Delta(N_{\alpha\alpha}) = \frac{2\mu(3\lambda + 2\mu)}{\lambda + 2\mu}\bigl\{2\Delta(\partial_\alpha\zeta_\alpha) + \Delta(\partial_\alpha\zeta_3\partial_\alpha\zeta_3)\bigr\}.$$

Thanks to the relations $\partial_\alpha N_{\alpha\beta} = 0$, which imply in particular that

$$0 = \partial_{\alpha\beta}N_{\alpha\beta} = \frac{8\mu(\lambda + \mu)}{\lambda + 2\mu}\Delta(\partial_\alpha\zeta_\alpha) + \frac{2\lambda\mu}{\lambda + 2\mu}\Delta(\partial_\alpha\zeta_3\partial_\alpha\zeta_3) + 2\mu\partial_{\alpha\beta}(\partial_\alpha\zeta_3\partial_\beta\zeta_3),$$

the expression $\Delta(\partial_\alpha\zeta_\alpha)$ in $\Delta^2\phi$ can be replaced by a function of ζ_3 only. In this fashion, we obtain

$$\Delta^2\phi = -\frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}[\zeta_3, \zeta_3],$$

and the proof is complete. □

Remarks. – (1) The regularity and compatibility conditions satisfied by the functions $\tilde{h}_\alpha^\varepsilon$ are consequences of the *assumption* of the existence of a solution (ζ_i^ε) with ad hoc *regularity* to the boundary value problem found in Theorem 6. There is otherwise no reason why these properties should be satisfied *in general*.

(2) Naturally, the classical *von Kármán equations* are recovered by letting $\gamma_1 = \gamma$.

(3) The situation is substantially more delicate if ω is not simply connected. In this direction, see in particular [11] and [16].

The next result is the converse to Theorem 7.

THEOREM 8. – *Assume that the functions* $\tilde{h}^\varepsilon_\alpha$ *defined as in Theorem* 7 *are in the space* $H^{3/2}(\gamma)$. *Let there be given a solution* $(\zeta^\varepsilon_3, \phi^\varepsilon)$ *of the generalized von Kármán equations of Theorem* 7 *with the regularity*

$$\zeta^\varepsilon_3 \in H^4(\omega) \quad \textit{and} \quad \phi^\varepsilon \in H^4(\omega).$$

Then the functions $\tilde{h}^\varepsilon_\alpha$ *necessarily satisfy the same compatibily relations as in Theorem* 7.

Next, define functions $N^\varepsilon_{\alpha\beta} \in H^2(\omega)$ *by letting*:

$$N^\varepsilon_{11} = \varepsilon\partial_{22}\phi^\varepsilon, \qquad N^\varepsilon_{12} = N^\varepsilon_{21} = -\varepsilon\partial_{12}\phi^\varepsilon, \qquad N^\varepsilon_{22} = \varepsilon\partial_{11}\phi^\varepsilon \quad \textit{in } \omega.$$

Then there exist functions $\zeta^\varepsilon_\alpha \in H^3(\omega)$ *such that*

$$N^\varepsilon_{\alpha\beta} = \varepsilon\left\{\frac{4\lambda^\varepsilon\mu^\varepsilon}{\lambda^\varepsilon + 2\mu^\varepsilon} E^0_{\sigma\sigma}(\boldsymbol{\zeta}^\varepsilon)\delta_{\alpha\beta} + 4\mu^\varepsilon E^0_{\alpha\beta}(\boldsymbol{\zeta}^\varepsilon)\right\},$$

where $\boldsymbol{\zeta}^\varepsilon = (\zeta^\varepsilon_i)$ *and*

$$E^0_{\alpha\beta}(\boldsymbol{\zeta}^\varepsilon) = \frac{1}{2}\left(\partial_\alpha\zeta^\varepsilon_\beta + \partial_\beta\zeta^\varepsilon_\alpha + \partial_\alpha\zeta^\varepsilon_3\partial_\beta\zeta^\varepsilon_3\right),$$

and the vector field $\boldsymbol{\zeta}^\varepsilon$ *satisfies the boundary value problem found in Theorem* 6.

Proof. – As the proof is essentially the same as that of Theorem 5.6-1(b) in [7] (see also [5, Theorem 5.1]), it is omitted. We simply mention that the field $(\zeta^\varepsilon_\alpha) \in \mathbf{H}^3(\omega)$ is uniquely determined up to the addition of fields (η_α) with components of the form $\eta_1 = a_1 - bx_2$, $\eta_2 = a_2 + bx_1$. □

6. Conclusions and commentary

We have thus generalized the asymptotic analysis of Ciarlet [5], by showing that a nonlinearly elastic plate may be again modeled by equations generalizing the von Kármán equations, even if the three-dimensional "von Kármán surface forces" are only applied to a *portion* of its lateral face, the remaining portion being free.

To this end, we established in particular the somewhat unexpected result that the *boundary conditions on the Airy function* ϕ^ε (which otherwise always exists; see the proof of Theorem 7) can still be determined on the entire boundary γ solely from the data h^ε_α on γ_1, a circumstance that in turn affords the possibility of writing a boundary value problem with ζ^ε_3 and ϕ^ε as sole unknowns (Theorem 7).

Other three-dimensional boundary conditions may surely lead to similar generalized von Kármán equations, for instance, boundary conditions corresponding to "live" von Kármán surface forces, as considered by Blanchard and Ciarlet [4], or boundary conditions of "simple support" on $\gamma_2 \times [-\varepsilon, \varepsilon]$, as considered by Schaeffer and Golubitsky [22] and Gratie [17]; see Ciarlet and Gratie [10].

However, there seem to be counter-examples. For instance, if the boundary γ of ω is partitioned as $\gamma = \gamma_0 \cup \gamma_1 \cup \gamma_2$, the three-dimensional boundary conditions being the same as here on $\gamma_1 \times [-\varepsilon, \varepsilon]$ and $\gamma_2 \times [-\varepsilon, \varepsilon]$, and of the form $u^\varepsilon_i = 0$ on $\gamma_0 \times [-\varepsilon, \varepsilon]$, it seems unlikely that the boundary conditions on the Airy function could still be determined along the entire boundary γ solely from the data of the three-dimensional problem; see again Ciarlet and Gratie [10].

The equivalence between the limit "displacement" boundary value problem of Theorem 6 and the generalized von Kármán equations of Theorem 7 is established under the assumption of existence of *smooth* solutions to either problem. Whereas such an assumption is not unduly restrictive when $\gamma_2 = \phi$ (because von Kármán equations have smooth solutions for smooth data; see [19, Theorem 4.4, p. 56]), it undoubtedly becomes a severe, but seemingly unavoidable, restriction in the more general case (treated here) where *length* $\gamma_2 > 0$.

This restriction does not prevent, however, a mathematical analysis of the generalized von Kármán equations "for themselves".

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