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On the nilpotency degree of the algebra with identity $x^n = 0$

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ABSTRACT

Denote by $C_{n,d}$ the nilpotency degree of a relatively free algebra generated by d elements and satisfying the identity $x^n = 0$. Under assumption that the characteristic p of the base field is greater than $n/2$, it is shown that $C_{n,d} < n^{\log_2(3d+2)+1}$ and $C_{n,d} < 4 \cdot 2^{\frac{n}{2}} d$. In particular, it is established that the nilpotency degree $C_{n,d}$ has a polynomial growth in case the number of generators d is fixed and $p > \frac{n}{2}$. For $p \neq 2$ the nilpotency degree $C_{4,d}$ is described with deviation 3 for all d. As an application, a finite generating set for the algebra $R^{GL(n)}$ of $GL(n)$-invariants of d matrices is established in terms of $C_{n,d}$. Several conjectures are formulated.

1. Introduction

We assume that $\mathbb{F}$ is an infinite field of arbitrary characteristic $p = \operatorname{char} \mathbb{F} \geqslant 0$. All vector spaces, algebras and modules are over $\mathbb{F}$ and all algebras are associative with unity unless otherwise stated.

We denote by $\mathcal{M} = \mathcal{M}(x_1, \dots, x_d)$ the semigroup (without unity) freely generated by *letters* $x_1, \dots, x_d$ and denote by $\mathcal{M}_{\mathbb{F}} = \mathcal{M}_{\mathbb{F}}(x_1, \dots, x_d)$ the vector space with the basis $\mathcal{M}$. Let

$$N_{n,d} = N_{n,d}(x_1, \dots, x_d) = \frac{\mathcal{M}_{\mathbb{F}}}{\mathrm{id}\{x^n \mid x \in \mathcal{M}_{\mathbb{F}}\}}$$

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be the relatively free algebra with the identity $x^n = 0$. The connection between this algebra and analogues of the Burnside problems for associative algebras suggested by Kurosh and Levitzky is discussed in recent survey [27] by Zelmanov.

We write

$$C_{n,d} = \min\{c > 0 \mid a_1 \cdots a_c = 0 \text{ for all } a_1, \ldots, a_c \in N_{n,d}\}$$

for the *nilpotency* degree of $N_{n,d}$. Since $C_{1,d} = 1$ and $C_{n,1} = n$, we assume that $n, d \geqslant 2$ unless otherwise stated. Obviously, $C_{n,d}$ depends only on n, d, and p.

We consider the following three cases:

(a) $p = 0$;
(b) $0 < p \leqslant n$;
(c) $p > n$.

By the well-known Nagata–Higman Theorem (see [22] and [12]), which at first was proved by Dubnov and Ivanov [9] in 1943, $C_{n,d} < 2^n$ in cases (a) and (c). As it was pointed out in [6], $C_{n,d} \geqslant d$ in case (b); in particular, $C_{n,d} \to \infty$ as $d \to \infty$. Thus, the case (b) is drastically different from cases (a) and (c). In 1974 Razmyslov [24] proved that $C_{n,d} \leqslant n^2$ in case (a). As about lower bounds on $C_{n,d}$, in 1975 Kuzmin [14] established that $C_{n,d} \geqslant \frac{1}{2}n(n+1)$ in cases (a) and (c) and conjectured that $C_{n,d}$ is actually equal to $\frac{1}{2}n(n+1)$ in these cases. A proof of the mentioned lower bound was reproduced in books [8] and [3] (see p. 341). Kuzmin's Conjecture is still unproven apart from some partial cases. Namely, the conjecture holds for $n = 2$ and $n = 3$ (for example, see [15]). In case (a) the conjecture was proved for $n = 4$ by Vaughan-Lee [26] and for $n = 5$, $d = 2$ by Shestakov and Zhukavets [25].

Using approach by Belov [2], Klein [13] obtained that for an arbitrary characteristic the inequalities $C_{n,d} < \frac{1}{6}n^6 d^n$ and $C_{n,d} < \frac{1}{(m-1)!}n^{n^3} d^m$ hold, where $m = [n/2]$. Here $[a]$ (where $a \in \mathbb{R}$) stands for the largest integer $b < a$. Recently, Belov and Kharitonov [4] established that $C_{n,d} \leqslant 2^{18} \cdot n^{12\log_3(n)+28} d$ (see Remark 4.8 for more details). Moreover, they proved that a similar estimation also holds for the Shirshov Height of a finitely generated PI-algebra. We can summarize the above mentioned bounds on the nilpotency degree as follows:

- if $p = 0$, then $\frac{1}{2}n(n+1) \leqslant C_{n,d} \leqslant n^2$;
- if $0 < p \leqslant n$, then $d \leqslant C_{n,d} < \frac{1}{6}n^6 d^n$ and $C_{n,d} \leqslant 2^{18} \cdot n^{12\log_3(n)+28} d$;
- if $p > n$, then $\frac{1}{2}n(n+1) \leqslant C_{n,d} < 2^n$.

For $d > 0$ and arbitrary characteristic of the field the nilpotency degree $C_{n,d}$ is known for $n = 2$ (for example, see [6]) and $n = 3$ (see [15] and [16]):

$$C_{2,d} = \begin{cases} 3, & \text{if } p = 0 \text{ or } p > 2, \\ d+1, & \text{if } p = 2 \end{cases} \quad \text{and} \quad C_{3,d} = \begin{cases} 6, & \text{if } p = 0 \text{ or } p > 3, \\ 6, & \text{if } p = 2 \text{ and } d = 2, \\ d+3, & \text{if } p = 2 \text{ and } d > 2, \\ 3d+1, & \text{if } p = 3. \end{cases}$$

In this paper we obtained the following upper bounds on $C_{n,d}$:

- $C_{n,d} < n^{\log_2(3d+2)+1}$ in case $p > \frac{n}{2}$ (see Corollary 3.1). Therefore, we establish a polynomial upper bound on $C_{n,d}$ under assumption that the number of generators d is fixed.
- $C_{n,d} < 4 \cdot 2^{\frac{n}{2}} d$ for $\frac{n}{2} < p \leqslant n$ (see Corollary 4.1). Modulo Conjecture 4.6, we prove that $C_{n,d} < n^2 \ln(n) d$ for $\frac{n}{2} < p \leqslant n$ (see Corollary 4.7).
- $C_{4,d}$ is described with deviation 3 for all d under assumption that $p \neq 2$ (see Theorem 5.1).

Note that even in the partial case of $p > n$ and $d = 2$ a polynomial bound on $C_{n,d}$ has not been known. If n is fixed and d is large enough, then the bound from Corollary 4.1 is better than that from Corollary 3.1. In Remark 4.8 we show that for $p > \frac{n}{2}$, $4 \leqslant n \leqslant 2000$, and all d the bound from Corollary 4.1 is at least 10^{20} times better than the bounds by Belov and Kharitonov [4].

As an application, we consider the algebra $R^{GL(n)}$ of $GL(n)$-invariants of several matrices and describe a finite generating set for $R^{GL(n)}$ in terms of $C_{n,d}$ (see Theorem 6.2). We conjecture that $R^{GL(n)}$ is actually generated by its elements of degree less or equal to $C_{n,d}$ (see Conjecture 6.3).

The paper is organized as follows. In Section 2 we establish a key recursive formula for an upper bound on $C_{n,d}$ that holds in case $p = 0$ or $p > \frac{n}{2}$ (see Theorem 2.5):

$$C_{n,d} \leqslant d \sum_{i=2}^{n} (i-1) C_{[n/i],d} + 1. \tag{1}$$

The main idea of proof of Theorem 2.5 is the following one. We introduce some partial order $>$ on $\mathcal{M}$ and the $\asymp$-equivalence on $\mathcal{M}_{\mathbb{F}}$ in such a way that $f \asymp h$ if and only if the image of $f - h$ in $N_{n,d}$ belongs to $\mathbb{F}$-span of elements that are bigger than $f - h$ with respect to $>$. Since $N_{n,d}$ is homogeneous with respect to degrees, there exists a $w \in \mathcal{M}$ satisfying $w \not\asymp 0$ and $C_{n,d} = \deg w + 1$. Thus we can deal with the $\asymp$-equivalence instead of the equality in $N_{n,d}$. Some relations of $N_{n,d}$ modulo $\asymp$-equivalence resembles relations of $N_{k,d}$ for $k < n$ (see formula (2)). This fact allows us to obtain the upper bound on $C_{n,d}$ in terms of $C_{k,d}$, where $k < n$. To illustrate the proof of Theorem 2.5, in Example 2.7 we consider the partial case of $n = 5$ and $p \neq 2$. Note that a similar approach to the problem of description of $C_{n,d}$ can be originated from every partial order on $\mathcal{M}$.

In Section 3 we apply recursive formula (1) several times to obtain the polynomial bound from Corollary 3.1. On the other hand, in Section 4 we use formula (1) together with the Nagata–Higman Theorem to establish Corollary 4.1. Formula (1) is applied to the partial case of $n \leqslant 9$ in Corollary 4.5.

In Section 5 we develop the approach from Section 2 for $n = 4$ to prove Theorem 5.1. We define a new partial order $\succ$ on $\mathcal{M}$, which is weaker than $>$, and obtain a new $\approx$-equivalence on $\mathcal{M}_{\mathbb{F}}$, which is stronger than $\asymp$-equivalence. Considering relations of $N_{4,d}$ modulo $\approx$-equivalence, we obtain the required bounds on $C_{4,d}$.

Section 6 is dedicated to the algebras of invariants of several matrices.

We end up this section with the following optimistic conjecture, which follows from Kuzmin's Conjecture. We write $C_{n,d,p}$ for $C_{n,d}$.

Conjecture 1.1. *For all $p > n$ we have $C_{n,d,0} = C_{n,d,p}$.*

This conjecture holds for $n = 2, 3$ (see above). Note that Conjecture 4.6 follows from Conjecture 1.1 by the above mentioned result by Razmyslov.

2. Recursive upper bound

We start with some notations. Let $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \sqcup \{0\}$, and $\mathbb{F}^* = \mathbb{F} \backslash \{0\}$. Denote $\mathcal{M}_1 = \mathcal{M} \sqcup \{1\}$, i.e., we endow $\mathcal{M}$ with the unity. Given a letter x, denote by $\mathcal{M}^{\neg x}$ the set of words $a_1 \cdots a_r \in \mathcal{M}$ such that neither letter a_1 nor letter a_r is equal to x and $r > 0$.

For $a \in \mathcal{M}_1$ and a letter x we denote by $\deg_x(a)$ the degree of a in the letter x and by $\mathrm{mdeg}(a) = (\deg_{x_1}(a), \ldots, \deg_{x_r}(a))$ the multidegree of a. For short, we write 1^r for $(1, \ldots, 1)$ (r times) and say that a is *multilinear* in case $\mathrm{mdeg}(a) = 1^r$.

Given $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}_0^r$, we set $\#\underline{\alpha} = r$, $|\underline{\alpha}| = \alpha_1 + \cdots + \alpha_r$, and $\underline{\alpha}^{\mathrm{ord}} = (\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(r)})$ for a permutation $\sigma \in S_r$ such that $\alpha_{\sigma(1)} \geqslant \cdots \geqslant \alpha_{\sigma(r)}$. If $r = 0$, then we say that $\underline{\alpha}$ is an empty vector and write $\underline{\alpha} = \emptyset$. Note that for $\underline{\alpha} = \emptyset$ we also have $\underline{\alpha}^{\mathrm{ord}} = \emptyset$.

Given $\underline{\theta} \in \mathbb{N}_0^r$ with $|\underline{\theta}| = n$ and $a_1, \ldots, a_r \in \mathcal{M}$, denote by $T_{\underline{\theta}}(a_1, \ldots, a_r)$ the coefficient of $\alpha_1^{\theta_1} \cdots \alpha_r^{\theta_r}$ in $(\alpha_1 a_1 + \cdots + \alpha_r a_r)^n$, where $\alpha_i \in \mathbb{F}$. Since the field $\mathbb{F}$ is infinite, standard Vandermonde arguments give that $T_{\underline{\theta}}(a_1, \ldots, a_r) = 0$ holds in $N_{n,d}$.

Definition 2.1 *(of* $\mathrm{pwr}_x(a)$*).* Let x be a letter and $a = a_1 x^{\alpha_1} \cdots a_r x^{\alpha_r} a_{r+1} \in \mathcal{M}$, where $r \geqslant 0$, $a_1, a_{r+1} \in \mathcal{M}_1$, $a_2, \ldots, a_r \in \mathcal{M}$, $\alpha_1, \ldots, \alpha_r > 0$, and $\deg_x(a_i) = 0$ for all i. Then we denote by $\mathrm{pwr}_x(a) = (\alpha_1, \ldots, \alpha_r)$ the *x-power* of a. In particular, if $\deg_x(a) = 0$, then $\mathrm{pwr}_x(a) = \emptyset$.

Let $\underline{\alpha} \in \mathbb{N}^r$, $\underline{\beta} \in \mathbb{N}^s$ $(r, s \geqslant 0)$ satisfy $\underline{\alpha} = \underline{\alpha}^{\mathrm{ord}}$ and $\underline{\beta} = \underline{\beta}^{\mathrm{ord}}$. Then we write $\underline{\alpha} > \underline{\beta}$ if one of the following conditions holds:

- $r < s$;
- $r = s$ and $\alpha_1 = \beta_1, \ldots, \alpha_l = \beta_l$, $\alpha_{l+1} > \beta_{l+1}$ for some $0 \leqslant l < r$.

As an example, $(2, 2, 2) < (3, 2, 1) < (4, 1, 1) < (3, 3) < (4, 2) < (5, 1) < (6) < \emptyset$.

Definition 2.2. Let x be a letter and $a, b \in \mathcal{M}$. Introduce the partial order $>$ and the $\gtrless$-equivalence on $\mathcal{M}$ as follows:

- $a > b$ if and only if $\mathrm{pwr}_x(a)^{\mathrm{ord}} > \mathrm{pwr}_x(b)^{\mathrm{ord}}$ for some letter x and $\mathrm{pwr}_y(a)^{\mathrm{ord}} \geqslant \mathrm{pwr}_y(b)^{\mathrm{ord}}$ for every letter y;
- $a \gtrless b$ if and only if $\mathrm{pwr}_y(a)^{\mathrm{ord}} = \mathrm{pwr}_y(b)^{\mathrm{ord}}$ for every letter y; in particular, $\mathrm{mdeg}\, a = \mathrm{mdeg}\, b$.

Remark 2.3. There is no an infinite chain $a_1 < a_2 < \cdots$ such that $a_i \in \mathcal{M}$ and $\deg(a_i) = \deg(a_j)$ for all i, j.

Definition 2.4 *(of the* $\asymp$*-equivalence).*

1. Let $f = \sum_i \alpha_i a_i \in \mathcal{M}_{\mathbb{F}}$, where $\alpha_i \in \mathbb{F}^*$, $a_i \in \mathcal{M}$, and $a_i \gtrless a_{i'}$ for all i, i'. Then $f \asymp 0$ if $f = 0$ in $N_{n,d}$ or $f = \sum_j \beta_j b_j$ in $N_{n,d}$ for some $\beta_j \in \mathbb{F}^*$, $b_j \in \mathcal{M}$ satisfying $b_j > a_i$ for all i, j.
2. If $f = \sum_k f_k \in \mathcal{M}_{\mathbb{F}}$ and $f_k \asymp 0$ satisfies conditions from part 1 for all k, then $f \asymp 0$.

Given $h \in \mathcal{M}_{\mathbb{F}}$, we write $f \asymp h$ if $f - h \asymp 0$.

It is not difficult to see that $\asymp$ is actually an equivalence on the vector space $\mathcal{M}_{\mathbb{F}}$, i.e., $\asymp$ have properties of transitivity and linearity over $\mathbb{F}$. Note that part 2 of Definition 2.4 is necessary for $\asymp$ to be an equivalence.

Theorem 2.5. *Let* $p = 0$ *or* $p > \frac{n}{2}$*. Then*

$$C_{n,d} \leqslant d \sum_{i=2}^{n} (i-1) C_{[n/i],d} + 1.$$

Proof. There exists a $w \in \mathcal{M}$ with $\deg(w) = C_{n,d} - 1$ and $w \neq 0$ in $N_{n,d}$. Moreover, by Remark 2.3 and $\mathbb{N}$-homogeneity of $N_{n,d}$ we can assume that $w \not\asymp 0$. Given a letter x, we write $d(x^i)$ for the number of ith in the x-power of w, i.e.,

$$\mathrm{pwr}_x(w)^{\mathrm{ord}} = (\alpha_1, \ldots, \alpha_r, \underbrace{i, \ldots, i}_{d(x^i)}, \beta_1, \ldots, \beta_s),$$

where $\alpha_r < i < \beta_1$. Obviously, $d(x^i) = 0$ for $i \geqslant n$.

Let $2 \leqslant i \leqslant n$ and x be a letter. Then $n = ki + r$ for $k = [n/i] \geqslant 1$ and $0 \leqslant r < i$. Consider elements $a_1, \ldots, a_k \in \mathcal{M}^{\neg x}$ and $\underline{\theta} = ((i-1)k + r, 1^k)$. Note that for $a_\sigma = x^{i-1} a_{\sigma(1)} \cdots x^{i-1} a_{\sigma(k)} x^{i-1}$, $\sigma \in S_k$, the following statements hold:

- $a_\sigma \geqslant a_\tau$ for all $\sigma, \tau \in S_k$.
- Let $i_1, \ldots, i_s > 0$ satisfy $i_1 + \cdots + i_s = (i-1)(k+1)$ and $e_0, \ldots, e_s \in \mathcal{M}_1$ be such products of $a_1, \ldots, a_k$ that for every $1 \leqslant j \leqslant k$, a_j is a factor of one and only element from the set $\{e_0, \ldots, e_s\}$. Moreover, we assume that $e_1, \ldots, e_{s-1} \in \mathcal{M}$. Define $e = e_0 x^{i_1} e_1 x^{i_2} \cdots x^{i_s} e_s \neq a_\sigma$ for all $\sigma \in S_k$. Then $e > a_\sigma$ for all $\sigma \in S_k$.

To prove the second claim, we notice that there are two cases. Namely, in the first case $s = k+1$, $e_0 = e_{k+1} = 1$, and $e_1 = a_{\tau(1)}, \ldots, e_k = a_{\tau(k)}$ for some $\tau \in S_k$; and in the second case $\#\mathrm{pwr}_x(e) < \#\mathrm{pwr}_x(a_\sigma)$ for all $\sigma \in S_k$. In both cases we have $\mathrm{pwr}_x(e)^{\mathrm{ord}} > \mathrm{pwr}_x(a_\sigma)^{\mathrm{ord}}$ and $\mathrm{pwr}_y(e)^{\mathrm{ord}} \geqslant \mathrm{pwr}_y(a_\sigma)^{\mathrm{ord}}$ for any letter $y \neq x$ and any $\sigma \in S_k$. The claim is proven.

Since $T_{\underline{\theta}}(x, a_1, \ldots, a_k)x^{i-r-1} = 0$ in $N_{n,d}$, we have $\sum_{\sigma \in S_k} a_\sigma \asymp 0$. Moreover,

$$\sum_{\sigma \in S_k} v a_\sigma w \asymp 0 \tag{2}$$

for all $v, w \in \mathcal{M}_1$ such that if $v \neq 1$ ($w \neq 1$, respectively), then its last (first, respectively) letter is not x.

Let $D = 2^k - 1$. Since $p = 0$ or $p > \frac{n}{2} \geqslant k$, the Nagata–Higman Theorem implies that $C_{k,D} \leqslant 2^k - 1$. For short, we write C for $C_{k,D}$. Thus $y_1 \cdots y_C = 0$ in $N_{k,D}(y_1, \ldots, y_D)$, where $y_1, \ldots, y_D$ are new letters. Since $y_1 \cdots y_C$ is multilinear, an equality

$$y_1 \cdots y_C = \sum_{\underline{u}} \alpha_{\underline{u}} u_0 T_{1^k}(u_1, \ldots, u_k) u_{k+1} \tag{3}$$

holds in $\mathcal{M}_{\mathbb{F}}(y_1, \ldots, y_C)$, where the sum ranges over $(k+2)$-tuples $\underline{u} = (u_0, \ldots, u_{k+1})$ such that $u_0, u_{k+1} \in \mathcal{M}_1(y_1, \ldots, y_C)$, $u_1, \ldots, u_k \in \mathcal{M}(y_1, \ldots, y_C)$, and the number of non-zero coefficients $\alpha_{\underline{u}} \in \mathbb{F}$ is finite.

Given $b_1, \ldots, b_C \in \mathcal{M}^{\neg x}$ and $0 \leqslant l \leqslant k+1$, denote by $v_l \in \mathcal{M}_1$ the result of substitution $y_j \to x^{i-1} b_j$ $(1 \leqslant j \leqslant C)$ in u_l. We apply these substitutions to equality (3) and multiply the result by x^{i-1}. Thus,

$$x^{i-1} b_1 \cdots x^{i-1} b_C x^{i-1} = \sum_{\underline{u}} \alpha_{\underline{u}} v_0 T_{1^k}(v_1, \ldots, v_k) v_{k+1} x^{i-1}$$

in $\mathcal{M}_{\mathbb{F}} = \mathcal{M}_{\mathbb{F}}(x_1, \ldots, x_d)$. For every $\underline{u}$ there exist $a_1, \ldots, a_k \in \mathcal{M}^{\neg x}$ satisfying $v_l = x^{i-1} a_l$ for all $1 \leqslant l \leqslant k$. If $u_{k+1} \neq 1$, then we also have $v_{k+1} = x^{i-1} a_{k+1}$ for some $a_{k+1} \in \mathcal{M}^{\neg x}$. Since $T_{1^k}(v_1, \ldots, v_k) = \sum_{\sigma \in S_k} v_{\sigma(1)} \cdots v_{\sigma(k)}$, we have

$$T_{1^k}(v_1, \ldots, v_k) v_{k+1} x^{i-1} = \sum_{\sigma \in S_k} a_\sigma f,$$

where f stands for 1 in case $u_{k+1} = 1$ and for $a_{k+1} x^{i-1}$ in case $u_{k+1} \neq 1$. Combining the previous two equalities with equivalence (2), we obtain

$$x^{i-1} b_1 \cdots x^{i-1} b_C x^{i-1} \asymp 0. \tag{4}$$

Hence, the equivalence $b_0 x^{i-1} b_1 \cdots x^{i-1} b_{C+1} \asymp 0$ holds for all $b_1, \ldots, b_C \in \mathcal{M}^{\neg x}$ and $b_0, b_{C+1} \in \mathcal{M}_1$ such that if $b_0 \neq 1$ ($b_{C+1} \neq 1$, respectively), then its last (first, respectively) letter is not x. Since $w \not\asymp 0$, we obtain

$$d\big(x^{i-1}\big) \leqslant C_{[n/i],d},$$

and therefore $\deg_x(w) \leqslant \sum_{1<i\leqslant n}(i-1)C_{[n/i],d}$ for every letter x. The proof is completed. □

Remark 2.6. Since $C_{1,d} = 1$, we can reformulate the statement of Theorem 2.5 as follows. Let $p = 0$ or $p > \frac{n}{2}$ and $m = [n/2]$. Then $C_{n,d} \leqslant A_n d + 1$, where

$$A_n = \sum_{i=2}^{m}(i-1)C_{[n/i],d} + \frac{1}{2}(n+m-1)(n-m).$$

Example 2.7. To illustrate the proof of Theorem 2.5, we repeat this proof in the partial case of $n = 5$ and $p \neq 2$. We write a, b, c for some elements from $\mathcal{M}^{\neg x}$.

Let $i = 2$. Then $k = [n/i] = 2$ and $r = 1$. Since $T_{311}(x, a, b) = 0$ in $N_{5,d}$, we have the following partial case of (2):

$$xaxbx + xbxax \asymp 0. \tag{5}$$

Note that $C_{2,D} = 3$ for all $D \geqslant 2$. We rewrite the proof of this fact, using formula (5) instead of the equality $uv + vu = 0$ in $N_{2,D}$:

$$xax \cdot bxc \cdot x \asymp -xb(xcxax) \asymp (xbxax)cx \asymp -xaxbxcx.$$

Here we use dots and parentheses to show how we apply (5). Thus we obtain the partial case of formula (4): $xaxbxcx \asymp 0$. Therefore, $d(x) \leqslant 3$.

Let $i = 3$. Then $k = [n/i] = 1$ and $r = 2$. Since $T_{41}(x, a) = 0$ in $N_{5,d}$, we have $x^2ax^2 \asymp 0$. Considering $i = 4, 5$, we can see that $x^3ax^3 \asymp 0$ and $x^4ax^4 \asymp 0$. Thus, $d(x^j) \leqslant C_{1,D} = 1$ for $j = 2, 3, 4$.

The obtained restrictions on $d(x^j)$ for $1 \leqslant j \leqslant 4$ imply that $\deg w \leqslant 12d$. Hence, $C_{5,d} \leqslant 12d + 1$.

3. Polynomial bound

This section is dedicated to the proof of the next result.

Corollary 3.1. *If $p > \frac{n}{2}$, then $C_{n,d} < n^{\log_2(3d+2)+1}$.*

Theorem 2.5 together with the inequality $C_{j-1,d} \leqslant C_{j,d}$ for all $j \geqslant 2$ implies that

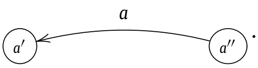
$$C_{n,d} \leqslant d\sum_{j=1}^{k}\gamma_j C_{[n/2^j],d} + 1$$

for $\gamma_j = (2^j - 1) + 2^j + \cdots + (2^{j+1} - 2) = 3(2^j - 1)2^{j-1}$ and $k > 0$ satisfying $1 \leqslant \frac{n}{2^k} < 2$. Thus,

$$C_{n,d} < \frac{3d}{2}\sum_{j=1}^{k}4^j C_{[n/2^j],d}, \tag{6}$$

where $\frac{n}{2} < 2^k \leqslant n$.

Let us fix some notations. If a is an arrow in an oriented graph, then we denote the head of a by a' and the tail of a by a'', i.e.,

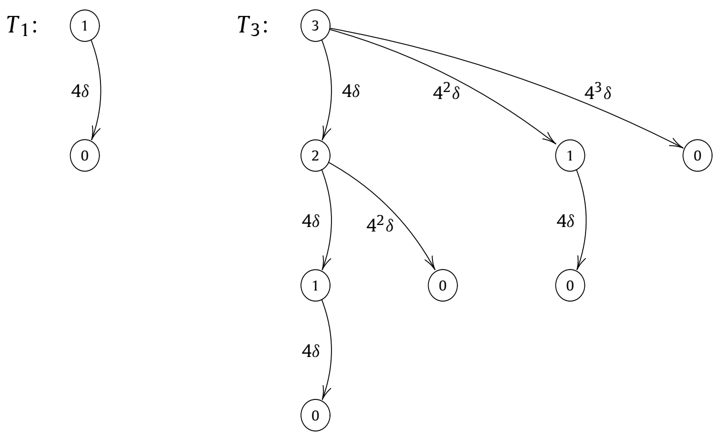


We say that a'' is a *predecessor* of a' and a' is a *successor* of a''.

For every $l \geqslant 1$ we construct an oriented tree T_l as follows.

- The underlying graph of T_l is a tree.
- Vertices of T_l are marked with $0, \dots, l$.
- Let a vertex v be marked with i. Then v has exactly i successors, marked with $0, 1, \dots, i-1$. If $i < l$, then v has exactly one predecessor. If $i = l$, then v does not have a predecessor and it is called the *root* of T_l.
- If a is an arrow of T_l and a', a'' are marked with i, j, respectively, then a is marked with $4^{j-i}\delta$, where $\delta = 3d/2$.

Example 3.2.



Here we write a number that is prescribed to a vertex (an arrow, respectively) in this vertex (near this arrow, respectively).

If b is an oriented path in T_l, then we write $\deg b$ for the number of arrows in b and $|b|$ for the product of numbers assigned to arrows of b. Denote by P_l the set of maximal (by degree) paths in T_l. Note that there is 1-to-1 correspondence between P_l and the set of leaves of T_l, i.e., vertices marked with 0. We claim that

$$C_{n,d} < \sum_{b \in P_k} |b|.$$

To prove this statement we use induction on $n \geqslant 2$. If $n = 2$, then $k = 1$ and $C_{2,d} < 4\delta$ by (6), and therefore the statement holds. For $n > 2$ formulas (6) and $[[n/2^{j_1}]/2^{j_2}] = [n/2^{j_1+j_2}]$ for all $j_1, j_2 > 0$ together with the induction hypothesis imply that

$$C_{n,d} < \sum_{j=1}^{k} \sum_{b \in P_{k-j}} 4^j \delta \, |b|.$$

The statement is proven.

Since the sum of exponents of 4 along every maximal path is k, we obtain that

$$C_{n,d} < \sum_{b \in P_k} 4^k \left(\frac{3d}{2}\right)^{\deg b}. \tag{7}$$

Given $1 \leqslant r \leqslant k$, denote by $P_{k,r}$ the set of $b \in P_k$ with $\deg b = r$. We claim that

$$\#P_{k,r} = \binom{k-1}{r-1}, \tag{8}$$

where $\#P_{k,r}$ stands for the cardinality of $P_{k,r}$. To prove the claim we notice that $P_{k,r}$ is the set of r-tuples $(j_1, \ldots, j_r)$ satisfying $j_1, \ldots, j_r \geqslant 1$ and $j_1 + \cdots + j_r = k$. Hence $\#P_{k,r}$ is equal to the cardinality of the set of all $(r-1)$-tuples $(q_1, \ldots, q_{r-1})$ such that $1 \leqslant q_1 < \cdots < q_{r-1} \leqslant k-1$ since we can set $j_1 = q_1, j_2 = q_2 - q_1, \ldots, j_r = k - q_{r-1}$. The claim is proven.

Applying (8) to inequality (7), we obtain

$$C_{n,d} < 4^k \sum_{r=1}^{k} \left(\frac{3d}{2}\right)^r \binom{k-1}{r-1} = 4^k \frac{3d}{2} \sum_{r=0}^{k-1} \left(\frac{3d}{2}\right)^r \binom{k-1}{r} = 4^k \frac{3d}{2}\left(1 + \frac{3d}{2}\right)^{k-1}.$$

Thus,

$$C_{n,d} < 4^k \left(1 + \frac{3d}{2}\right)^k.$$

Since $2^k \leqslant n$, we have

$$C_{n,d} < n^2 \left(1 + \frac{3d}{2}\right)^{\log_2(n)} = n^{\log_2(1+\frac{3d}{2})+2} = n^{\log_2(3d+2)+1}.$$

Corollary 3.1 is proven.

4. Corollaries

Corollary 4.1. *Let $p > \frac{n}{2}$. Then $C_{n,d} < 4 \cdot 2^{n/2} d$. Moreover, if $n \geqslant 30$, then $C_{n,d} < 2 \cdot 2^{n/2} d$.*

We split the proof of Corollary 4.1 into several lemmas. Let $m = [n/2]$. For $2 \leqslant i \leqslant m$ denote $\gamma_i = (i-1)2^{n/i}$ and $\delta_n = 2^{n/2} + 2^{n/3}(n-4) + \frac{1}{4}(n+1)^2$.

Lemma 4.2. *For $3 \leqslant i \leqslant m$ the inequality $\gamma_i \leqslant \gamma_3$ holds.*

Proof. The required inequality is equivalent to the following one:

$$i - 1 \leqslant 2 \cdot 2^{n\frac{i-3}{3i}}. \tag{9}$$

Let $i = 4$. Then $n \geqslant 8$ and it is not difficult to see that the inequality $3 \leqslant 2 \cdot 2^{n/12}$ holds.

Let $i \geqslant 5$. Then inequality (9) follows from $i - 1 \leqslant 2 \cdot 2^{2n/15}$. Since $i - 1 \leqslant \frac{n}{2}$, the last inequality follows from $n \leqslant 4 \cdot 2^{2n/15}$, which holds for all $n \geqslant 2$. □

Lemma 4.3. *For $n \geqslant 2$ the inequality $\delta_n \leqslant 4 \cdot 2^{n/2} - 1$ holds. Moreover, $\delta_n \leqslant 2 \cdot 2^{n/2} - 1$ in case $n \geqslant 30$.*

Proof. Let $n \geqslant 30$. Then it is not difficult to see that $2 \cdot 2^{n/2} - 1 - \delta_n = (2^{n/2} - n \cdot 2^{n/3}) + (4 \cdot 2^{n/3} - \frac{1}{4}(n+1)^2 - 1) \geqslant 0$. If $2 \leqslant n < 30$, then performing calculations we can see that the claim of the lemma holds. □

Now we can prove Corollary 4.1:

Proof of Corollary 4.1. If $n = 2$ or $n = 3$, respectively, then $C_{n,d} \leqslant \max\{3, d\}$ or $C_{n,d} \leqslant 3d + 1$, respectively (see Section 1), and the required is proven.

Assume that $n \geqslant 4$. By Remark 2.6, $C_{n,d} \leqslant A_n d + 1$. Since $p > [n/i]$ for $2 \leqslant i \leqslant m$, the Nagata–Higman Theorem implies $C_{[n/i],d} \leqslant 2^{n/i} - 1$. Thus,

$$A_n \leqslant \sum_{2 \leqslant i \leqslant m} \gamma_i + \beta_n,$$

where $\beta_n = \frac{1}{2}(-m(m-1) + (m+n-1)(n-m))$. Separately considering the cases of n even and odd, we obtain that $\beta_n \leqslant (n+1)^2/4$. Since $m \geqslant 2$, Lemma 4.2 implies that

$$\sum_{2 \leqslant i \leqslant m} \gamma_i \leqslant \gamma_2 + \gamma_3(m-2).$$

It follows from the above mentioned upper bound on β_n and the inequality $m \leqslant \frac{n}{2}$ that $A_n \leqslant \delta_n$. Lemma 4.3 completes the proof. □

To prove Corollary 4.5 (see below) we need the following slight improvement of the upper bound from Nagata–Higman Theorem.

Lemma 4.4. *If $p > n$, then $C_{n,d} < 7 \cdot 2^{n-3}$ for all $n \geqslant 3$.*

Proof. If $n = 3$, then the claim of the lemma follows from $C_{3,d} = 6$ (see Section 1).

It is well known that

$$nx^{n-1}ay^{n-1} = 0 \tag{10}$$

in $N_{n,d}$ for all a, x, y (see [10]). Thus, $C_{n,d} \leqslant 2C_{n-1,d} + 1$. Applying this formula recursively, we obtain that $C_{n,d} \leqslant 2^{n-3}C_{3,d} + \sum_{i=0}^{n-4} 2^i$ for $n \geqslant 4$. Since $p > 4$, the equality $C_{3,d} = 6$ concludes the proof. □

Corollary 4.5. *Let $4 \leqslant n \leqslant 9$ and $\frac{n}{2} < p \leqslant n$. Then $C_{n,d} \leqslant a_n d + 1$, where $a_4 = 8$, $a_5 = 12$, $a_6 = 24$, $a_7 = 30$, $a_8 = 50$, $a_9 = 64$.*

Proof. We have $C_{2,d} = 3$ in case $p > 2$ and $C_{3,d} = 6$ in case $p > 3$ (see Section 1). By Lemma 4.4, $C_{4,d} \leqslant 13$ in case $p > 4$. Applying the upper bound on $C_{n,d}$ from Theorem 2.5 recursively and using the above given estimations on $C_{k,d}$ for $k = 2, 3, 4$, we obtain the required. □

The following conjecture is a generalization of Razmyslov's upper bound to the case of $p > n$ and it holds for $n = 2, 3$:

Conjecture 4.6. *For all $n, d \geqslant 2$ and $p > n$ we have $C_{n,d} \leqslant n^2$.*

Corollary 4.7. *Assume that Conjecture 4.6 holds. Then $C_{n,d} < n^2 \ln(n) d$ for $\frac{n}{2} < p \leqslant n$.*

Proof. For $n=2,3$ the claim holds by Section 1.

Assume that $n\geqslant 4$. By Remark 2.6, $C_{n,d}\leqslant A_n d+1$. Since $p>[n/i]$, Conjecture 4.6 implies

$$A_n\leqslant \sum_{2\leqslant i\leqslant m}(i-1)\frac{n^2}{i^2}+\beta'_n,$$

where $\beta'_n=\frac{1}{2}(m+n-1)(n-m)$. Separately considering the cases of n even and odd, we obtain that $\beta'_n\leqslant 3n^2/8$. Denote by ξ_m the mth harmonic number $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}$. We have

$$A_n<n^2(\xi_m-1)+\frac{3}{8}n^2-1.$$

Since $\xi_m<\ln m+\gamma+\frac{1}{2m}$, where $\gamma<1$ is Euler's constant (for example, see pages 73 and 79 of [11]),

$$A_n<n^2\left(\ln m+\frac{5}{8}\right)-1<n^2\ln(n)-1$$

and we obtain the required inequality. □

Remark 4.8. Using another approach, in recent paper [4] Belov and Kharitonov obtained the following upper bounds on $C_{n,d}$ for all p:

(1) $C_{n,d}\leqslant 4^{\log_3(64)+5}\cdot(n^{12})^{\log_3(4n)+1}d$ (Corollary 1.16 from [4]);
(2) $C_{n,d}\leqslant 256\cdot n^{8\log_2(n)+22}d$ (see Theorem 1.17 from [4]);

where the second estimation is better for small n. These bounds are linear with respect to d and subexponential with respect to n.

Let us compare bounds (1) and (2) with the bound from Corollary 4.1 in case $p>\frac{n}{2}$: $C_{n,d}<4\cdot 2^{n/2}d$. If $n\gg 0$ is large enough, then bounds (1) and (2) are essentially better than the bound from Corollary 4.1. On the other hand, for $4\leqslant n\leqslant 2000$ the bound from Corollary 4.1 is at least 10^{20} times better than bounds (1) and (2). This claim follows from straightforward computations.

5. The case of $n=4$

Theorem 5.1. *For $d\geqslant 2$ we have*

- $C_{4,d}=10$, *if* $p=0$;
- $3d<C_{4,d}$, *if* $p=2$;
- $3d+1\leqslant C_{4,d}\leqslant 3d+4$, *if* $p=3$;
- $10\leqslant C_{4,d}\leqslant 13$, *if* $p>3$.

In what follows we assume that $n=4$ and $p\neq 2$ unless otherwise stated. To prove Theorem 5.1 (see the end of the section), we introduce a new $\approx$-equivalence on $\mathcal{M}_{\mathbb{F}}$ as follows. Given $\underline{\alpha}\in\mathbb{N}^r$ and $\underline{\beta}\in\mathbb{N}^s$ $(r,s\geqslant 0)$, we write

$$\underline{\alpha}\succ\underline{\beta}\quad\text{if } r<s.$$

Using $\succ$ instead of $>$, we introduce the partial order $\succ$ on $\mathcal{M}$ similarly to Definition 2.2. Then, using the partial order $\succ$ on $\mathcal{M}$ instead of $>$, we introduce the $\approx$-equivalence on $\mathcal{M}_{\mathbb{F}}$ similarly to the $\asymp$-equivalence (see Definition 2.4). The resulting definition of $\approx$ is the following one:

Definition 5.2 *(of the $\approx$-equivalence on $\mathcal{M}_{\mathbb{F}}$).*

1. Let $f=\sum_i \alpha_i a_i \in \mathcal{M}_{\mathbb{F}}$, where $\alpha_i \in \mathbb{F}^*$, $a_i \in \mathcal{M}$, and $\#\mathrm{pwr}_y(a_i)=\#\mathrm{pwr}_y(a_{i'})$ for every letter y and all i, i'. Then $f \approx 0$ if $f=0$ in $N_{n,d}$ or $f=\sum_j \beta_j b_j$ in $N_{n,d}$ for $\beta_j \in \mathbb{F}^*$, $b_j \in \mathcal{M}$ satisfying
   - $\#\mathrm{pwr}_x(a_i) > \#\mathrm{pwr}_x(b_j)$ for some letter x,
   - $\#\mathrm{pwr}_y(a_i) \geqslant \#\mathrm{pwr}_y(b_j)$ for every letter y,

   for all i, j;
2. If $f=\sum_k f_k \in \mathcal{M}_{\mathbb{F}}$ and $f_k \approx 0$ satisfies conditions from part 1 for all k, then $f \approx 0$.

Given $h \in \mathcal{M}_{\mathbb{F}}$, we write $f \approx h$ if $f-h \approx 0$.

Remark 5.3. Note that the partial order $>$ on $\mathcal{M}$ is stronger than $\succ$. Namely, for $a, b \in \mathcal{M}$ we have

- if $a \succ b$, then $a > b$;
- if $a > b$, then $a \succ b$ or $a \approx b$.

Therefore, $\asymp$-equivalence on $\mathcal{M}_{\mathbb{F}}$ is weaker than $\approx$-equivalence. Namely, for $f, h \in \mathcal{M}_{\mathbb{F}}$ the equality $f \approx h$ implies $f \asymp h$, but the converse statement does not hold.

Let $a, b, c, a_1, \ldots, a_4$ be elements of $\mathcal{M}$. By definition,

- $T_4(a)=a^4$,
- $T_{31}(a,b)=a^3b+a^2ba+aba^2+ba^3$,
- $T_{211}(a,b,c)=a^2bc+a^2cb+ba^2c+ca^2b+bca^2+cba^2+abca+acba+abac+acab+baca+cabc$,
- $T_{22}(a,b)=a^2b^2+b^2a^2+abab+baba+ab^2a+ba^2b$,
- $T_{1^4}(a_1,\ldots,a_4)=\sum_{\sigma\in S_4} a_{\sigma(1)}\cdots a_{\sigma(4)}$

(see Section 2). Then

$$T_4(a)=0, \qquad T_{31}(a,b)=0, \qquad T_{211}(a,b,c)=0, \qquad T_{22}(a,b)=0, \qquad T_{1^4}(a_1,\ldots,a_4)=0$$

are relations for $N_{4,d}$, which generate the ideal of relations for $N_{4,d}$. Multiplying $T_{31}(a,b)$ by a several times we obtain that equalities

$$a^3ba+a^2ba^2+aba^3=0, \tag{11}$$

$$a^3ba^2+a^2ba^3=0, \tag{12}$$

$$a^3ba^3=0 \tag{13}$$

hold in $N_{4,d}$.

Remark 5.4. Let $f \in \mathcal{M}_{\mathbb{F}}$. Denote by $\mathrm{inv}(f)$ the element of $\mathcal{M}_{\mathbb{F}}$ that we obtain by reading f from right to left. As an example, for $f=x_1^2x_2-x_3$ we have $\mathrm{inv}(f)=-x_3+x_2x_1^2$.

Obviously, if $f=0$ in $N_{n,d}$, then $\mathrm{inv}(f)=0$ in $N_{n,d}$. Similar result also holds for $\approx$-equivalence.

Lemma 5.5. *Let x be a letter and $a, b, c \in \mathcal{M}^{\neg x}$. Then the next relations are valid in $N_{4,d}$:*

$$x^3axbx^2=-x^3ax^2bx, \qquad xax^3bx^2=x^3ax^2bx. \tag{14}$$

Moreover, the following equivalences hold:

$$xax^2 \approx -x^2ax, \tag{15}$$

$$x^iaxbx \approx 0, \qquad xax^ibx \approx 0, \qquad xaxbx^i \approx 0 \tag{16}$$

for $i = 2, 3$,

$$xaxbxcx \approx 0. \tag{17}$$

Proof. We have

$$x^3aT_{31}(x, b) = x^3ax^3b + x^3ax^2bx + x^3axbx^2 + x^3abx^3 = 0$$

in $N_{4,d}$. By equality (13), $x^3axbx^2 = -x^3ax^2bx$ in $N_{4,d}$. Similarly we can see that

$$T_{31}\bigl(x, ax^3b\bigr) = x^3ax^3b + x^2ax^3bx + xax^3bx^2 + ax^3bx^3 = x^2ax^3bx + xax^3bx^2 = 0$$

in $N_{4,d}$. By (12), $x^2ax^3bx = -x^3ax^2bx$ in $N_{4,d}$ and equalities (14) are proven.

Since $T_{31}(x, a) = 0$ in $N_{4,d}$, equivalence (15) is proven.

Let $i = 2$. By (15), $xaxbx^2 \approx -xax^2bx \approx x^2axbx$. On the other hand, (15) implies $xaxbx^2 \approx -x^2axbx$. Equivalences (16) for $i = 2$ are proven.

Let $i = 3$. Since $T_{211}(x, a, x^3b) = 0$ and $x^3T_{211}(x, a, b) = 0$ in $N_{4,d}$, we have

$$xax^3bx + x^3bxax \approx 0 \quad \text{and} \quad x^3axbx + x^3bxax \approx 0,$$

respectively. Thus, $x^3axbx \approx xax^3bx$. Using Remark 5.4, we obtain

$$x^3axbx \approx xax^3bx \approx xaxbx^3. \tag{18}$$

The equality $x^2aT_{31}(x, a) = 0$ implies

$$x^2axbx^2 + x^2ax^2bx \approx 0.$$

Applying relation (11), we obtain

$$x^3axbx + xaxbx^3 + x^3axbx + xax^3bx \approx 0.$$

Equivalences (18) complete the proof of (16).

Since $T_{211}(x, a, bxc)x = 0$ and $T_{211}(x, a, b)xcx = 0$ in $N_{4,d}$, we obtain

$$xaxbxcx + xbxcxax \approx 0 \quad \text{and} \quad xaxbxcx + xbxaxcx \approx 0,$$

respectively. The equality $xbT_{211}(x, a, c)x = 0$ in $N_{4,d}$ implies

$$xbxcxax + xbxaxcx \approx 0,$$

and therefore $xaxbxcx \approx 0$. □

If $\underline{\alpha} \in \mathbb{N}^r$, $\underline{\beta} \in \mathbb{N}^s$, then we write $\underline{\alpha} \subset \underline{\beta}$ and say that $\underline{\alpha}$ is a subvector of $\underline{\beta}$ if there are $1 \leqslant i_1 < \cdots < i_r$ such that $\alpha_1 = \beta_{i_1}, \ldots, \alpha_r = \beta_{i_r}$.

Lemma 5.6. *If $f \in \mathcal{M}_{\mathbb{F}}$, then $f = 0$ in $N_{4,d}$ or $f = \sum_i \alpha_i a_i$ in $N_{4,d}$ for some $\alpha_i \in \mathbb{F}^*$, $a_i \in \mathcal{M}$ such that for every letter x $\mathrm{pwr}_x(a_i)$ belongs to the following list*:

- $\emptyset$, (1), $(1,1)$, $(1,1,1)$,
- (2), $(2,1)$,
- (3), $(3,1)$, $(1,3)$, $(3,2)$, $(3,2,1)$.

Moreover, we can assume that for all pairwise different letters x, y, z and all i the following conditions do not hold:

(a) $\mathrm{pwr}_x(a_i) = (3,2,1)$ *and* $(3) \subset \mathrm{pwr}_y(a_i)$;
(b) (3) *is a subvector of* $\mathrm{pwr}_x(a_i)$, $\mathrm{pwr}_y(a_i)$, *and* $\mathrm{pwr}_z(a_i)$;
(c) $(3,2)$ *is a subvector of* $\mathrm{pwr}_x(a_i)$ *and* $\mathrm{pwr}_y(a_i)$.

Proof. Let x be a letter and $f = \sum_{j \in J} \beta_j b_j$ for $\beta_j \in \mathbb{F}^*$ and $b_j \in \mathcal{M}$. We claim that the statement of the lemma holds for f for the given letter x. To prove the claim we use induction on $k = \max\{\#\mathrm{pwr}_x(b_j) \mid j \in J\}$.

If $k = 0, 1$, then the claim holds.

If $b_j = b_{1j} x^2 b_{2j} x^2 b_{3j}$ for some $b_{1j}, b_{2j}, b_{3j} \in \mathcal{M}^{\neg x}$, then $a_j = -b_{1j} x^3 b_{2j} x b_{3j} - b_{1j} x b_{2j} x^3 b_{3j}$ in $N_{4,d}$ by relation (11). Note that $\#\mathrm{pwr}_x(b_j) = \#\mathrm{pwr}_x(b_{1j} x^3 b_{2j} x b_{3j}) = \#\mathrm{pwr}_x(b_{1j} x b_{2j} x^3 b_{3j})$. Moreover, if $(2, \dots, 2) \subset \mathrm{pwr}_x(b_j)$, then we apply (11) several times. Therefore, without loss of generality can assume that $(2,2)$ is not a subvector of $\mathrm{pwr}_x(b_j)$ for all j.

If one of the vectors

$$(r), \quad r > 3; \quad (3,3); \quad (s,1,1), (1,s,1), (1,1,s), \quad s \in \{2,3\}; \quad (1,1,1,1)$$

is a subvector of $\mathrm{pwr}_x(b_j)$, then $b_j \approx 0$ by the equality $x^4 = 0$ in $N_{4,d}$ and formulas (13), (16), (17), respectively. Thus, $f \approx 0$ or $f \approx \sum_{j \in J_0} \beta_j b_j$ for such $J_0 \subset J$ that for every $j \in J_0$ the vector $\mathrm{pwr}_x(b_j)$ up to permutation of its entries belongs to the following list:

$$\emptyset, (1), (1,1), (1,1,1), (2), (2,1), (3), (3,1), (3,2), (3,2,1).$$

Let $j \in J_0$. If $\mathrm{pwr}_x(b_j) = (\sigma(1), \sigma(2), \sigma(3))$ for some $\sigma \in S_3$, then applying relations (12) and (14) we obtain that $b_j = \pm c_j$ in $N_{4,d}$ for a monomial $c_j \in \mathcal{M}$ satisfying $\mathrm{pwr}_x(c_j) = (3,2,1)$. If $\mathrm{pwr}_x(b_j)$ is $(1,2)$ or $(2,3)$, then we apply formulas (15) or (12), respectively, to obtain that $b_j \approx -c_j$ for a monomial $c_j \in \mathcal{M}$ with $\mathrm{pwr}_x(c_j) \in \{(2,1), (3,2)\}$. So we get that $f \approx h$ for such $h \in \mathcal{M}_{\mathbb{F}}$ that the claim holds for h. The induction hypothesis and Definition 5.2 complete the proof of the claim.

Let y be a letter different from x. Relations from the proof of the claim do not affect y-powers. Therefore, applying the claim to f for all letters subsequently, we complete the proof of the first part of the lemma.

Consider an $a \in \mathcal{M}$. If a satisfies condition (a), then relations (12) and (14) together with relation (10) imply that $a = 0$ in $N_{4,d}$. If a satisfies condition (b) or (c), then relations (10) and (12) imply that $a = 0$ in $N_{4,d}$. Thus, the second part of the lemma is proven. □

The following lemma resembles Lemma 3.3 from [19].

Lemma 5.7. *Let $p = 2$ and $1 \leqslant k \leqslant d$. For every homogeneous $f \in \mathcal{M}_{\mathbb{F}}$ of multidegree $(\theta_1, \dots, \theta_d)$ with $\theta_k \leqslant 3$ and $\theta_1 + \dots + \theta_{k-1} + \theta_{k+1} + \dots + \theta_d > 0$ we define $\pi_k(f) \in \mathcal{M}_{\mathbb{F}}$ as the result of the substitution $x_k \to 1$ in a, where 1 stands for the unity of $\mathcal{M}_1$.*

Then $f = 0$ in $N_{4,d}$ implies $\pi_k(f) = 0$ in $N_{4,d}$.

Proof. Let $a, b, c, u \in \mathcal{M}$. By definition, $\pi_k(ab) = \pi_k(a)\pi_k(b)$. Then by straightforward calculations we can show that $\pi_k(T_{31}(a,b)) = 0$, $\pi_k(T_{211}(a,b,c)) = 0$, $\pi_k(T_{22}(a,b)) = 0$, and $\pi_k(T_{1^4}(a,b,c,u)) = 0$ in $N_{4,d}$. The proof is completed. □

We now can prove Theorem 5.1:

Proof of Theorem 5.1. If $p = 0$, then the required was proven by Vaughan-Lee in [26]. If $p > 3$, then the claim follows from Kuzmin's low bound (see Section 1) and Lemma 4.4.

Let $p = 2$ and $a = x_1^3 \cdots x_d^3$. Assume that $a = 0$ in $N_{4,d}$. Applying $\pi_1, \dots, \pi_{d-1}$ from Lemma 5.7 to a we obtain that $x_d^3 = 0$ in $N_{4,d}$; a contradiction. Thus, $C_{4,d} > \deg a = 3d$.

Assume that $p = 3$. Consider an $a \in \mathcal{M}$ such that $a \neq 0$ in $N_{4,d}$. Applying Lemma 5.6 to a, without loss of generality we can assume that a satisfies all conditions from Lemma 5.6. Denote $t_i = \deg_{x_i}(a)$ and $r = \#\{i \mid (3) \text{ is subvector of } \mathrm{pwr}_{x_i}(a)\}$. Then

(a) $t_i \leqslant 6$;
(b) if $t_i \geqslant 4$, then $(3) \subset \mathrm{pwr}_{x_i}(a)$

for all $1 \leqslant i \leqslant d$.

If $r = 0$, then $\deg(a) \leqslant 3d$ by part (b). If $r = 1$, then $\deg(a) \leqslant 6 + 3(d-1) = 3d + 3$ by parts (a) and (b).

Let $r = 2$. Then without loss of generality we can assume that (3) is a subvector of $\mathrm{pwr}_{x_1}(a)$ and $\mathrm{pwr}_{x_2}(a)$. Since condition (a) of Lemma 5.6 does not hold for a, $(3, 2, 1)$ is not a subvector of $\mathrm{pwr}_{x_i}(a)$ for $i = 1, 2$. Hence, $t_1, t_2 < 6$. If $t_1 = t_2 = 5$, then condition (c) of Lemma 5.6 holds for a; a contradiction. Therefore, $t_1 + t_2 \leqslant 9$. By part (b), $t_i \leqslant 3$ for $3 \leqslant i \leqslant d$. Finally, we obtain that $\deg(a) \leqslant 3d + 3$.

If $r \geqslant 3$, then a satisfies condition (b) of Lemma 5.6; a contradiction.

So, we have shown that $\deg(a) \leqslant 3d + 3$, and therefore $C_{4,d} \leqslant 3d + 4$. On the other hand, $C_{4,d} \geqslant C_{3,d} = 3d + 1$ by [16]. The proof is completed. □

Remark 5.8. Assume that $n = 4$ and $p = 3$. Let us compare the upper bound $C_{4,d} \leqslant 3d + 3$ from Theorem 5.1 with the known upper bounds on $C_{4,d}$:

- Corollary 4.5 implies that $C_{4,d} < 8d + 1$;
- bounds by Belov and Kharitonov [4] imply that $C_{4,d} \leqslant B_4 d$, where $B_4 > 10^{20}$ (see Remark 4.8 for details);
- bounds by Klein [13] imply that $C_{4,d} < \frac{2^{11}}{3} d^4$ and $C_{4,d} < 2^{128} d^2$ (see Section 1 for details).

6. *GL*(*n*)-invariants of matrices

The general linear group $GL(n)$ acts on d-tuples $V = (\mathbb{F}^{n \times n})^{\oplus d}$ of $n \times n$ matrices over $\mathbb{F}$ by the diagonal conjugation, i.e.,

$$g \cdot (A_1, \dots, A_d) = \left(gA_1g^{-1}, \dots, gA_dg^{-1}\right), \tag{19}$$

where $g \in GL(n)$ and $A_1, \dots, A_d$ lie in $\mathbb{F}^{n \times n}$. The coordinate algebra of the affine variety V is the algebra of polynomials $R = \mathbb{F}[V] = \mathbb{F}[x_{ij}(k) \mid 1 \leqslant i, j \leqslant n,\ 1 \leqslant k \leqslant d]$ in $n^2 d$ variables. Denote by

$$X_k = \begin{pmatrix} x_{11}(k) & \cdots & x_{1n}(k) \\ \vdots & & \vdots \\ x_{n1}(k) & \cdots & x_{nn}(k) \end{pmatrix}$$

the kth *generic* matrix. The action of $GL(n)$ on V induces the action on R as follows:

$$g \cdot x_{ij}(k) = (i, j)\text{th} \quad \text{entry of } g^{-1} X_k g$$

for all $g \in GL(n)$. The algebra of *$GL(n)$-invariants of matrices* is

$$R^{GL(n)} = \{ f \in \mathbb{F}[V] \mid g \cdot f = f \text{ for all } g \in GL(n) \}.$$

Denote coefficients in the characteristic polynomial of an $n \times n$ matrix X by $\sigma_t(X)$, i.e.,

$$\det(X + \lambda E) = \sum_{t=0}^{n} \lambda^{n-t} \sigma_t(X). \tag{20}$$

In particular, $\sigma_0(X) = 1$, $\sigma_1(X) = \mathrm{tr}(X)$, and $\sigma_n(X) = \det(X)$.

Given $a = x_{i_1} \cdots x_{i_r} \in \mathcal{M}$, we set $X_a = X_{i_1} \cdots X_{i_r}$. It is known that the algebra $R^{GL(n)} \subset R$ is generated over $\mathbb{F}$ by $\sigma_t(X_a)$, where $1 \leqslant t \leqslant n$ and $a \in \mathcal{M}$ (see [7]). Note that in the case of $p = 0$ the algebra $R^{GL(n)}$ is generated by $\mathrm{tr}(X_a)$, where $a \in \mathcal{M}$. Relations between the mentioned generators were established in [28].

Remark 6.1. If G belongs to the list $O(n)$, $Sp(n)$, $SO(n)$, $SL(n)$, then we can define the algebra of invariants R^G in the same way as for $G = GL(n)$. A generating set for the algebra R^G is known, where we assume that $\mathrm{char}\,\mathbb{F} \neq 2$ in the case of $O(n)$ and $SO(n)$ (see [29,18]). In case $p = 0$ and $G \neq SO(n)$ relations between generators of R^G were described in [23]. In case $p \neq 2$ relations for $R^{O(n)}$ were described in [20,21].

By the Hilbert–Nagata Theorem on invariants, $R^{GL(n)}$ is a finitely generated $\mathbb{N}_0$-graded algebra by degrees, where $\deg \sigma_t(X_a) = t \deg a$ for $a \in \mathcal{M}$. But the above mentioned generating set is not finite. In [5] the following finite generating set for $R^{GL(n)}$ was established:

- $\sigma_t(X_a)$, where $1 \leqslant t \leqslant \frac{n}{2}$, $a \in \mathcal{M}$, $\deg a \leqslant C_{n,d}$;
- $\sigma_t(X_i)$, where $\frac{n}{2} < t \leqslant n$, $1 \leqslant i \leqslant d$.

We obtain a smaller generating set.

Theorem 6.2. *The algebra $R^{GL(n)}$ is generated by the following finite set*:

- $\sigma_t(X_a)$, *where $t = 1$ or $p \leqslant t \leqslant \frac{n}{2}$, $a \in \mathcal{M}$, $\deg a \leqslant C_{[n/t],d}$*;
- $\sigma_t(X_i)$, *where $\frac{n}{2} < t \leqslant n$, $p \leqslant t$, $1 \leqslant i \leqslant d$.*

To prove the theorem, we need the following notions. Let $1 \leqslant t \leqslant n$. For short, we write $\sigma_t(a)$ for $\sigma_t(X_a)$, where $a \in \mathcal{M}$. Amitsur's formula [1] enables us to consider $\sigma_t(a)$ with $a \in \mathcal{M}_{\mathbb{F}}$ as an invariant from $R^{GL(n)}$ for all $t \in \mathbb{N}$. Zubkov [28] established that the ideal of relations for $R^{GL(n)}$ is generated by $\sigma_t(a) = 0$, where $t > n$ and $a \in \mathcal{M}_{\mathbb{F}}$. More details can be found, for example, in [20]. Denote by $I(t)$ the $\mathbb{F}$-span of elements $\sigma_{t_1}(a_1) \cdots \sigma_{t_r}(a_r)$, where $r > 0$, $1 \leqslant t_1, \ldots, t_r \leqslant t$, and $a_1, \ldots, a_r \in \mathcal{M}$. For short, we write I for $I(n) = R^{GL(n)}$. Denote by I^+ the subalgebra generated by $\mathbb{N}_0$-homogeneous elements of I of positive degree. Obviously, the algebra I is generated by a set $\{f_k\} \subset I$ if and only if $\{\overline{f_k}\}$ is a basis of $\bar{I} = I/(I^+)^2$. Given an $f \in I$, we write $f \equiv 0$ if $\overline{f} = 0$ in $\bar{I}$, i.e., f is equal to a polynomial in elements of strictly lower degree.

Proof of Theorem 6.2. Let $1 \leqslant t \leqslant n$, $m = [n/t]$, and $a, b \in \mathcal{M}_{\mathbb{F}}$. We claim that

$$\text{there exists an } f \in I(t-1) \text{ such that } \sigma_t(ab^m) \equiv f. \tag{21}$$

To prove the claim we notice that the inequality $(m+1)t > n$ and the description of relations for $R^{GL(n)}$ imply $\sigma_{(m+1)t}(a+b) = 0$. Taking homogeneous component of degree t with respect to a and degree mt with respect to b, we obtain that $\sigma_t(ab^m) \equiv 0$ or $\sigma_t(ab^m) \equiv \sum_i \alpha_i \sigma_{t_i}(a_i)$, where $\alpha_i \in \mathbb{F}^*$, $1 \leqslant t_i < t$, and a_i is a monomial in a and b for all i. By Amitsur's formula, $\sigma_{t_i}(a_i) \equiv \sum_j \beta_{ij} \sigma_{r_{ij}}(b_{ij})$ for some $\beta_{ij} \in \mathbb{F}^*$, $1 \leqslant r_{ij} \leqslant t_i$, $b_{ij} \in \mathcal{M}$. Thus, $\sum_i \alpha_i \sigma_{t_i}(a_i) \in I(t-1)$ and the claim is proven.

Consider a monomial $c \in \mathcal{M}$ satisfying $\deg c > C_{m,d}$. Then $c = c'x$ for some letter x and $c' \in \mathcal{M}$. Since $c' = 0$ in $N_{m,d}$, we have $c' = \sum_i \gamma_i u_i v_i^m w_i$ for some $u_i, w_i \in \mathcal{M}_1$, $v_i \in \mathcal{M}_{\mathbb{F}}$, $\gamma_i \in \mathbb{F}$. Thus $\sigma_t(c) = \sigma_t(\sum_i \alpha_i u_i v_i^m w_i x)$. Applying Amitsur's formula, we obtain that $\sigma_t(c) - \sum_i \alpha_i^t \sigma_t(u_i v_i^m w_i x) \in I(t-1)$. Statement (21) implies

$$\sigma_t(c) \equiv h \quad \text{for some } h \in I(t-1). \tag{22}$$

Consecutively applying (22) to $t = n, n-1, \ldots, 2$ we obtain that $R^{GL(n)}$ is generated by $\sigma_t(a)$, where $1 \leqslant t \leqslant n$, $a \in \mathcal{M}$, $\deg a \leqslant C_{[n/t],d}$. Note that if $t > \frac{n}{2}$, then $m = 1$ and $C_{m,d} = 1$. If $t < p \leqslant n$, then the Newton formulas imply that $\sigma_t(a)$ is a polynomial in $\mathrm{tr}(a^i)$, $i > 0$ (the explicit expression can be found, for example, in Lemma 10 of [17]). The last two remarks complete the proof. □

Conjecture 6.3. *The algebra $R^{GL(n)}$ is generated by elements of degree less or equal to $C_{n,d}$.*

Remark 6.4. Theorem 6.2 and the inequality $C_{n,d} \geqslant n$ imply that to prove Conjecture 6.3 it is enough to show that

$$tC_{[n/t],d} \leqslant C_{n,d}$$

for all t satisfying $p \leqslant t \leqslant \frac{n}{2}$. Thus it is not difficult to see that Conjecture 6.3 holds for $n \leqslant 5$. Moreover, as it was proven in [5] (and also follows from Theorem 6.2), Conjecture 6.3 holds in case $p = 0$ or $p > \frac{n}{2}$.

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