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Compact embedding from $W_0^{1,2}(\Omega)$ to $L^{q(x)}(\Omega)$ and its application to nonlinear elliptic boundary value problem with variable critical exponent

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Abstract

We study the compact embedding from $W_0^{1,2}(\Omega)$ to $L^{q(x)}(\Omega)$ with a variable critical exponent $1 \leqslant q(x) \leqslant 2N/(N-2)$, $N \geqslant 3$ if there exist a point $x_0 \in \Omega$, a small $\eta > 0$, $0 < l < 1$ and $C_0 > 0$ such that $q(x_0) = 2N/(N-2)$ and $q(x) \leqslant 2N/(N-2) - C_0/(\log(1/|x-x_0|))^l$ for $|x-x_0| \leqslant \eta$. As an application, we show an existence of a positive solution to the nonlinear elliptic boundary value problem $-\Delta u = u^{q(x)-1}$ in Ω, $u(x) = 0$ on $\partial\Omega$.

Keywords: Compact embedding; Sobolev spaces; Variable exponent; Nonlinear elliptic boundary value problem

1. Introduction and main results

There are many studies on properties of the generalized Lebesgue–Sobolev spaces $W^{k,p(x)}(\Omega)$ with variable exponent $p(x)$, especially the embedding theorem from $W^{k,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ with $1 \leqslant p(x) \leqslant q(x) \leqslant Np(x)/(N-kp(x))$ under certain assumptions on the domain $\Omega \subset \mathbf{R}^N$ and $p(x)$. Here $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$ are defined by

$$L^{p(x)}(\Omega) = \left\{u\colon\ u \text{ is a real-valued measurable function on } \Omega,\ \int_\Omega |u(x)|^{p(x)}\,dx < +\infty\right\},$$

$$W^{k,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega)\colon\ D^\alpha u \in L^{p(x)}(\Omega),\ |\alpha| \leqslant k\right\}.$$

It is known that $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$ become Banach spaces with the following norm:

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$$|u|_{p(x)} = \inf\left\{\lambda > 0\colon \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leqslant 1\right\},$$

$$\|u\|_{W^{k,p(x)}} = |u|_{p(x)} + \sum_{|\alpha|\leqslant k} \left|D^\alpha u\right|_{p(x)}.$$

Suppose Ω is a bounded domain with smooth boundary for simplicity. Then, it is known that for a Lipschitz continuous function $p(x)$ and a measurable function $q(x)$ satisfying $1 < \inf_{x\in\Omega} p(x) \leqslant \sup_{x\in\Omega} p(x) < N$ and $p(x) \leqslant q(x) \leqslant Np(x)/(N-p(x))$ for $x \in \Omega$, there is a continuous embedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ (see [11, Theorem 1.1]). Actually, the Lipschitz continuity of $p(x)$ has been weekend to the local uniform continuity condition

$$\left|p(x) - p(y)\right| \leqslant \frac{C}{|\log|x-y||}, \quad x, y \in \Omega,$$

where C is a positive constant (see [5, Corollary 5.3]). Furthermore, if $\operatorname{ess\,inf}_{x\in\Omega}(Np(x)/(N-p(x)) - q(x)) > 0$, then for a continuous function $p(x)$ there exists a continuous compact embedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ (see [11, Theorem 1.3]). For other properties of the generalized Lebesgue–Sobolev spaces, we refer the reader to papers [5,7,11,16] and references therein.

However, as far as we know, even in the simplest case $p(x) \equiv 2$ and $N \geqslant 3$ there are no results on the compact embedding from $W_0^{1,2}(\Omega)$ to $L^{q(x)}(\Omega)$ when $1 \leqslant q(x) \leqslant 2N/(N-2)$, $x \in \Omega$ and $q(x_0) = 2N/(N-2)$ at some point $x_0 \in \Omega$. In this paper we study this problem and give a condition for $q(x)$ to assure the compact embedding from $W_0^{1,2}(\Omega)$ to $L^{q(x)}(\Omega)$. As an application, we discuss an existence of a positive solution $u \in W_0^{1,2}(\Omega)$ to the following nonlinear elliptic boundary value problem with variable exponent $2 < q(x) \leqslant 2N/(N-2)$,

$$-\Delta u(x) = u(x)^{q(x)-1}, \quad x \in \Omega, \qquad u(x) = 0, \quad x \in \partial\Omega. \tag{1}$$

When $2 < \operatorname{ess\,inf}_{x\in\Omega} q(x) \leqslant \operatorname{ess\,sup}_{x\in\Omega} q(x) < 2N/(N-2)$, the existence of a positive solution to (1) can be shown by using the standard mountain pass theorem for any bounded domain Ω even if the exponent $q(x)$ is variable (see e.g., [22]). However, when $q(x) \equiv 2N/(N-2)$ in (1), the existence of nontrivial solutions to (1) depends on the geometry and topology of the domain Ω and there are many works on the solvability of (1) (see e.g., [3,14,17,19–21] and references therein). For example, when $q(x) = 2N/(N-2)$ and Ω is star-shaped then it is known that there exists no nontrivial solutions to (1). On the other hand, if Ω is an annulus, more generally if Ω has a nontrivial topology [3], then there exists a positive solution. Recently, under the uniform subcritical condition, i.e. $\operatorname{ess\,inf}_{x\in\Omega}(Np(x)/(N-p(x)) - q(x)) > 0$, nonlinear elliptic boundary value problems of the type $-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = |u|^{q(x)-2}u$ with variable exponents has been studied by using the critical point theory (see [4,6,8–10,12] and references therein). In [2], Alves and Souto studied the existence of nonnegative solutions of $-\nabla(|\nabla u|^{p(x)-2}\nabla u) = u^{q(x)-1}$ in $\mathbf{R}^N$ under the following conditions on $p(x)$ and $q(x)$: $p(x)$ and $q(x)$ are radially symmetric, $1 < \operatorname{ess\,inf}_{x\in\mathbf{R}^N} p(x) \leqslant \operatorname{ess\,sup}_{x\in\mathbf{R}^N} p(x) < N$, $p(x) \leqslant q(x) \leqslant 2N/(N-2)$, and that there exist positive constants δ and R such that $\delta < R$, $q(x) = 2N/(N-2)$ for $|x| \leqslant \delta$ and $|x| \geqslant R$, and $p(x) = 2$ for $|x| \leqslant \delta$ and $|x| \geqslant R$. However, even for the case $p(x) \equiv 2$ there are no other results for the critical case, i.e. $\operatorname{ess\,inf}_{x\in\Omega}(Np(x)/(N-p(x)) - q(x)) = 0$.

By using our main result (Theorem 2) on the compact embedding, we give a partial answer to the existence of positive solutions $u \in W_0^{1,2}(\Omega)$ of (1) with a critical variable exponent $p(x)$ (Theorem 3).

Theorem 1. *Let Ω be a bounded domain in $\mathbf{R}^N$ with $N \geqslant 3$. Let $q(x)$ be a measurable function on Ω satisfying $1 \leqslant q(x) \leqslant 2N/(N-2)$ for almost every $x \in \Omega$. If there exist a point $x_0 \in \Omega$ and constants $C_0 > 0$, $\eta > 0$ such that*

$$q(x) \geqslant \frac{2N}{N-2} - \frac{C_0}{\log(1/|x-x_0|)} \quad \text{for a.e. } x \in \Omega \text{ with } |x-x_0| \leqslant \eta, \tag{2}$$

then the embedding from $W_0^{1,2}(\Omega)$ to $L^{q(x)}(\Omega)$ is not compact.

The condition above is sharp in the following sense.

Theorem 2. *Let N, Ω and $q(x)$ be as in Theorem* 1. *Suppose that there exist a point $x_0 \in \Omega$ and constants $C_0 > 0$, $\eta > 0$ and $0 < l < 1$ such that*

$$\operatorname*{ess\,sup}_{x\in\Omega,\,|x-x_0|\geqslant\eta} q(x) < \frac{2N}{N-2}$$

and

$$q(x) \leqslant \frac{2N}{N-2} - \frac{C_0}{(\log(1/|x-x_0|))^l} \quad \text{for a.e. } x \in \Omega \text{ with } |x-x_0| \leqslant \eta.$$

Then the embedding from $W_0^{1,2}(\Omega)$ to $L^{q(x)}(\Omega)$ is compact.

Remark 1. The logarithmic condition in Theorem 1 also appears in other situations. For example, for the mapping properties of fractional integral operators in the Hölder spaces of variable order, see [15,23]; for regularity theories in the study of partial differential equations with $p(x)$-growth, see [1,13]; for the Lavrentev phenomenon and an energy concentration in variational integrals with $p(x)$-growth conditions, see [18,24].

As a consequence of Theorem 2, we obtain the following result.

Theorem 3. *Let N, Ω and $q(x)$ be as in Theorem* 1 *and suppose all hypotheses in Theorem* 2. *Suppose also that $\partial\Omega$ is smooth and $\operatorname{ess\,inf}_{x\in\Omega} q(x) > 2$. Then there exists a positive solution $u \in W_0^{1,2}(\Omega)$ to* (1). *Moreover, the solution u satisfies $u \in W^{2,r}(\Omega) \cap C^1(\overline{\Omega})$ for any $r > 1$.*

Note that we have $2 < \operatorname{ess\,inf}_{x\in\Omega} q(x) \leqslant \operatorname{ess\,sup}_{x\in\Omega} q(x) \leqslant 2N/(N-2)$ and $\operatorname{ess\,sup}_{x\in\Omega\cap\{|x-x_0|\geqslant\eta\}} q(x) < 2N/(N-2)$ for each small $\eta > 0$, and that we may have $q(x) \to 2N/(N-2)$ as $x \to x_0$. This paper seems a first attempt to consider the existence of a positive solution to (1) with variable exponent $q(x)$ which coincides with the critical exponent $2N/(N-2)$ at some point $x_0 \in \Omega$.

Remark 2. If we assume the cone property on Ω, one can see that the embedding from $W^{1,2}(\Omega)$ to $L^{q(x)}(\Omega)$ is also compact under the same assumptions on $q(x)$ from the proof of Theorem 2 and the following fact. Under the cone property on Ω, it is known that the embedding from $W^{1,2}(\Omega)$ to $L^{q(x)}(\Omega)$ is compact if $1 \leqslant q(x) \leqslant \operatorname{ess\,sup}_{x\in\Omega} q(x) < 2N/(N-2)$ (see, e.g., [11]). Furthermore, from the proofs of Theorems 1 and 2, it is rather straightforward to generalize Theorems 1 and 2 for embeddings from $W_0^{1,p}(\Omega)$ or $W^{1,p}(\Omega)$ to $L^{q(x)}(\Omega)$ with $1 < p < N$, when $q(x)$ satisfies same assumptions with replacing all $2N/(N-2)$ by $pN/(N-p)$ in Theorems 1 and 2, respectively.

Remark 3. It is easy to see from the proof that Theorems 2 and 3 also hold when $q(x)$ coincides with the critical exponent $2N/(N-2)$ at finite number of points in Ω and satisfies the growth condition above near a neighborhood of the points.

In Section 2, we show Theorems 1–3. In Appendix A, by using a blow-up analysis we give another proof of Theorem 3, when Ω is a ball and $p(x)$ is radially symmetric for future reference.

2. Proofs of Theorems 1–3

Theorem 1 can be shown as in a similar way as in the well-known case $q(x) \equiv 2N/(N-2)$.

Proof of Theorem 1. Suppose that there exist a point $x_0 \in \Omega$ and constants $C_0 > 0$, $\eta > 0$ satisfying (2). We may assume $x_0 = 0$. We set $r(x) = 2N/(N-2) - q(x)$ for $x \in \Omega$. Let $\phi \in C_0^\infty(\mathbf{R}^N)$ be a function satisfying $\phi(x) = 1$ for $|x| \leqslant 1/2$ and $\operatorname{supp}\phi \subset \{x \in \mathbf{R}^N\colon |x| \leqslant 1\}$. For $n \in \mathbf{N}$, define $\phi_n(x) = n^{(N-2)/2}\phi(nx)$. Then, for large n we have $\phi_n \in C_0^\infty(\Omega)$,

$$\int_\Omega \left|\nabla\phi_n(x)\right|^2 dx = \int_{|y|<1} \left|\nabla\phi(y)\right|^2 dy$$

and

$$\int_\Omega \left|\phi_n(x)\right|^{q(x)}dx = n^N \int_{|x|<1/n} \left|\phi(nx)\right|^{q(x)} n^{-\frac{N-2}{2}r(x)}\,dx$$
$$= \int_{|y|<1} \left|\phi(y)\right|^{q(\frac{y}{n})} n^{-\frac{N-2}{2}r(\frac{y}{n})}\,dy.$$

Since $\phi(y)=1$ on $|y|\leqslant 1/2$ and $r(x)\leqslant C_0/|\log|x||$ for small $|x|$ by the assumption, we obtain

$$\int_{|y|<1} \left|\phi(y)\right|^{q(\frac{y}{n})} n^{-\frac{N-2}{2}r(\frac{y}{n})}\,dy \geqslant \int_{|y|\leqslant 1/2} \left(\frac{1}{n^{(N-2)/2}}\right)^{r(\frac{y}{n})} dy$$
$$\geqslant \int_{|y|\leqslant 1/2} \left(\frac{1}{n^{(N-2)/2}}\right)^{\frac{C_0}{|\log(|y|/n)|}} dy.$$

For $1/4\leqslant|y|\leqslant 1/2$, it follows $|\log(|y|/n)| = |\log n - \log|y|| \geqslant (1/2)\log n$ for large n. Therefore we arrive at

$$\int_\Omega \left|\phi_n(x)\right|^{q(x)}dx \geqslant \int_{1/4\leqslant|y|\leqslant 1/2} e^{-\frac{N-2}{2}\frac{C_0}{|\log(|y|/n)|}\log n}\,dy$$
$$\geqslant \int_{1/4\leqslant|y|\leqslant 1/2} e^{-\frac{N-2}{2}\frac{2C_0\log n}{\log n}}\,dy$$
$$= e^{-(N-2)C_0}\left|\left\{y\colon 1/4\leqslant|y|\leqslant 1/2\right\}\right| \equiv \delta > 0.$$

Since $\int_\Omega |\nabla\phi_n|^2\,dx \leqslant C$, there exist a subsequence $\{\phi_{n_k}\}$ and $\psi\in W_0^{1,2}(\Omega)$ such that $\{\phi_{n_k}\}$ converges to ψ weakly in $W_0^{1,2}(\Omega)$. But, since $\int_\Omega |\phi_n(x)|^2\,dx = n^{-2}\int_{\mathbf{R}^N}|\phi|^2\,dy \to 0$ as $n\to+\infty$, we have $\psi\equiv 0$. On the other hand, $\int_\Omega |\phi_n(x)|^{q(x)}\,dx \geqslant \delta > 0$ for large n. This concludes the proof of Theorem 1. □

Proof of Theorem 2. We may assume $x_0 = 0$ and $\eta>0$ is small enough. For the sake of simplicity, we set $2^* = 2N/(N-2)$. First, we note that if A is a measurable subset of Ω and $q(x)\leqslant\bar{q}<2^*$ on A, then

$$\int_A \left|v(x)\right|^{q(x)}dx \leqslant |A| + C_1^{2^*}\|v\|^{\bar{q}}_{W_0^{1,2}(\Omega)}|A|^{\frac{2^*-\bar{q}}{2^*}} \quad \text{for each } v\in W_0^{1,2}(\Omega), \tag{3}$$

where C_1 is a constant such that $C_1>1$ and $\|w\|_{L^{2^*}(\Omega)}\leqslant C_1\|w\|_{W_0^{1,2}(\Omega)}$ for each $w\in W_0^{1,2}(\Omega)$. Indeed, we have

$$\int_A \left|v(x)\right|^{q(x)}dx = \int_{A\cap\{v<1\}} \left|v(x)\right|^{q(x)}dx + \int_{A\cap\{v\geqslant 1\}} \left|v(x)\right|^{q(x)}dx$$
$$\leqslant |A| + \int_A \left|v(x)\right|^{\bar{q}}dx \leqslant |A| + \left(\int_A \left|v(x)\right|^{2^*}dx\right)^{\frac{\bar{q}}{2^*}}|A|^{\frac{2^*-\bar{q}}{2^*}}$$
$$\leqslant |A| + C_1^{2^*}\left(\int_\Omega \left|\nabla v(x)\right|^2 dx\right)^{\frac{\bar{q}}{2}}|A|^{\frac{2^*-\bar{q}}{2^*}}$$

for each $v\in W_0^{1,2}(\Omega)$. Let $C_2>1$. We will show

$$\lim_{\varepsilon\to+0}\sup\left\{\int_{B_\varepsilon(0)} \left|v(x)\right|^{q(x)}dx\colon\ v\in W_0^{1,2}(\Omega),\ \|v\|_{W_0^{1,2}(\Omega)}\leqslant C_2\right\} = 0. \tag{4}$$

Let $\varepsilon \in (0,\eta)$ and define $\xi\ (=\xi(\varepsilon))$ by $(\xi/2)^{1/N} = \varepsilon$. We set $a_n = (\xi/2)^{n/N}$ for $n \in \mathbf{N}$. We note that $B_\varepsilon(0) \setminus \{0\} = \bigcup_{n=1}^{\infty}(B_{a_n}(0) \setminus B_{a_{n+1}}(0))$ and $q(x) \leqslant 2^* - C_0/(\log|1/a_{n+1}|)^l$ on $B_{a_n}(0) \setminus B_{a_{n+1}}(0)$. We denote by σ_N the volume of the unit ball $B_1(0)$. Then for each sufficiently small $\xi > 0$ and for each $n \in \mathbf{N}$, we have

$$\left|B_{a_n}(0) \setminus B_{a_{n+1}}(0)\right|^{\frac{C_0}{2^*(\log|1/a_{n+1}|)^l}} = \left[\sigma_N\left\{\left(\frac{\xi}{2}\right)^n - \left(\frac{\xi}{2}\right)^{n+1}\right\}\right]^{\frac{C_0 N^l}{2^*(n+1)^l(\log 2 - \log\xi)^l}}$$
$$\leqslant \sigma_N^{\frac{C_0 N^l}{2^*(n+1)^l(-2\log\xi)^l}}\left(\frac{\xi}{2}\right)^{\frac{C_0 N^l}{2^*(-2\log\xi)^l}(\frac{n}{n+1})^l n^{1-l}} \leqslant C_3\delta^{n^{1-l}}, \tag{5}$$

where C_3 is a positive constant satisfying $\sigma_N^{C_0 N^l/(2^* 2^l(n+1)^l(-\log\xi)^l)} \leqslant C_3$ for each sufficiently small $\xi > 0$ and for each $n \in \mathbf{N}$, and $\delta\ (=\delta(\xi))$ is defined by $(\xi/2)^{C_0 N^l/(2^* 2^{2l}(-\log\xi)^l)}$. Using (3) and (5), we have

$$\int_{B_\varepsilon(0)} \left|v(x)\right|^{q(x)}dx = \sum_{n=1}^{\infty}\int_{B_{a_n}(0)\setminus B_{a_{n+1}}(0)}\left|v(x)\right|^{q(x)}dx$$
$$\leqslant \sum_{n=1}^{\infty}\left(\left|B_{a_n}(0)\setminus B_{a_{n+1}}(0)\right| + C_1^{2^*}C_2^{2^*}\left|B_{a_n}(0)\setminus B_{a_{n+1}}(0)\right|^{\frac{C_0}{2^*(\log|1/a_{n+1}|)^l}}\right)$$
$$= \left|B_\varepsilon(0)\right| + C_1^{2^*}C_2^{2^*}\sum_{n=1}^{\infty}\left|B_{a_n}(0)\setminus B_{a_{n+1}}(0)\right|^{\frac{C_0}{2^*(\log|1/a_{n+1}|)^l}}$$
$$\leqslant \left|B_\varepsilon(0)\right| + C_1^{2^*}C_2^{2^*}C_3\sum_{n=1}^{\infty}\delta^{n^{1-l}}.$$

Since $\sum_{n=1}^{\infty}\delta^{n^{1-l}} \leqslant \delta + \int_1^\infty \delta^{x^{1-l}}dx < \infty$ for each $\delta \in (0,1)$ and $\delta = \delta(\xi(\varepsilon)) \to 0$ as $\varepsilon \to +0$, we have $\sum_{n=1}^{\infty}\delta^{n^{1-l}} \to 0$ as $\varepsilon \to +0$. Hence, we have shown (4). Let $\{v_n\}$ be a sequence in $W_0^{1,2}(\Omega)$ which converges weakly to $v \in W_0^{1,2}(\Omega)$. Since $W_0^{1,2}(\Omega)$ is compactly embedded into $L^2(\Omega)$, there exists a subsequence $\{v_{n_i}\}$ which converges to v almost everywhere. By (4) and the compactness of the embedding from $W_0^{1,2}(\Omega)$ into $L^{\bar q}(\Omega)$ for each $\bar q \in [1, 2^*)$, we have $\int_\Omega |v_{n_i}(x) - v(x)|^{q(x)}dx \to 0$, which means $|v_{n_i} - v|_{q(x)} \to 0$. This completes the proof. □

Remark 4. We remark that the numerical series construction in Theorem 2 is close to the one appears in the proof of Theorem 8.2 in [18].

Once we have Theorem 2, it is easy to show Theorem 3 by using the standard mountain pass theorem in the critical point theory and elliptic regularity theorems. For the reader's convenience, we recall an abstract statement of the mountain pass theorem and give a proof of Theorem 3.

Let E be a Banach space and let J be a C^1 functional on E. We say $u \in E$ is a critical point of J if the Fréchet derivative $J'(u)$ of J at u is zero. We also say J satisfies (PS) condition, if any sequence $\{u_n\}_{n=1}^{\infty} \subset E$ such that $\{J(u_n)\}_{n=1}^{\infty}$ is bounded and $J'(u_n) \to 0$ as $n \to +\infty$ in the dual space E', has a convergent subsequence.

Theorem 4 *(Mountain pass theorem). Let E be a Banach space and let J be a C^1 functional on E satisfying* (PS) *condition. Suppose $J(0) = 0$ and*

(i) *there exist positive constants α, r such that $J(u) \geqslant 0$ for any $u \in E$ satisfying $\|u\| \leqslant r$ and $J(u) \geqslant \alpha$ for any $u \in E$ satisfying $\|u\| = r$;*
(ii) *there exists an element $e \in E$ such that $J(e) < 0$ and $\|e\| > r$.*

Define $c = \inf_{\gamma\in\Gamma}\max_{0\leqslant t\leqslant 1} J(\gamma(t))$, where $\Gamma = \{\gamma \in C([0,1]; E)\colon\ \gamma(0) = 0,\ J(\gamma(1)) < 0\}$. Then, there exists a critical point $u \in E$ with $J(u) = c$.

As in the proof of [22, Theorem 2.2], we can easily give a proof of Theorem 4.

Proof of Theorem 3. We define a functional J from $W_0^{1,2}(\Omega)$ into $\mathbf{R}$ by

$$J(u)=\int_\Omega \frac{1}{2}|\nabla u|^2\,dx-\int_\Omega \frac{(u_+(x))^{q(x)}}{q(x)}\,dx,\quad u\in W_0^{1,2}(\Omega),$$

where $u_+=\max(u,0)$. Since $2<\operatorname{ess\,inf}_{x\in\Omega} q(x)\leqslant q(x)\leqslant 2N/(N-2)$ and $W_0^{1,2}(\Omega)$ is continuously embedded into $L^{2N/(N-2)}(\Omega)$, it is easy to check that $J\in C^2(W_0^{1,2}(\Omega);\mathbf{R})$ and (i) and (ii) in Theorem 4 hold. We will show that J satisfies (PS) condition. Let $\{u_n\}$ be a sequence in $W_0^{1,2}(\Omega)$ such that $\{J(u_n)\}$ is bounded and $J'(u_n)\to 0$ as $n\to+\infty$. Since we can easily see that $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$, there are a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ and an element u of $W_0^{1,2}(\Omega)$ such that $\{u_{n_i}\}$ converges weakly to u in $W_0^{1,2}(\Omega)$. Noting

$$\int_\Omega \left|\nabla(u_{n_i}-u)\right|^2 dx=\int_\Omega \nabla u_{n_i}\nabla(u_{n_i}-u)\,dx-\int_\Omega \nabla u\nabla(u_{n_i}-u)\,dx,$$

it is sufficient to show $\int_\Omega \nabla u_{n_i}\nabla(u_{n_i}-u)\,dx\to 0$ as $i\to+\infty$. Using $J'(u_{n_i})\to 0$ and Theorem 2, we have

$$\begin{aligned}\lim_{i\to\infty}\left|\int_\Omega \nabla u_{n_i}\nabla(u_{n_i}-u)\,dx\right|&=\lim_{i\to\infty}\left|\int_\Omega (u_{n_i})_+^{q(x)-1}(u_{n_i}-u)\,dx\right|\\&\leqslant C\lim_{i\to\infty}\left|(u_{n_i})_+^{q(x)-1}\right|_{q(x)/(q(x)-1)}|u_{n_i}-u|_{q(x)}\\&=C\lim_{i\to\infty}\left|(u_{n_i})_+\right|_{q(x)}|u_{n_i}-u|_{q(x)}=0,\end{aligned}$$

where C is a positive constant due to the generalized Hölder inequality (see e.g. [16, Theorem 2.1]). Thus, we have shown that J satisfies (PS) condition. Now, we can apply Theorem 4 to assure the existence of a nontrivial critical point $u\in W_0^{1,2}(\Omega)$. Then u is a weak solution to

$$-\Delta u=u_+^{q(x)-1},\quad x\in\Omega,\qquad u(x)=0,\quad \partial\Omega.$$

By the maximum principle we have $u(x)>0$ for $x\in\Omega$ and $u\in W^{2,r}(\Omega)\cap C^1(\overline{\Omega})$ for any $r>1$ by elliptic regularity theorems. □

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Appendix A

We give another proof of Theorem 3 by using a blow-up analysis, when Ω is a ball and $q(x)$ is radially symmetric. Here we just use the compact embedding from $W_0^{1,2}(\Omega)$ to $L^{q(x)}(\Omega)$ with $\operatorname{ess\,sup}_{x\in\Omega} q(x)<2N/(N-2)$.

Assume Ω is a ball and $q(x)=2N/(N-2)-r(x)$ with radially symmetric $r(x)$ satisfying $r(x)\geqslant C_0/(\log(1/|x|))^l$ near $x=0$ for some $0<l<1$. For $k\in\mathbf{N}$, define $r_k(x)=r(x)$ for $|x|\geqslant 1/k$ and $r_k(x)=r(1/k)$ for $|x|\leqslant 1/k$. Put $q_k(x)=2N/(N-2)-r_k(x)$. Then $q_k(x)$ satisfies $2<\operatorname{ess\,inf}_{x\in\Omega} q_k(x)\leqslant\operatorname{ess\,sup}_{x\in\Omega} q_k(x)<2N/(N-2)$. So for each k, by using Theorem 4 on radially symmetric space $W_{0,r}^{1,2}(\Omega)=\{u\in W_0^{1,2}(\Omega)\colon u(x)=u(|x|)\}$, we obtain a positive solution $u_k\in W_{0,r}^{1,2}(\Omega)$ to the approximated problem

$$-\Delta u=u^{q_k(x)-1},\quad x\in\Omega,\qquad u(x)=0,\quad x\in\partial\Omega.$$

Actually, u_k satisfies $J_k'(u_k)=0$ in $(W_{0,r}^{1,2}(\Omega))'$ and $J_k(u_k)=c_k$, where

$$J_k(u)=\int_\Omega \frac{1}{2}|\nabla u|^2\,dx-\int_\Omega \frac{(u_+(x))^{q_k(x)}}{q_k(x)}\,dx,\quad u\in W^{1,2}_{0,r}(\Omega),$$
$$c_k=\inf_{\gamma\in\Gamma_k}\max_{0\leqslant t\leqslant 1} J_k\big(\gamma(t)\big).$$

Here, $\Gamma_k=\{\gamma\in C([0,1];W^{1,2}_{0,r}(\Omega))\colon\ \gamma(0)=0,\ J_k(\gamma(1))<0\}$. Choose a function $u_0\in W^{1,2}_{0,r}(\Omega)\cap C_0^\infty(\Omega)$ such that $\operatorname{supp} u_0\subset\{x\in\Omega\colon\ \kappa\leqslant|x|\}$ for small $\kappa>0$. Let $\gamma_0(t)=t(s_0u_0(x))$ for sufficiently large $s_0>0$. Then we have $\gamma_0\in\Gamma_k$ and that, by the definition of $q_k(x)$,

$$0<c_k\leqslant\max_{0\leqslant t\leqslant 1}J_k\big(\gamma_0(t)\big)=\max_{0\leqslant t\leqslant 1}J\big(\gamma_0(t)\big)\equiv d_1$$

for any $k\geqslant 1/\kappa$. Combining $\int_\Omega|\nabla u_k|^2\,dx-\int_\Omega u_k^{q_k(x)}\,dx=0$ with $\int_\Omega|\nabla u_k|^2/2\,dx-\int_\Omega u_k^{q_k(x)}/q_k(x)\,dx=c_k$, we have

$$\int_\Omega\left(\frac{1}{2}-\frac{1}{q_k(x)}\right)u_k^{q_k(x)}\,dx=c_k\leqslant d_1.$$

This implies

$$\int_\Omega|\nabla u_k|^2\,dx=\int_\Omega u_k^{q_k(x)}\,dx\leqslant C \tag{A.1}$$

for some constant C. Since $u_k(x)$ is radially symmetric, $u_k(r)$, $r=|x|$, satisfies

$$-\big(r^{N-1}u_k'(r)\big)'=r^{N-1}u_k(r)^{q_k(r)-1}.$$

Integrating over $(0,r)$, we have

$$-r^{N-1}u_k'(r)=\int_0^r s^{N-1}u_k(s)^{q_k(s)-1}\,ds>0$$

which implies $u_k'(r)<0$ for $r>0$. Thus $u_k(0)=\max_{x\in\Omega}u_k(x)=\|u_k\|_{L^\infty(\Omega)}$.

We claim that there exists a positive constant K independent of k such that $\|u_k\|_{L^\infty(\Omega)}\leqslant K$. If not, we may assume $\|u_k\|_{L^\infty(\Omega)}\to+\infty$ as $k\to+\infty$. Define

$$\epsilon_k=\big(\|u_k\|_{L^\infty(\Omega)}\big)^{-\frac{2}{N-2}},\qquad v_k(y)=\frac{u_k(y\epsilon_k)}{\|u_k\|_{L^\infty(\Omega)}},\quad y\in\Omega_k\equiv\Omega/k.$$

Then $\epsilon_k\to 0$ as $k\to+\infty$ and $v_k(0)=\max_{x\in\Omega_k}v_k(x)=\|v_k\|_{L^\infty(\Omega_k)}=1$. Then we have

$$-\Delta v_k(y)=\frac{\epsilon_k^2}{\|u_k\|_{L^\infty(\Omega)}}\|u_k\|_{L^\infty(\Omega)}^{q_k(y\epsilon_k)-1}v_k(y)^{q_k(y\epsilon_k)-1}$$
$$=\|u_k\|_{L^\infty(\Omega)}^{-r_k(y\epsilon_k)}v_k(y)^{q_k(y\epsilon_k)-1}\equiv h_k(y)$$

for $y\in\Omega_k$. Note that

$$r_k(y\epsilon_k)\geqslant\frac{C_0}{(\log|1/(y\epsilon_k)|)^l}$$

for any k, even if $|y\epsilon_k|\leqslant 1/k$. Thus it follows that

$$\|u_k\|_{L^\infty(\Omega)}^{-r_k(y\epsilon_k)}=e^{-r_k(y\epsilon_k)\log(\|u_k\|_{L^\infty(\Omega)})}\leqslant e^{-\frac{C_0}{(\log|1/(y\epsilon_k)|)^l}\log(\|u_k\|_{L^\infty(\Omega)})}$$
$$=e^{-\frac{C_0\log\|u_k\|_{L^\infty(\Omega)}}{(\frac{2}{N-2}\log\|u_k\|_{L^\infty(\Omega)}-\log|y|)^l}}\to 0$$

as $k\to+\infty$ for any $y\neq 0$. Now, for any bounded open subset $G\subset\mathbf{R}^N$ such that $0\in G$, since $0\leqslant v_k(y)\leqslant 1$ we have

$$\big|h_k(y)\big|\leqslant\|u_k\|_{L^\infty(\Omega)}^{-r_k(y\epsilon_k)}\leqslant 1\quad\text{for } y\in G \text{ and sufficiently large } k.$$

This implies $|\Delta v_k(y)| \leqslant 1$ on G and $\|v_k\|_{L^\infty(G)} = 1$. By the interior elliptic $W^{2,t}$-estimate, we have for some $\alpha \in (0,1)$ that $\|v_k\|_{C^{1,\alpha}(G)} \leqslant C = C(G)$. Therefore there exist a subsequence $\{v_{k_l}\}$ and v such that $\{v_{k_l}\}$ converges to v in $C^1_{\mathrm{loc}}(\mathbf{R}^N)$. It follows

$$\int_G \nabla v \cdot \nabla \psi \, dy = 0$$

for any $\psi \in C_0^\infty(G)$ and $v(0) = 1 = \|v\|_{L^\infty(G)}$. Since G is arbitrary, v is a bounded harmonic function and $v(0) = \|v\|_{L^\infty(\mathbf{R}^N)} = 1$. This implies $v(x)$ is constant and hence $v(x) \equiv 1$. On the other hand, by Sobolev's inequality and (A.1) we have

$$S\biggl(\int_{\Omega_k} |v_k|^{\frac{2N}{N-2}}\biggr)^{\frac{N-2}{N}} dy \leqslant \int_{\Omega_k} |\nabla v_k|^2 \, dy = \int_\Omega |\nabla u_k|^2 \, dx \leqslant C$$

for some positive constants C and S independent of k and G. This yields

$$S\biggl(\int_G |v|^{\frac{2N}{N-2}} \, dy\biggr)^{\frac{N-2}{N}} \leqslant C$$

for any G and hence

$$S\biggl(\int_{\mathbf{R}^N} |v|^{\frac{2N}{N-2}} \, dy\biggr)^{\frac{N-2}{N}} \leqslant C.$$

This contradicts $v = 1$.

Now, let $\{u_k\}$ converge weakly to $u \in W_0^{1,2}(\Omega)$. It is easy to see that the mountain pass value c_k also has a uniform lower bound: $0 < d_2 \leqslant c_k \leqslant d_1$ for $k \geqslant 1$. Indeed, if $c_k \to 0$ as $k \to +\infty$, then it is easy to see that $\int_\Omega |\nabla u_k|^2 \, dx \to 0$. Note that $t^{q_k(x)} \leqslant t^{q_0} + t^{2N/(N-2)}$ for $t \geqslant 0$, where $q_0 = \operatorname{ess\,inf}_{x \in \Omega} q_k(x) > 2$. Thus we have for large k

$$\begin{aligned}\int_\Omega |\nabla u_k|^2 \, dx &= \int_\Omega u_k^{q_k(x)} \, dx \leqslant \int_\Omega u_k^{q_0} \, dx + \int_\Omega u_k^{\frac{2N}{N-2}} \, dx \\ &\leqslant C\biggl(\biggl(\int_\Omega |\nabla u_k|^2 \, dx\biggr)^{\frac{q_0}{2}} + \biggl(\int_\Omega |\nabla u_k|^2 \, dx\biggr)^{\frac{N}{N-2}}\biggr) \leqslant 2C\biggl(\int_\Omega |\nabla u_k|^2 \, dx\biggr)^{\frac{q_0}{2}}.\end{aligned}$$

It follows $\int_\Omega |\nabla u_k|^2 \, dx \geqslant \delta > 0$ for some constant δ independent of k, which contradicts $\int_\Omega |\nabla u_k|^2 \, dx \to 0$.

Now, there exist positive constants C, C' such that

$$0 < C \leqslant \int_\Omega \bigl(u_k(x)\bigr)^{q_k(x)} \, dx \leqslant C'.$$

Then, by using $\|u_k\|_{L^\infty(\Omega)} \leqslant K$, it is easy to see that there exist a constant $\delta > 0$ and a compact subset $G_0 \subset \Omega \setminus \{0\}$ such that $|\Omega| - |G_0|$ is small enough and that

$$\int_{G_0} u^{q(x)} \, dx \geqslant \delta > 0.$$

It follows $u \neq 0$. Since $\int_\Omega \nabla u_k \cdot \nabla \phi \, dx = \int_\Omega u_k^{q_k(x)} \phi \, dx$ for every $\phi \in C_0^\infty(\Omega)$, we have $\int_\Omega \nabla u \cdot \nabla \phi \, dx = \int_\Omega u^{q(x)} \phi \, dx$ and that u is a desired positive solution by the maximum principle.

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