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On Local Control of the Number of Conjugacy Classes

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INTRODUCTION

In this paper, we extend the ideas of [4] to significantly greater generality and, in particular, we derive a new general statement about local control of the number of nontrivial conjugacy classes of a finite group. We anticipate that the ideas and methods used in the proof, especially those of Section 1, will find further applications beyond the result presented here.

Let G be a finite group and $\mathscr{S}(G)$ denote the simplicial complex associated to the poset of non-trivial solvable subgroups of G. For a chain, $\sigma \in \mathscr{S}(G)$, V_σ denotes the initial subgroup of σ. We let $\mathscr{S}^{\#}(G)$ denote the set of non-empty chains in $\mathscr{S}(G)$. When a group X acts by conjugation on another group Y, we will denote the number of X-conjugacy classes of Y by $k_X(Y)$ (as usual, we abbreviate $k_X(X)$ to $k(X)$). We will denote the number of X-conjugacy classes of $Y^{\#}$ by $k^{\#}_X(Y)$ If Y is normal in X, and μ is an irreducible (complex) character of Y, we let $k(X, \mu)$ denote the number of irreducible (complex) characters of X which have μ as an irreducible constituent of their restriction to Y. For a simplex $\sigma \in \mathscr{S}(G)$. we let $|\sigma|$ denote the number of non-trivial subgroups in σ.

We let $\mathscr{S}_m(G)$ denote the subcomplex of $\mathscr{S}(G)$ consisting of those chains in which every subgroup occurring is the intersection of the maximal solvable subgroups containing it (together with the empty chain). We remark that every subgroup with this last property contains $\mathrm{sol}(G)$, the largest solvable normal subgroup of G. Our main result is:

THEOREM. *Let G be any finite group. Then we have*

$$k^{\#}(G) = \sum_{\sigma \in \mathscr{S}^{\#}(G)/G} (-1)^{|\sigma|+1} k^{\#}_{G_\sigma}(V_\sigma).$$

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Furthermore, in the expression on the right hand side, $\mathscr{S}(G)$ *can be replaced by* $\mathscr{S}_m(G)$.

1. FINESSING COCYCLES VIA SUBGROUP COMPLEXES

Let V be a non-trivial solvable subgroup of our group chosen finite group G. Let H be a finite group with a central subgroup, Z, such that $\tilde{H} = H/Z$ is a subgroup of $N_G(V)/V$. Let $\mathscr{S}(G,V)$ denote the set of chains in $\mathscr{S}(G)$ whose initial subgroup is V. This may be identified with the subcomplex $\mathscr{S}(G)_{>V}$ with appropriate adjustment of the length of chains, and this identification is compatible with the action of $N_G(V)$.

Let p be a prime divisor of $|H|$, and let R be a complete discrete valuation ring of characteristic 0, with unique maximal ideal π such that $F = R/\pi$ is algebraically closed of characteristic p, where H acts on $\mathscr{S}(G,V)$ via the action of $\tilde{H}$, and H_σ acts on RH_σ via conjugation. Let λ be a linear character of Z, and let $E_\lambda = \sum_{z \in Z} \lambda(z^{-1})z$. Then (by an argument similar to that used in [4]), $E_\lambda RH$ is a monomial module when viewed as an $R\tilde{H}$-module via conjugation action, and $\{E_\lambda t : t \in T\}$ is a "monomial basis," where T is a fixed transversal to Z in H.

We wish to consider the virtual module

$$\sum_{\sigma \in \mathscr{S}(G,V)/H} (-1)^{|\sigma|} \mathrm{Ind}_{H_\sigma}^{H}(E_\lambda RH_\sigma)$$

(in the Green ring for $R\tilde{H}$), where H acts on $\mathscr{S}(G,V)$ via the action of $\tilde{H}$.

If possible, let Q be a p-subgroup of H which strictly contains $O_p(Z)$. Let Q_0 be the full pre-image of $\tilde{Q}$ in $N_G(V)$. Then $\mathscr{S}(G)^Q_{>V}$ is $N_H(Q)$-contractible via $X \to XQ_0 \to Q_0$. We explain how this implies that the virtual module we are considering involves no summand with vertex Q. The ideas of the proof can be found in [4], but since this situation is somewhat different, we give the proof in detail here. First, it suffices, by the Burry−Carlson−Puig theorem, to show that the restriction of the virtual module to $N_H(Q)$ involves no summand with vertex Q. The restriction in question is

$$\sum_{\sigma \in \mathscr{S}(G,V)/N_H(Q)} (-1)^{|\sigma|} \mathrm{Ind}_{N_H(Q)_\sigma}^{N_H(Q)}(E_\lambda RH_\sigma),$$

and it is clear that only Q-stable chains can give any summands with vertex Q, so we need to determine the summands with vertex Q which occur in

$$\sum_{\sigma \in S(G,V)^Q/H_H(Q)} (-1)^{|\sigma|} \mathrm{Ind}_{H_H(Q)_\sigma}^{N_H(Q)}(E_\lambda RH_\sigma).$$

We note that (arguing as in Section 1 of [4]), for each $\sigma \in \mathscr{S}(G,V)^Q$, the summands of $\mathrm{Ind}_{N_H(Q)_\sigma}^{N_H(Q)}(RH_\sigma)$ with vertex Q are the summands of $\mathrm{Ind}_{N_H(Q)_\sigma}^{N_H(Q)}(RC_{H_\sigma}(Q/O_p(Z)))$ with vertex Q. We note that if h is an element of H such that $h^x \in hZ$ for all $x \in Q$, then h normalizes $QZ = Q \times O_{p'}(Z)$, so that h normalizes Q (and centralizes $Q/O_p(Z)$).

Now we choose a chain $\sigma \in \mathscr{S}(G,V)^Q$, say $\sigma = V < V_1 < \cdots < V_n$, and we construct a chain $\sigma' \neq \sigma$ as follows: choose m maximal such that $Q_0 \not\leq V_m$ (regarding V as V_0, there is always such an m). If $V_{m+1} = Q_0V_m$, let $\sigma' = V < \cdots < V_m < V_{m+2} < \cdots < V_n$ (if $m = n-1$, just omit V_n). Otherwise, let $\sigma' = V < \cdots < V_m < Q_0V_m < V_{m+1} < \cdots < V_n$ (if $m = n$, just insert Q_0V_n after V_n). Then it is easy to see that $\sigma'' = \sigma$, and that $N_H(Q)_\sigma = N_H(Q)_{\sigma'}$. Thus $C_{H_\sigma}(Q/O_p(Z)) = C_{H_{\sigma'}}(Q/O_p(Z))$, and the summands with vertex Q from the chains σ and σ' cancel each other. Thus the virtual module in question involves no indecomposable module with vertex Q.

As Q is arbitrary, we conclude that the virtual module we are considering is a difference of $O_p(Z)$-projective RH-modules (so is really a virtual projective $R\tilde{H}$-module, as Z acts trivially on all modules involved). We deduce that the virtual character afforded by

$$\sum_{\sigma \in \mathscr{S}(G,V)/H} (-1)^{|\sigma|} \mathrm{Ind}_{H_\sigma}^{H}(E_\lambda \mathbb{C} H_\sigma)$$

is an integer multiple of the regular character of $\tilde{H}$, since the above argument shows that it vanishes on p-singular elements of $\tilde{H}$ for every prime divisor p of $|H|$. It is easy to see that $\dim_{\mathbb{C}}(E_\lambda \mathbb{C} H_\sigma) = [H_\sigma : Z]$, so counting dimension tells us that the given virtual character is $\sum_{\sigma \in \mathscr{S}(G,V)/H} (-1)^{|\sigma|}$ times the regular character of $\tilde{H}$. On the other hand, counting the multiplicity of the trivial module in the above virtual module by Frobenius reciprocity shows that the multiplicity of the trivial module is

$$\sum_{\sigma \in \mathscr{S}(G,V)/H} (-1)^{|\sigma|} \dim_{\mathbb{C}}\big(E_\lambda Z(\mathbb{C} H_\sigma)\big).$$

This last integer is

$$\sum_{\sigma \in \mathscr{S}(G,V)/H} (-1)^{|\sigma|} k(H_\sigma, \lambda).$$

Hence we have

$$\sum_{\sigma \in \mathscr{S}(G,V)/H} (-1)^{|\sigma|} \big(k(H_\sigma, \lambda) - 1\big) = 0.$$

2. PROOF OF THE MAIN THEOREM

Let G, V be as in Section 1. Let N denote $N_G(V)$. We wish to compute the alternating sum $\sum_{\sigma \in \mathscr{S}(G,V)/N}(-1)^{|\sigma|}k(G_\sigma)$. Using elementary Clifford theory, this may be written as

$$\sum_{\sigma \in \mathscr{S}(G,V)/N} (-1)^{|\sigma|} \sum_{\mu \in Irr(V)/G_\sigma} k\big(I_{G_\sigma}(\mu), \mu\big).$$

We may change the order of summation, and re-write the last double sum as

$$\sum_{\mu \in Irr(V)/N} \sum_{\sigma \in \mathscr{S}(G,V)/I_N(\mu)} (-1)^{|\sigma|} k\big(I_{G_\sigma}(\mu), \mu\big).$$

Now we fix a choice of μ for a while, and let $\tilde{H} = I_N(\mu)/V$. By standard Clifford theory, there are a central extension, H, of $\tilde{H}$, with a cyclic central subgroup Z, and a linear character λ of Z, such that $\tilde{H} \cong H/Z$ and such that there is a bijection between irreducible characters of $I_N(\mu)$ which lie over μ and irreducible characters of H which lie over λ. Then H acts on $\mathscr{S}(G,V)$ via the action of $\tilde{H}$, with Z acting trivially. Furthermore, for each chain $\sigma \in \mathscr{S}(G,V)$, we have $k(I_{G_\sigma}(\mu), \mu) = k(H_\sigma, \lambda)$. Thus we have

$$\sum_{\sigma \in \mathscr{S}(G,V)/I_N(\mu)} (-1)^{|\sigma|} k\big(I_{G_\sigma}(\mu), \mu\big) = \sum_{\sigma \in \mathscr{S}(G,V)/H} (-1)^{|\sigma|} k(H_\sigma, \lambda).$$

By the results of Section 1, this last expression is $\sum_{\sigma \in \mathscr{S}(G,V)/H}(-1)^{|\sigma|}$.

Applying this argument to each $\mu \in \mathrm{Irr}(V)$, we conclude that

$$\begin{aligned} &\sum_{\sigma \in \mathscr{S}(G,V)/N} (-1)^{|\sigma|} k(G_\sigma) \\ &\quad = \sum_{\mu \in Irr(V)/N} \sum_{\sigma \in \mathscr{S}(G,V)/I_N(\mu)} (-1)^{|\sigma|} k\big(I_{G_\sigma}(\mu), \mu\big) \\ &\quad = \sum_{\mu \in Irr(V)/N} \sum_{\sigma \in \mathscr{S}(G,V)/I_N(\mu)} (-1)^{|\sigma|}. \end{aligned}$$

But we may change the order of summation again, and we find that the last double sum is $\sum_{\sigma \in \mathscr{S}(G,V)/N} \sum_{\mu \in Irr(V)/G_\sigma} (-1)^{|\sigma|} = \sum_{\sigma \in \mathscr{S}(G,V)/N}(-1)^{|\sigma|} k_{G_\sigma}(V)$.

Let $\mathscr{S}_0$ denote the set of non-trivial solvable subgroups of G. We may apply the above argument for V running through a set of representatives of a non-trivial conjugacy classes of solvable subgroups of G to conclude that

$$\sum_{\sigma \in \mathscr{S}(G)/G} (-1)^{|\sigma|} k(G_\sigma)$$
$$= k(G) - \sum_{V \in \mathscr{S}_0/G} \sum_{\sigma \in \mathscr{S}(G,V)/N_G(V)} (-1)^{|\sigma|+1} k(G_\sigma)$$
$$= k(G) - \sum_{V \in \mathscr{S}_0/G} \sum_{\sigma \in \mathscr{S}(G,V)/N_G(V)} (-1)^{|\sigma|+1} k_{G_\sigma}(V).$$

It was proved in [3] that $\sum_{\sigma \in \mathscr{S}(G)/G} (-1)^{|\sigma|}(k(G_\sigma) - 1) = 0$ (this result may also be recovered by modifying the arguments of Section 1 of this paper to treat the case $V = 1_G$, and taking $H = G$, $Z = 1_G$). Substituting this in the previous expression, we obtain

$$k(G) - 1 = \sum_{V \in \mathscr{S}_0/G} \sum_{\sigma \in \mathscr{S}(G,V)/N_G(V)} (-1)^{|\sigma|+1} \left(k_{G_\sigma}(V) - 1\right),$$

which is easily seen to be equivalent to the first formula in the main theorem.

To prove the last assertion of the main theorem, we consider the inclusion-preserving map $f : \mathscr{S}_0 \to \mathscr{S}_0$ defined by $f(X) = \cap_{M \in \mathscr{M}(X)} M$, where $\mathscr{M}(X)$ is the set of maximal solvable subgroups of G which contain X. Then $f(X^g) = f(X)^g$ for all $X \in \mathscr{S}_0$, all $g \in G$, so that $N_G(X)$ normalizes $f(X)$ for each such X. We show that the contribution to the above alternating sum from chains in $\mathscr{S}(G)^\#$ which are not f-stable is 0 (and by definition the f-stable chains in $\mathscr{S}(G)^\#$ are precisely the chains in $\mathscr{S}_m(G)^\#$). As usual, given a chain σ which is not f-stable, we produce another chain σ' which is also not f-stable, such that $\sigma'' = \sigma$ and $G_\sigma = G_{\sigma'}$, so that the contributions from σ and σ' cancel each other. Furthermore, we do this in such a way that σ and σ' have the same initial subgroup.

Suppose, then, that $\sigma = V_0 < \cdots < V_n$ is not f-stable. Choose m maximal so that $V_m \neq f(V_m)$. If $V_{m+1} \neq f(V_m)$, let $\sigma' = V_0 < \cdots < V_m < f(V_m) < V_{m+1} < \cdots < V_n$ (insert $f(V_m)$ at the end if $n = m$ in this case). If $V_{m+1} = f(V_m)$, let $\sigma' = V_0 < \cdots < V_m < V_{m+2} < \cdots < V_n$ (delete V_n if $m = n - 1$ in this case). This construction yields the required cancellations, and completes the proof of the main theorem.

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